ON THE AVERAGE JOINT CYCLE INDEX AND THE AVERAGE JOINT WEIGHT ENUMERATOR

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Abstract. In this paper, we introduce the concept of the complete joint cycle index and the average complete joint cycle index, and discuss a relation with the complete joint weight enumerator and the average complete joint weight enumerator respectively in coding theory.

Key Words and Phrases. Cycle index, Complete weight enumerator.

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1. Introduction

Let G be a permutation group on a set Ω, where $|\Omega| = n$. For each element $h \in G$, we can decompose the permutation h into a product of disjoint cycles; let $c(h,i)$ be the number of i-cycles occurring by the action of h.

Definition 1.1 (Cameron [2])**.** The *cycle index* of G is the polynomial $Z(G; s_1, \ldots, s_n)$ in indeterminates $s_1, \ldots, s_n$ defined as

$$Z(G; s_1, \ldots, s_n) = \sum_{g \in G} s_1^{c(g,1)} \cdots s_n^{c(g,n)}.$$

Example 1.1. Let G be the symmetric group of degree 4. Each partition of 4 is the cycle type of some element of G. We have the following number of elements of G corresponding to each partition:

Partition	4	31	22	211	1111
# of elements	6	8	3	6	1

Table 1. Partitions and element numbers in G.

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Therefore, the cycle index of G is

$$Z(G; s_1, s_2, s_3, s_4) = 6s_4 + 8s_1s_3 + 3s_2^2 + 6s_2s_1^2 + s_1^4.$$

Definition 1.2 (Miezaki-Oura [6])**.** The *complete cycle index* of G is the polynomial $Z(G; s(h,i) : h \in G, i \in \mathbb{N})$ in indeterminates $\{s(h,i) \mid h \in G, i \in \mathbb{N}\}$ given by

$$Z(G; s(h,i) : h \in G, i \in \mathbb{N}) = \sum_{h \in G} \prod_{i \in \mathbb{N}} s(h,i)^{c(h,i)},$$

where $\mathbb{N} := \{x \in \mathbb{Z} \mid x > 0\}$.

Let $\mathbb{F}_q$ be the finite field of order q, where q is a prime power. An $\mathbb{F}_q$-linear code C is a vector subspace of $\mathbb{F}_q^n$. The *dual code* of a code C is

$$C^\perp := \{\mathbf{v} \in \mathbb{F}_q^n \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{u} \in C\},$$

where $\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^n u_i v_i$ denotes the *inner product* of $\mathbf{u}$ and $\mathbf{v}$. If $C = C^\perp$, then C is called *self-dual*. The *weight* of $\mathbf{u} \in C$ is denoted by $\mathrm{wt}(\mathbf{u}) := \#\{i \mid u_i \neq 0\}$. For $\mathbf{u} \in C$, we denote the *composition* of $\mathbf{u}$ by $s(\mathbf{u}) := (s_a(\mathbf{u}) : a \in \mathbb{F}_q)$, where $s_a(\mathbf{u}) := \#\{i \mid u_i = a\}$.

Definition 1.3. Let C be an $\mathbb{F}_q$-linear code of length n. Then the *weight enumerator* of C is the homogeneous polynomial

$$w_C(x,y) := \sum_{\mathbf{u} \in C} x^{n-\mathrm{wt}(\mathbf{u})} y^{\mathrm{wt}(\mathbf{u})} \in \mathbb{C}[x,y],$$

and the *complete weight enumerator* of C is defined as:

$$\mathbf{cwe}_C(x_a : a \in \mathbb{F}_q) := \sum_{\mathbf{u} \in C} \prod_{a \in \mathbb{F}_q} x_a^{s_a(\mathbf{u})} \in \mathbb{C}[x_a : a \in \mathbb{F}_q].$$

Definition 1.4. Let C be an $[n,k]$ *linear code* over $\mathbb{F}_q$. We construct a *permutation group* $G(C)$ from C whose *cycle index* is the *weight enumerator*. The group we construct is the *additive* group of C. We let it act on the set

$$\{1, \ldots, n\} \times \mathbb{F}_q \quad \text{(a set of cardinality } nq)$$

in the following way: the codeword $(u_1, \ldots, u_n)$ acts as the permutation

$$(i, x) \mapsto (i, x + u_i)$$

of the set $\{1, \ldots, n\} \times \mathbb{F}_q$. The group $G(C)$ is an elementary *abelian group* of order q^k. We call the cycle index

$$Z(G(C); s_1, \ldots, s_n)$$

the *cycle index for a code* C. We call the complete cycle index

$$Z(G(C); s(g,i) : g \in G(C), i \in \mathbb{N})$$

the *complete cycle index for a code* C.

Example 1.2. Let $C := \{(0,0),(0,1),(1,0),(1,1)\} = \mathbb{F}_2^2$ be a code. Again let $G(C)$ be the permutation groups on

$$\{1,2\} \times \mathbb{F}_2 = \{(1,0),(1,1),(2,0),(2,1)\}.$$

For $\mathbf{u} = (u_1, u_2) \in C$ acts as a permutation on $\{1,2\} \times \mathbb{F}_2$ as follows:

$$(i,x) \mapsto (i, x+u_i).$$

Now let $\mathbf{u} = (0,1) \in C$. Then

$$\begin{aligned}
&(1,0) \mapsto (1,0+0) = (1,0) \Leftarrow \text{1-cycle},\\
&(1,1) \mapsto (1,1+0) = (1,1) \Leftarrow \text{1-cycle},\\
&(2,0) \mapsto (2,0+1) = (2,1) \mapsto (2,1+1) = (2,0) \Leftarrow \text{2-cycle}.
\end{aligned}$$

Therefore the partition is 211.

