Some Binary Minimal Clones on a Finite Set

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1 Introduction

According to Type Theorem of I. G. Rosenberg ([Ro86]), minimal functions, i.e., generators of minimal clones with minimal arity, on a finite set are classified into five types.

(1) unary function
(2) binary idempotent function
(3) ternary majority function
(4) ternary minority function
(5) semiprojection

While the minimal functions of type (1) and type (4) have been completely determined, the problem of characterizing minimal functions of types (2), (3) and (5) still remain unsolved. In fact, the problem of determining all minimal clones is considered to be one of the most difficult problems in clone theory.

In ([BM20]), we considered some specific kind of minimal functions of type (2); binary idempotent functions. This article is a brief report of the results obtained there.

> Behrisch, M. and Machida, H., "On minimality of some binary clones related to unary functions", to appear in *Proceedings 50th International Symposium on Multiple-Valued Logic*, IEEE, 2020.

This paper will appear in a symposium proceedings, which is not easily accessible to those who do not attend the symposium. So, it would be of some value to present the results here.

2 Preliminaries

For $k > 1$, let $E_k = \{0, 1, \ldots, k-1\}$. Denote by $\mathcal{O}_k^{(n)}$, for any $n > 0$, the set of n-variable functions on E_k and by $\mathcal{O}_k$ the set of all functions on E_k, i.e., $\mathcal{O}_k = \bigcup_{n>0} \mathcal{O}_k^{(n)}$. The n-variable i-th *projection* e_i^n, $1 \le i \le n$, is the function in $\mathcal{O}_k^{(n)}$ defined by $e_i^n(x_1, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n \in E_k$. Denote by $\mathcal{J}_k$ the set of projections on E_k.

A subset C of $\mathcal{O}_k$ is a *clone* on E_k if C contains all the projections, i.e., $\mathcal{J}_k \subseteq C$, and is closed under (functional) composition. The set $\mathcal{L}_k$ of clones on E_k forms a lattice with respect to inclusion. It is well known that the cardinality of $\mathcal{L}_k$ is of continuum for every $k \ge 3$.

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For a clone $C \in \mathcal{L}_k$ and a subset F of C, F is said to *generate* C if C is the smallest clone containing F. When F generates C, we write $C = \langle F \rangle$. If F is a singleton, i.e., $F = \{f\}$, we simply write $\langle f \rangle$ instead of $\langle F \rangle$.

A clone C is called a *binary clone* if it is generated by a binary function, i.e., if $C = \langle f \rangle$ for some $f \in \mathcal{O}_k^{(2)}$.

A clone C in $\mathcal{L}_k$ is a *minimal clone* if it is an atom of $\mathcal{L}_k$. Equivalently, $C\,(\neq \mathcal{J}_k)$ is a minimal clone if $\mathcal{J}_k \subset C' \subseteq C$ implies $C' = C$ for any C' in $\mathcal{L}_k$. A minimal clone C is generated by a single function, i.e., $C = \langle f \rangle$ for some f in $\mathcal{O}_k \setminus \mathcal{J}_k$. A function f is a *minimal function* if it generates a minimal clone C and its arity is minimum among functions generating C.

As stated in Section 1, a minimal function is of one of the five types, and minimal functions of type (1) and (4) have been completely characterized. The case of type (1) is easy; a unary function s is minimal if and only if either it is a permutation of prime order or a non-surjective function satisfying $s^2 = s$. The case of type (4) was solved in [Ro86]; a minority function m is minimal if and only if $k = 2^r$ for some $r > 0$ and $m(x, y, z) = x + y + z$ for all $x, y, z \in E_k$ where $(E_2^r; +)$ is the elementary 2-group. (The operation $+$ is the component-wise addition over $\mathrm{GF}(2)$.)

For $k = 2$, i.e., the case of Boolean functions, there are 7 minimal clones, which are easily obtained from the Post lattice ([Po41]). For $k = 3$, it is known ([Cs83]) that there are 84 minimal clones. Among them, 48 are generated by binary idempotent functions.

In this article, we consider binary minimal clones, i.e., minimal clones generated by binary idempotent functions. A binary function $f \in \mathcal{O}_k^{(2)}$ is *idempotent* if $f(x, x) = x$ for all $x \in E_k$.

A unary function s on E_k naturally induces a directed graph (V, A) where $V = E_k$ and $A = \{\, (x, s(x)) \mid x \in E_k \,\}$. This graph will be denoted by $\Gamma(s)$. For every vertex $x \in V$, either x is in a cycle or there is a simple path connecting x and some vertex in some cycle.