$\mathbf{u} \in C$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
Partitions	1111	211	211	22

TABLE 2. Elements and Partitions in $G(C)$.

Therefore, the cycle index, $Z(G(C), s_1, s_2) = s_1^4 + 2s_1^2 s_2 + s_2^2$. Then the complete cycle index,

$$\begin{aligned}
Z(G(C);&s(h,i) : h \in G(C), i \in \mathbb{N})\\
&= s((0,0),1)^2 s((0,0),1)^2 + s((0,1),1)^2 s((0,1),2)^1\\
&+ s((1,0),2)^1 s((1,0),1)^2 + s((1,1),2)^1 s((1,1),2)^1.
\end{aligned}$$

Theorem 1.1 ([2, 6])**.** *Let C be a linear code over $\mathbb{F}_q$ of length n, where q is a power of the prime number p. Then we have the following results.*

(i) $W_C(x,y) = Z(G(C); s_1 \leftarrow x^{1/q}, s_p \leftarrow y^{p/q})$.

(ii) *Let T be a map defined as: for each $g = (u_1, \ldots, u_n) \in C$ and $i \in \{1, \ldots, n\}$, if $u_i = 0$, then $s(g,1) \mapsto x_{u_i}^{1/q}$; if $u_i \neq 0$, then $s(g,p) \mapsto x_{u_i}^{p/q}$. Then $\mathbf{cwe}_C(x_a : a \in \mathbb{F}_q) = T(Z(G(C); s(g,i) : g \in G(C), i \in \mathbb{N}))$.*

The notion of the joint weight enumerator of two $\mathbb{F}_q$-linear codes was introduced in [5]. Further, the notion of the g-fold complete joint weight enumerator of g linear codes over $\mathbb{F}_q$ was given in [7].

Definition 1.5 ([7]). Let C and D be two linear codes of length n over $\mathbb{F}_q$. The *complete joint weight enumerator* of codes C and D is defined as follows:

$$\mathcal{J}_{C,D}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^2) := \sum_{\mathbf{u}\in C, \mathbf{v}\in D} \prod_{\mathbf{a}\in\mathbb{F}_q^2} x_{\mathbf{a}}^{n_{\mathbf{a}}(\mathbf{u},\mathbf{v})},$$

where $n_{\mathbf{a}}(\mathbf{u}, \mathbf{v})$ denotes the number of i such that $\mathbf{a} = (u_i, v_i)$.

Let G and H be two permutation groups on a set Ω, where $|\Omega| = n$. Again let $\mathcal{G}_{G,H} := G \times H$ be the direct product of G and H. For each element $(g, h) \in \mathcal{G}_{G,H}$, where $g \in G$ and $h \in H$, we can decompose each permutation of the pair (g, h) into a product of disjoint cycles. Let $c(gh, i)$ be the number of i-cycles occurring by the action of gh, where gh denotes the *product of permutations* g and h which acts on Ω as $(gh)(\alpha) = h(g(\alpha))$ for any $\alpha \in \Omega$.

Definition 1.6. The *complete joint cycle index* of permutation groups G and H is the polynomial

$$\mathcal{Z}_{G,H}(s((g, h), i)) := \mathcal{Z}(\mathcal{G}_{G,H}; s((g, h), i) : (g, h) \in \mathcal{G}_{G,H}, i \in \mathbb{N})$$

in indeterminates $s((g, h), i)$, where $(g, h) \in \mathcal{G}_{G,H}$ and $i \in \mathbb{N}$, given by

$$\mathcal{Z}_{G,H}(s((g, h), i)) := \sum_{(g,h)\in\mathcal{G}_{G,H}} \prod_{i\in\mathbb{N}} s((g, h), i)^{c(gh,i)}.$$

The concept of the complete joint cycle index is used in Theorem 2.1. Theorem 2.1 gives a relation between complete joint cycle index and complete joint weight enumerator. This generalizes the earlier work Theorem 1.1. Further, we give the notion of the r-fold complete joint cycle index and the (ℓ, r)-fold complete joint weight enumerator. In this paper we also give a link between the r-fold complete joint cycle index and the (ℓ, r)-fold complete joint weight enumerator. The link is a generalization of Theorem 2.1. This result presents us a new application of constructing the average r-fold complete joint cycle index and a motivation to establish a relation with the average (ℓ, r)-fold complete joint weight enumerator.

2. The Relation

In this section, from any two $\mathbb{F}_q$-linear codes, we construct two permutation groups, whose complete joint cycle index is essentially the complete joint weight enumerator of codes.

Definition 2.1. Let C and D be two linear codes of length n over $\mathbb{F}_q$. We construct from C and D two permutation groups $G(C)$ and $H(D)$

respectively. The groups $G(C)$ and $H(D)$ are the additive group of C and D respectively. We let each group act on the set $\{1, \dots, n\} \times \mathbb{F}_q$ in the following way: the codeword $(u_1, \dots, u_n)$ acts as the permutation

$$(i, x) \mapsto (i, x + u_i)$$

of the set $\{1, \dots, n\} \times \mathbb{F}_q$. We define the *product* of two permutations $(u_1, \dots, u_n) \in C$ and $(v_1, \dots, v_n) \in D$ as follows:

$$(i, x) \mapsto (i, x + u_i + v_i)$$

of a set $\{1, \dots, n\} \times \mathbb{F}_q$. Let $\mathcal{G}_{C,D} := G(C) \times H(D)$. We call the complete joint cycle index

$$\mathcal{Z}_{C,D}(s((g,h),i)) := \mathcal{Z}(\mathcal{G}_{C,D}; s((g,h),i) : (g,h) \in \mathcal{G}_{C,D}, i \in \mathbb{N})$$

the complete joint cycle index for codes C and D.

Example 2.1. Let

$$C := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, D := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Then the complete joint weight enumerator is

$$x_{00}^2 + x_{01}^2 + x_{00}x_{10} + x_{01}x_{11} + x_{10}x_{00} + x_{11}x_{01} + x_{10}^2 + x_{11}^2.$$

Let $G(C)$ and $H(D)$ are the permutation groups on $\{1, 2\} \times \mathbb{F}_2$. In the following calculation, for $g \in G(C)$ and $h \in H(D)$, we prefer to write the indeterminates $s((g,h),i)$ as

$$s\left(\begin{pmatrix} g \\ h \end{pmatrix}, i\right).$$

Then the joint cycle index is

$$\begin{aligned}
&s\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1\right)^2 + s\left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, 2\right)^1 \\
&+ s\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 2\right)^1 + s\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 1\right)^2 \\
&+ s\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1\right)^2 + s\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 2\right)^1 \\
&+ s\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, 2\right)^1 s\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, 1\right)^2 + s\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 1\right)^2 s\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 1\right)^2.
\end{aligned}$$

Now we have the following result.

Theorem 2.1. *Let C and D be two codes over $\mathbb{F}_q$ of length n, where q is a power of the prime number p. Let $\mathcal{J}_{C,D}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^2)$ be the complete joint weight enumerator and $\mathcal{Z}(\mathcal{G}_{C,D}; s((g,h),i) : (g,h) \in \mathcal{G}_{C,D}, i \in \mathbb{N})$ be the complete joint cycle index.*

Let T be a map defined as follows: for each $g = (u_1, \dots, u_n) \in C$ and $h = (v_1, \dots, v_n) \in D$, and for $i \in \{1, \dots, n\}$,
if $u_i + v_i = 0$, then

$$s((g,h),1) \mapsto x_{u_i v_i}^{1/q};$$

if $u_i + v_i \neq 0$, then

$$s((g,h),p) \mapsto x_{u_i v_i}^{p/q}.$$

Then we have

$$\mathcal{J}_{C,D}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^2) = T(\mathcal{Z}(\mathcal{G}_{C,D}; s((g,h),i) : (g,h) \in \mathcal{G}_{C,D}, i \in \mathbb{N})).$$

3. r-fold Complete Joint Cycle Index

Let $G_1, G_2, \dots, G_r$ be r permutation groups on a set Ω, where $|\Omega| = n$. Again let $\mathcal{G}_{G_1,\dots,G_r} := G_1 \times \cdots \times G_r$ be the direct product of $G_1, G_2, \dots, G_r$. For any element $(g_1, g_2, \dots, g_r) \in \mathcal{G}_{G_1,\dots,G_r}$, where $g_k \in G_k$ for $k \in \{1, 2, \dots, r\}$, we can decompose each permutation g_k into a product of disjoint cycles. Let $c(g_k, i)$ be the number of i-cycles occurring by the action of g_k. Now the *r-fold complete joint cycle index* of $G_1, G_2, \dots, G_r$ is the polynomial

$$\begin{aligned}&\mathcal{Z}_{G_1,\dots,G_r}(s((g_1,\dots,g_r),i)) \\ &:= \mathcal{Z}(\mathcal{G}_{G_1,\dots,G_r}; s((g_1,\dots,g_r),i) : (g_1,\dots,g_r) \in \mathcal{G}_{G_1,\dots,G_r}, i \in \mathbb{N})\end{aligned}$$

in indeterminates $s((g_1,\dots,g_r),i)$, where $(g_1,\dots,g_r) \in \mathcal{G}_{G_1,\dots,G_r}$ and $i \in \mathbb{N}$, given by

$$\begin{aligned}&\mathcal{Z}_{G_1,\dots,G_r}(s((g_1,\dots,g_r),i)) \\ &:= \sum_{(g_1,\dots,g_r) \in \mathcal{G}_{G_1,\dots,G_r}} \prod_{i \in \mathbb{N}} s((g_1,\dots,g_r),i)^{c(g_1 \cdots g_r, i)}.\end{aligned}$$

where $g_1 \cdots g_r$ denotes the *product of permutations* $g_1, \dots, g_r$ which acts on Ω as $(g_1 \cdots g_r)(\alpha) = g_r(\cdots g_1(\alpha) \cdots)$ for any $\alpha \in \Omega$. If $G_1 = \cdots = G_r = G$, then we call $\mathcal{Z}(\mathcal{G}_{G,\dots,G}; s((g_1,\dots,g_r),i) : (g_1,\dots,g_r) \in \mathcal{G}_{G,\dots,G}, i \in \mathbb{N})$ the *r-fold complete multi-joint cycle index* of G.