3 Two Classes of Binary Minimal Functions

Two particular classes $\mathcal{F}_1$ and $\mathcal{F}_2$ of binary idempotent functions are introduced, and we shall consider minimal functions in these classes.

A unary function $s \in \mathcal{O}_k^{(1)}$ is *reflective* (or *retractive*) if it satisfies $s^2\,(=\ s \circ s) = s$, equivalently, s is the identity on $\mathrm{Im}(s)$. (Note: A more standard term may be idempotent which, however, is used for another meaning in this article.)

3.1 The Class $\mathcal{F}_1$

For a unary function $s \in \mathcal{O}_k^{(1)}$, we define a binary function $\varphi_s \in \mathcal{O}_k^{(2)}$ by

$$\varphi_s(x, y) = \begin{cases} s(y) & \text{if} \quad x = 0, \\ x & \text{if} \quad x \in E_k \setminus \{0\}. \end{cases}$$

When s is 0-preserving, i.e., $s(0) = 0$, φ_s is idempotent. The Cayley table of φ_s is shown below.

$x\backslash y$	0	1	2	$\cdots$	$k-1$
0	0	$s(1)$	$s(2)$	$\cdots$	$s(k-1)$
1	1	1	1	$\cdots$	1
2	2	2	2	$\cdots$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$k-1$	k–1	k–1	k–1	$\cdots$	k–1

Now, $\mathcal{F}_1$ is defined to be the set of binary idempotent functions φ_s with 0-preserving unary functions s, i.e.,

$$\mathcal{F}_1 \;=\; \{\, \varphi_s \in \mathcal{O}_k^{(2)} \mid s \in \mathcal{O}_k^{(1)},\ s(0) = 0 \,\}.$$

For any $f \in \mathcal{F}_1$, if $f = \varphi_s$ we denote s by s_f. In other words, s_f is the unary function satisfying $s_f(y) = f(0, y)$ for all $y \in E_k$.

The following characterizes the minimal functions in $\boldsymbol{\mathcal{F}_1}$. Here, $c_0\,(\in\, \mathcal{O}_k^{(1)})$ denotes the unary constant function taking the value 0.

Proposition 3.1 *For any $f \in \mathcal{F}_1$, f is a minimal function if and only if either of the following conditions is satisfied.*

(1) s_f is reflective, i.e., $s_f^2 = s_f$, and $s_f \neq c_0$,
(2) $s_f^2 = c_0$ and $s_f \neq c_0$.

Let us describe the conditions (1) and (2) in terms of the associated graph $\Gamma(s_f)$ of s_f. Let $f \in \mathcal{F}_1$ be a function satisfying (1) or (2). Since $\Gamma(s_f)$ has a loop leaving and entering the vertex 0, we have $\#_{\mathrm{loop}}(\Gamma(s_f)) \geq 1$, where $\#_{\mathrm{loop}}(\Gamma(s_f))$ denotes the number of loops in $\Gamma(s_f)$. Moreover, it is easy to see that $\Gamma(s_f)$ is cycle-free, except loops. Define the *height* of $\Gamma(s_f)$, denoted by $\mathrm{height}(\Gamma(s_f))$, as the length of a longest simple path connecting some vertex and some loop-vertex.

The conditions given in Proposition 3.1 can be expressed with respect to the graph $\Gamma(s_f)$ as follows.

(1) $\Gamma(s_f)$ has no cycles except loops, $\#_{\mathrm{loop}}(\Gamma(s_f)) \geq 2$ and $\mathrm{height}(\Gamma(s_f)) \leq 1$.
(2) $\Gamma(s_f)$ has no cycles except loops, $\#_{\mathrm{loop}}(\Gamma(s_f)) = 1$ and $\mathrm{height}(\Gamma(s_f)) = 2$.

Examples of graphs satisfying these conditions are depicted in Figure 1.

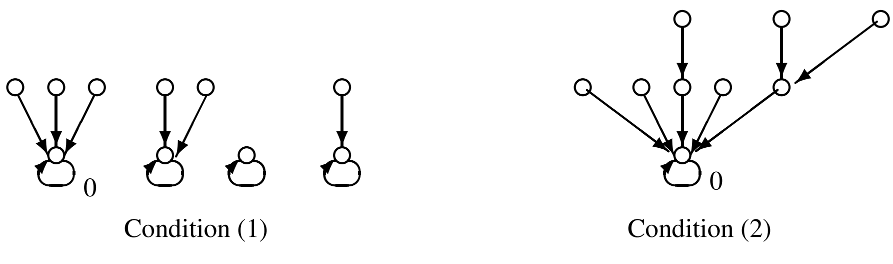

Figure 1: Examples of $\Gamma(s_f)$ for Conditions (1) and (2)