Definition 3.1. We denote, $\Pi^\ell := C_1 \times \cdots \times C_\ell$, where $C_1, \dots, C_\ell$ be the linear codes of length n over $\mathbb{F}_q$. We call Π^ℓ as the *ℓ-fold joint code* of $C_1, \ldots, C_\ell$. We denote an element of Π^ℓ by

$$\tilde{\mathbf{c}} := (\mathbf{c}_1, \ldots, \mathbf{c}_n) := \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & \ldots & \vdots \\ a_{\ell 1} & \ldots & a_{\ell n} \end{pmatrix},$$

where $\mathbf{c}_i := {}^t(a_{1i}, \ldots, a_{\ell i}) \in \mathbb{F}_q^\ell$ and $\mu_j(\tilde{\mathbf{c}}) := (a_{j1}, \ldots, a_{jn}) \in C_j$.

Now let $\Pi_1^\ell, \ldots, \Pi_r^\ell$ be the ℓ-fold joint codes (not necessarily the same) over $\mathbb{F}_q$. For $k \in \{1, \ldots, r\}$, we denote, $\Pi_k^\ell := C_{k1} \times \cdots \times C_{k\ell}$, where $C_{k1}, \ldots, C_{k\ell}$ be the linear codes of length n over $\mathbb{F}_q$. An element of Π_k^ℓ is denoted by

$$\tilde{\mathbf{c}}_k := (\mathbf{c}_{k1}, \ldots, \mathbf{c}_{kn}) := \begin{pmatrix} a_{11}^{(k)} & \ldots & a_{1n}^{(k)} \\ a_{21}^{(k)} & \ldots & a_{2n}^{(k)} \\ \vdots & \ldots & \vdots \\ a_{\ell 1}^{(k)} & \ldots & a_{\ell n}^{(k)} \end{pmatrix},$$

where $\mathbf{c}_{ki} := {}^t(a_{1i}^{(k)}, \ldots, a_{\ell i}^{(k)}) \in \mathbb{F}_q^\ell$. and $\mu_j(\tilde{\mathbf{c}}_k) := (a_{j1}^{(k)}, \ldots, a_{jn}^{(k)}) \in C_{kj}$. Then the (ℓ, r)*-fold complete joint weight enumerator* of $\Pi_1^\ell, \ldots, \Pi_r^\ell$ is defined as follows:

$$\mathcal{J}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r}) := \sum_{\tilde{\mathbf{c}}_1 \in \Pi_1^\ell, \ldots, \tilde{\mathbf{c}}_r \in \Pi_r^\ell} \prod x_{\mathbf{a}}^{n_{\mathbf{a}}(\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_r)},$$

where $n_{\mathbf{a}}(\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_r)$ denotes the number of i such that $\mathbf{a} = (\mathbf{c}_{1i}, \ldots, \mathbf{c}_{ri})$. For $r = 2$ and $\ell = 1$ the complete (ℓ, r)-fold joint weight enumerator coincide with complete joint weight enumerator (Definition 1.5).

Definition 3.2. We construct from Π^ℓ a permutation group $G(\Pi^\ell)$. The group we construct is the additive group of Π^ℓ. We let it act on the set $\{1, \ldots, n\} \times \mathbb{F}_q^\ell$ in the following way: $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ acts as the permutation

$$\left(i, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{pmatrix}\right) \mapsto \left(i, \begin{pmatrix} x_1 + a_{1i} \\ x_2 + a_{2i} \\ \vdots \\ x_\ell + a_{\ell i} \end{pmatrix}\right)$$

of the set $\{1, \ldots, n\} \times \mathbb{F}_q^\ell$. Now let $G_1(\Pi_1^\ell), \ldots, G_r(\Pi_r^\ell)$ be r permutation groups. We define the *product* of r permutations $(\mathbf{c}_{11}, \ldots, \mathbf{c}_{1n}) \in$

$\Pi_1^\ell, \dots, (\mathbf{c}_{r1}, \dots, \mathbf{c}_{rn}) \in \Pi_r^\ell$ as follows:

$$\left(i, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{pmatrix}\right) \mapsto \left(i, \begin{pmatrix} x_1 + \sum_{k=1}^r a_{1i}^{(k)} \\ x_2 + \sum_{k=1}^r a_{2i}^{(k)} \\ \vdots \\ x_\ell + \sum_{k=1}^r a_{\ell i}^{(k)} \end{pmatrix}\right)$$

of a set $\{1, \dots, n\} \times \mathbb{F}_q^\ell$. Let $\mathcal{G}_{\Pi_1^\ell, \dots, \Pi_r^\ell} := G_1(\Pi_1^\ell) \times \cdots \times G_r(\Pi_r^\ell)$. We call the r-fold complete joint cycle index

$$\begin{aligned} &\mathcal{Z}_{\Pi_1^\ell, \dots, \Pi_r^\ell}(s((g_1, \dots, g_r), i)) \\ &\quad := \mathcal{Z}(\mathcal{G}_{\Pi_1^\ell, \dots, \Pi_r^\ell}; s((g_1, \dots, g_r), i) : (g_1, \dots, g_r) \in \mathcal{G}_{\Pi_1^\ell, \dots, \Pi_r^\ell}, i \in \mathbb{N}) \end{aligned}$$

the r-fold complete joint cycle index for $\Pi_1^\ell, \dots, \Pi_r^\ell$.

Remark 3.1. Let $\Pi_1^\ell = \cdots = \Pi_r^\ell = \Pi^\ell$, where

$$\Pi^\ell := C_1 \times \cdots \times C_\ell,$$

for the $\mathbb{F}_q$-linear codes $C_1, \dots, C_\ell$ of length n. Then we call

$$\mathcal{Z}(\mathcal{G}_{\Pi^\ell, \dots, \Pi^\ell}; s((g_1, \dots, g_r), i) : (g_1, \dots, g_r) \in \mathcal{G}_{\Pi^\ell, \dots, \Pi^\ell}, i \in \mathbb{N})$$

the *r-fold complete multi-joint cycle index for* Π^ℓ.

Again let $C_1 = \cdots = C_\ell = C$, for some $\mathbb{F}_q$-linear code C of length n. Then we denote Π^ℓ by C^ℓ, that is,

$$C^\ell := \underbrace{C \times \cdots \times C}_{\ell}.$$

We call $\mathcal{Z}(\mathcal{G}_{C^\ell, \dots, C^\ell}; s((g_1, \dots, g_r), i) : (g_1, \dots, g_r) \in \mathcal{G}_{C^\ell, \dots, C^\ell}, i \in \mathbb{N})$ the *r-fold complete multi-joint cycle index for C^ℓ*. Note that if $r = 1$, the r-fold complete multi-joint cycle index for C^ℓ coincide with the *complete cycle index of genus* ℓ for code C in the sense of Miezaki-Oura [6].

Now we give a generalization of Theorem 2.1 as follows.

Theorem 3.1. *For $k \in \{1, \dots, r\}$ and $j \in \{1, \dots, \ell\}$, let C_{kj} be an $\mathbb{F}_q$-linear code of length n, where q is a power of the prime number p. Again let Π_k^ℓ be the ℓ-fold joint code of $C_{k1}, \dots, C_{k\ell}$. Let $\mathcal{J}_{\Pi_1^\ell, \dots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r})$ be the (ℓ, r)-fold complete joint weight enumerator of $\Pi_1^\ell, \dots, \Pi_r^\ell$, and*

$$\mathcal{Z}(\mathcal{G}_{\Pi_1^\ell, \dots, \Pi_r^\ell}; s((g_1, \dots, g_r), i) : (g_1, \dots, g_r) \in \mathcal{G}_{\Pi_1^\ell, \dots, \Pi_r^\ell}, i \in \mathbb{N})$$

be the r-fold complete joint cycle index for $\Pi_1^\ell, \dots, \Pi_r^\ell$.

Let T be a map defined as follows: for each $g_1 = (\mathbf{c}_{11}, \ldots, \mathbf{c}_{1n}) \in \Pi_1^\ell, \ldots, g_r = (\mathbf{c}_{r1}, \ldots, \mathbf{c}_{rn}) \in \Pi_r^\ell$, and for $i \in \{1, \ldots, n\}$,
if $\sum_{k=1}^r \mathbf{c}_{ki} = \mathbf{0}$, then

$$s((g_1, \ldots, g_r), 1) \mapsto x_{\mathbf{c}_{1i} \cdots \mathbf{c}_{ri}}^{1/q^\ell};$$

if $\sum_{k=1}^r \mathbf{c}_{ki} \neq \mathbf{0}$, then

$$s((g_1, \ldots, g_r), p) \mapsto x_{\mathbf{c}_{1i} \cdots \mathbf{c}_{ri}}^{p/q^\ell}.$$

Then we have

$$\mathcal{J}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r}) = T(\mathcal{Z}(\mathcal{G}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}; s((g_1, \ldots, g_r), i) : (g_1, \ldots, g_r) \in \mathcal{G}_{\Pi_1^\ell, \ldots, \Pi_r^\ell}, i \in \mathbb{N})).$$

4. Main Results

In [8], the notion of the average joint weight enumerators was given. Further, the notion of the average r-fold complete joint weight enumerators was given in [4]. In this section, we give the concept of the average complete joint cycle index and provide a relation with average complete joint weight enumerator of codes. We also give an analogy of Theorem 3.1 for the average complete joint cycle index. For two permutation groups G and G' on Ω, where $|\Omega| = n$, we write $G' \cong G$ if G and G' are isomorphic as permutation groups.

Definition 4.1. Let $G_1, \ldots, G_r$ be r permutation groups on a set Ω, where $|\Omega| = n$. Then the $(G_1, \ldots, G_r)$-*average r-fold complete joint cycle index* of $G_1, \ldots, G_r$ is the polynomial

$$\mathcal{Z}^{av}_{G_1, \ldots, G_r}(s((g'_1, \ldots, g'_r), i)) := \mathcal{Z}^{av}(\mathcal{G}_{G'_1, \ldots, G'_r}; s((g'_1, \ldots, g'_r), i) : G'_1 \cong G_1, \ldots, G'_r \cong G_r, (g'_1, \ldots, g'_r) \in \mathcal{G}_{G'_1, \ldots, G'_r}, i \in \mathbb{N}),$$

in indeterminates $s((g', \ldots, g'_r), i)$ where $g'_1 \in G'_1, \ldots, g'_r \in G'_r$, and $i \in \mathbb{N}$ defined by

$$\mathcal{Z}^{av}_{G_1, \ldots, G_r}(s((g'_1, \ldots, g'_r), i)) := \frac{1}{\prod_{k=1}^r N_{\cong}(G_k)} \sum_{G'_1 \cong G_1} \cdots \sum_{G'_r \cong G_r} \mathcal{Z}_{G'_1, \ldots, G_r}(s((g'_1, \ldots, g'_r), i)),$$

where $N_{\cong}(G_k) := \sharp\{G'_k \mid G'_k \cong G_k\}$.