4 The Class $\mathcal{F}_2$

For a unary function $s \in \mathcal{O}_k^{(1)}$ we define a binary function $\psi_s \in \mathcal{O}_k^{(2)}$ by

$$\psi_s(x,y) = \begin{cases} s(y) & \text{if } (x,y) \in \{0\} \times E_k, \\ s(x) & \text{if } (x,y) \in E_k \times \{0\}, \\ \min\{x,y\} & \text{if } (x,y) \in (E_k \backslash \{0\})^2. \end{cases}$$

Furthermore, we assume s to be 0-preserving, i.e., $s(0) = 0$. As ψ_s is commutative, the Cayley table of ψ_s is "symmetric" as shown below.

$x \backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	$s(1)$	$s(2)$	$s(3)$	$\cdots$	$s(k-1)$
1	$s(1)$	1	1	1	$\cdots$	1
2	$s(2)$	1	2	2	$\cdots$	2
1	$s(3)$	1	2	3	$\cdots$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$k-1$	$s(k-1)$	1	2	3	$\cdots$	k–1

We define $\mathcal{F}_2$ to be the collection of such binary idempotent commutative functions ψ_s.

$$\mathcal{F}_2 \;=\; \{\; \psi_s \in \mathcal{O}_k^{(2)} \mid s \in \mathcal{O}_k^{(1)},\; s(0) = 0 \;\}$$

For any $f \in \mathcal{F}_2$, the unary function $s \in \mathcal{O}_k^{(1)}$ such that $f = \psi_s$ will be denoted by s_f. In other words, s_f is a function which satisfies $s_f(z) = f(0,z)$ $(= f(z,0))$ for all $z \in E_k$.

Assuming E_k as the initial segment of the set $\mathbf{N}$ of natural numbers, we introduce the order $\leq$ on E_k induced from the natural order on $\mathbf{N}$, i.e., $0 < 1 < 2 < \cdots < k-1$ is assumed. With respect to this ordering, we say a unary function $s \in \mathcal{O}_k^{(1)}$ is *intensional* if $s(z) \leq z$ for all $z \in E_k$.

The main result for $f \in \mathcal{F}_2$ is the following.

Proposition 4.1 *For any $f \in \mathcal{F}_2$, if s_f is reflective and intensional, then f is a minimal function. On the other hand, if f is a minimal function, then s_f is intensional.*

As an example, take a unary function $s \in \mathcal{O}_{10}^{(1)}$ given by the following table.

x	0	1	2	3	4	5	6	7	8	9
$s(x)$	0	0	0	3	4	0	3	7	7	3

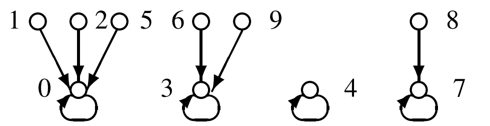

Figure 2: An example of $\Gamma(s)$ for Proposition 4.1

The graph $\Gamma(s)$ is shown in Figure 2. Clearly, s is reflective and intensional, implying that $\psi_s \ (\in \mathcal{O}_{10}^{(2)})$ is a minimal function.

Remark Let $\widehat{\mathcal{F}_2}$ be the set of binary idempotent functions f satisfying $f(x,y) = \min\{x,y\}$ for all $(x,y) \in (E_k \backslash \{0\})^2$. The values $f(0,y)$ and $f(x,0)$ for $x,y \in E_k \backslash \{0\}$ are arbitrary. Clearly, $\mathcal{F}_2$ is a proper subset of $\widehat{\mathcal{F}_2}$. For each $f \in \widehat{\mathcal{F}_2}$, define unary functions $r_f, c_f \in \mathcal{O}_k^{(1)}$ by

$$\begin{cases} r_f(y) &= f(0,y) \qquad \text{for all } y \in E_k, \\ c_f(x) &= f(x,0) \qquad \text{for all } x \in E_k. \end{cases}$$

Let $\widehat{\mathcal{F}_2}^{\nabla}$ be the subset of $\widehat{\mathcal{F}_2}$ consisting of functions $f \in \widehat{\mathcal{F}_2}$ for which both r_f and c_f are intensional.

Now, in general, reflectivity is not "preserved" by composition. However, it is proved that, for any $f \in \mathcal{F}_2$, there exists $g \in \langle f \rangle \cap \widehat{\mathcal{F}_2}^{\nabla}$ for which r_g and c_g are reflective. Hence, every minimal clone generated by a function in $\mathcal{F}_2$ contains a generator $g \in \widehat{\mathcal{F}_2}^{\nabla}$ for which r_g and c_g are reflective.

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