In this paper we only consider the case G_1-average complete joint cycle index. The G_1-*average r-fold complete joint cycle index* of $G_1, \ldots, G_r$ is the polynomial

$$\mathcal{Z}^{av}_{G_1, \ldots, G_r}(s((g'_1, \ldots, g_r), i)) := \mathcal{Z}^{av}(\mathcal{G}_{G'_1, \ldots, G_r}; s((g'_1, \ldots, g_r), i) :$$

$$G'_1 \cong G_1, (g'_1, \ldots, g_r) \in \mathcal{G}_{G'_1,\ldots,G_r}, i \in \mathbb{N}),$$

in indeterminates $s((g', \ldots, g_r), i)$ where $g'_1 \in G'_1$, $g_2 \in G_2, \ldots, g_r \in G_r$, and $i \in \mathbb{N}$ defined by

$$\mathcal{Z}^{av}_{G_1,\ldots,G_r}(s((g'_1, \ldots, g_r), i)) := \frac{1}{N_{\cong}(G_1)} \sum_{G'_1 \cong G_1} \mathcal{Z}_{G'_1,\ldots,G_r}(s((g'_1, \ldots, g_r), i)),$$

where $N_{\cong}(G_1) := \sharp\{G'_1 \mid G'_1 \cong G_1\}$.

Example 4.1. Let S_3 be the symmetric group on $\{1, 2, 3\}$. Again let G_1 and G_2 be two subgroup of S_3 such that $G_1 = \langle(1, 2)\rangle$ and $G_2 = \langle(1, 3, 2)\rangle$. Then the subgroups of S_3 that are isomorphic as permutation group to G_1 are $\langle(1, 2)\rangle, \langle(1, 3)\rangle, \langle(2, 3)\rangle$. That is $N_{\cong}(G_1) = 3$. Therefore

$$\begin{aligned}
&\mathcal{Z}^{av}_{G_1,G_2}(s((g'_1, g_2), i)) \\
&= \frac{1}{3}(\mathcal{Z}_{\langle(1,2)\rangle,G_2}(s((g'_1, g_2), i)) + \mathcal{Z}_{\langle(1,3)\rangle,G_2}(s((g'_1, g_2), i)) \\
&\quad + \mathcal{Z}_{\langle(2,3)\rangle,G_2}(s((g'_1, g_2), i))) \\
&= \frac{1}{3}(s(((1), (1)), 1)^3 + s(((1), (1, 2, 3)), 3)^1 + s(((1), (1, 3, 2)), 3)^1 \\
&\quad + s(((1, 2), (1)), 1)^1 s(((1, 2), (1)), 2)^1 \\
&\quad + s(((1, 2), (1, 2, 3)), 1)^1 s(((1, 2), (1, 2, 3)), 2)^1 \\
&\quad + s(((1, 2), (1, 3, 2)), 1)^1 s(((1, 2), (1, 3, 2)), 2)^1 \\
&\quad + s(((1), (1)), 1)^3 + s(((1), (1, 2, 3)), 3)^1 + s(((1), (1, 3, 2)), 3)^1 \\
&\quad + s(((1, 3), (1)), 1)^1 s(((1, 3), (1)), 2)^1 \\
&\quad + s(((1, 3), (1, 2, 3)), 1)^1 s(((1, 3), (1, 2, 3)), 2)^1 \\
&\quad + s(((1, 2), (1, 3, 2)), 1)^1 s(((1, 2), (1, 3, 2)), 2)^1 \\
&\quad + s(((1), (1)), 1)^3 + s(((1), (1, 2, 3)), 3)^1 + s(((1), (1, 3, 2)), 3)^1 \\
&\quad + s(((2, 3), (1)), 1)^1 s(((2, 3), (1)), 2)^1 \\
&\quad + s(((2, 3), (1, 2, 3)), 1)^1 s(((2, 3), (1, 2, 3)), 2)^1 \\
&\quad + s(((2, 3), (1, 3, 2)), 1)^1 s(((2, 3), (1, 3, 2)), 2)^1)
\end{aligned}$$

Definition 4.2. We write S_n for the symmetric group acting on the set $\{1, 2, \ldots, n\}$. Let C be any linear code of length n over $\mathbb{F}_q$, and $\mathbf{u} = (u_1, \ldots, u_n) \in C$. Then $\sigma(\mathbf{u}) := (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$ for a permutation $\sigma \in S_n$. Now the code $C' := \sigma(C) := \{\sigma(\mathbf{u}) \mid \mathbf{u} \in C\}$ for $\sigma \in S_n$ is called *permutationally equivalent* to C, and denoted by $C \sim C'$. Then the *average r-fold complete joint weight enumerator* of codes $C_1, \ldots, C_r$

over $\mathbb{F}_q$ are defined in [4] as:

$$\mathcal{J}^{av}_{C_1,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r) := \frac{1}{N_\sim(C_1')} \sum_{C_1' \sim C_1} \mathcal{J}_{C_1',C_2,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r),$$

where $N_\sim(C_1') := \sharp\{C_1' \mid C_1' \sim C_1\}$.

We call the G_1-average r-fold complete joint cycle index

$$\mathcal{Z}^{av}_{C_1,\ldots,C_r}(s((g_1', g_2, \ldots, g_r), i)) := \mathcal{Z}^{av}(\mathcal{G}_{C_1',C_2,\ldots,C_r}; \\ s((g_1', g_2, \ldots, g_r), i) : C_1' \sim C_1, (g_1', g_2, \ldots, g_r) \in \mathcal{G}_{C_1',C_2,\ldots,C_r}, i \in \mathbb{N})$$

the G_1-average r-fold complete joint cycle index for codes $C_1, \ldots, C_r$.

The following theorem gives a connection between the G_1-average of r-fold complete joint cycle index and the average of r-fold complete joint weight enumerator.

Theorem 4.1. *Let $C_1, \ldots, C_r$ be the linear codes of length n over $\mathbb{F}_q$, where q is a power of the prime number p. Let $\mathcal{J}^{av}_{C_1,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r)$ be the average r-fold complete joint weight enumerator and*

$$\mathcal{Z}^{av}(\mathcal{G}_{C_1',C_2,\ldots,C_r}; s((g_1', g_2, \ldots, g_r), i) : C_1' \sim C_1, (g_1', g_2, \ldots, g_r) \in \\ \mathcal{G}_{C_1',C_2,\ldots,C_r}, i \in \mathbb{N})$$

be the G_1-average complete joint cycle index for $C_1, \ldots, C_r$.

Let T be a map defined as follows: for $\sigma \in S_n$, and $g_1 = (u_{11}, \ldots, u_{1n}) \in C_1, g_2 = (u_{21}, \ldots, u_{2n}) \in C_2, \ldots, g_r = (u_{r1}, \ldots, u_{rn}) \in C_r$, and for $i \in \{1, \ldots, n\}$, if $u_{1\sigma(i)} + u_{2i} + \cdots + u_{ri} = 0$, then

$$s((g_1', g_2, \ldots, g_r), 1) \mapsto x^{1/q}_{u_{1\sigma(i)}u_{2i}\ldots u_{ri}};$$

if $u_{1\sigma(i)} + u_{2i} + \cdots + u_{ri} \neq 0$, then

$$s((g_1', g_2, \ldots, g_r), p) \mapsto x^{p/q}_{u_{1\sigma(i)}u_{2i}\ldots u_{ri}}.$$

Then we have

$$\mathcal{J}^{av}_{C_1,\ldots,C_r}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r) = T(\mathcal{Z}^{av}(\mathcal{G}_{C_1',C_2,\ldots,C_r}; s((g_1', g_2, \ldots, g_r), i) : \\ C_1' \sim C_1, (g_1', g_2, \ldots, g_r) \in \mathcal{G}_{C_1',C_2,\ldots,C_r}, i \in \mathbb{N})).$$

Definition 4.3. For $S_n^\ell := \underbrace{S_n \times \cdots \times S_n}_{\ell}$, we define the semidirect product of S_ℓ and S_n^ℓ as

$$S_\ell \rtimes S_n^\ell := \{\iota := (\pi; \sigma_1, \ldots, \sigma_\ell) \mid \pi \in S_\ell \text{ and } \sigma_1, \ldots, \sigma_\ell \in S_n\}.$$

We recall the ℓ-fold joint code, Π^ℓ and for $\tilde{\mathbf{c}} = (\mathbf{c}_1, \dots, \mathbf{c}_n) \in \Pi^\ell$, the group $S_\ell \rtimes S_n^\ell$ acts on Π^ℓ as

$$\iota(\tilde{\mathbf{c}}) := (\iota(\mathbf{c}_1), \dots, \iota(\mathbf{c}_n)) := \begin{pmatrix} a_{\pi(1)\sigma_1(1)} & \cdots & a_{\pi(1)\sigma_1(n)} \\ a_{\pi(2)\sigma_2(1)} & \cdots & a_{\pi(2)\sigma_2(n)} \\ \vdots & \cdots & \vdots \\ a_{\pi(\ell)\sigma_\ell(1)} & \cdots & a_{\pi(\ell)\sigma_\ell(n)} \end{pmatrix},$$

where $\iota(\mathbf{c}_i) := {}^t(a_{\pi(1)\sigma_1(i)}, \dots, a_{\pi(\ell)\sigma_\ell(i)}) \in \mathbb{F}_q^\ell$. Then we call $\Pi^{\ell\prime} := \iota(\Pi^\ell) := \{\iota(\tilde{\mathbf{c}}) \mid \tilde{\mathbf{c}} \in \Pi^\ell\}$ an *equivalent ℓ-fold joint code* to Π^ℓ, and denoted by $\Pi^{\ell\prime} \sim \Pi^\ell$. Now the *average (ℓ, r)-fold complete joint weight enumerator* of $\Pi_1^\ell, \dots, \Pi_r^\ell$ is defined by

$$\mathcal{J}^{av}_{\Pi_1^\ell, \Pi_2^\ell, \dots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r}) := \frac{1}{N_\sim(\Pi_1^{\ell\prime})} \sum_{\Pi_1^{\ell\prime} \sim \Pi_1^\ell} \mathcal{J}_{\Pi_1^{\ell\prime}, \Pi_2^\ell, \dots, \Pi_r^\ell}(x_{\mathbf{a}}),$$

where $N_\sim(\Pi_1^{\ell\prime}) := \sharp\{\Pi_1^{\ell\prime} \mid \Pi_1^{\ell\prime} \sim \Pi_1^\ell\}$. We call the G_1-average r-fold complete joint cycle index

$$\begin{aligned} &\mathcal{Z}^{av}_{\Pi_1^\ell, \dots, \Pi_r^\ell}(s((g_1', g_2, \dots, g_r), i)) := \mathcal{Z}^{av}(\mathcal{G}_{\Pi_1^{\ell\prime}, \Pi_2^\ell, \dots, \Pi_r^\ell}; \\ &\quad s((g_1', g_2, \dots, g_r), i) : \Pi_1^{\ell\prime} \sim \Pi_1^\ell, (g_1', g_2, \dots, g_r) \in \mathcal{G}_{\Pi_1^{\ell\prime}, \Pi_2^\ell, \dots, \Pi_r^\ell}, i \in \mathbb{N}) \end{aligned}$$

the G_1-average r-fold complete joint cycle index for ℓ-fold joint codes $\Pi_1^\ell, \dots, \Pi_r^\ell$.

In the following theorem, we give a relationship between the average (ℓ, r)-fold complete joint weigh enumerator and the G_1-average r-fold complete joint cycle index for ℓ-fold joint codes as a generalization of Theorem 4.1.

Theorem 4.2. *For $k \in \{1, \dots, r\}$ and $j \in \{1, \dots, \ell\}$, let C_{kj} be an $\mathbb{F}_q$-linear code of length n, where q is a power of the prime number p. Again let Π_k^ℓ be an ℓ-fold joint code of $C_{k1}, \dots, C_{k\ell}$. Let $\mathcal{J}^{av}_{\Pi_1^\ell, \dots, \Pi_r^\ell}(x_{\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^{\ell \times r})$ be the average (ℓ, r)-fold complete joint weight enumerator of $\Pi_1^\ell, \dots, \Pi_r^\ell$, and*

$$\mathcal{Z}^{av}(\mathcal{G}_{\iota\Pi_1^\ell, \dots, \Pi_r^\ell}; s((g_1', \dots, g_r), i) : \Pi_1^{\ell\prime} \sim \Pi_1^\ell, (g_1', \dots, g_r) \in \mathcal{G}_{\Pi_1^{\ell\prime}, \dots, \Pi_r^\ell}, i \in \mathbb{N})$$

be the G_1-average r-fold complete joint cycle index for $\Pi_1^\ell, \dots, \Pi_r^\ell$.

Let T be a map defined as follows: for $\iota = (\pi; \sigma_1, \dots, \sigma_\ell) \in S_\ell \rtimes S_n^\ell$, and $g_1 = (\mathbf{c}_{11}, \dots, \mathbf{c}_{1n}) \in \Pi_1^\ell, \dots, g_r = (\mathbf{c}_{r1}, \dots, \mathbf{c}_{rn}) \in \Pi_r^\ell$, and for $i \in \{1, \dots, n\}$, if $\iota(\mathbf{c}_{1i}) + \mathbf{c}_{2i} + \cdots + \mathbf{c}_{ri} = \mathbf{0}$, then

$$s((g_1', \dots, g_r), 1) \mapsto x^{1/q^\ell}_{\iota(\mathbf{c}_{1i})\mathbf{c}_{2i} \dots \mathbf{c}_{ri}};$$

if $\iota(\mathbf{c}_{1i}) + \mathbf{c}_{2i} + \cdots + \mathbf{c}_{ri} \neq \mathbf{0}$*, then*

$$s((g'_1, \dots, g_r), p) \mapsto x^{p/q^\ell}_{\iota(\mathbf{c}_{1i})\mathbf{c}_{2i}\dots\mathbf{c}_{ri}}.$$

Then we have

$$\mathcal{J}^{av}_{\Pi^\ell_1,\dots,\Pi^\ell_r}(x_\mathbf{a} : \mathbf{a} \in \mathbb{F}_q^{\ell\times r}) = T(\mathcal{Z}^{av}(\mathcal{G}_{\Pi^{\ell\prime}_1,\dots,\Pi^\ell_r}; s((g'_1, \dots, g_r), i) : \Pi^{\ell\prime}_1 \sim \Pi^\ell_1, (g'_1, \dots, g_r) \in \mathcal{G}_{\Pi^{\ell\prime}_1,\dots,\Pi^\ell_r}, i \in \mathbb{N})).$$

Future Research

We would like to study with the concept of the joint Jacobi polynomial for codes over $\mathbb{F}_q$. We are also interested in studying average joint Jacobi polynomials of codes over $\mathbb{F}_q$. Further we would like to investigate the average Jacobi intersection number of codes.

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