Fixed point and its iteration theorems of new mappings in Banach spaces

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Abstract

In a Hilbert space, concepts of attractive point and acute point studied by many reseachers. Moreover, these concepts extended to Banach space. In this paper, we introduce a new class of mappings on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. Moreover, we introduce some extensions of acute point and prove some acute point theorems and some extensions of weak convergence theorems.

1 Introduction

In [24] Takahashi and Takeuchi introduced a concept of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. $x \in H$ is called an attractive point of T if $\|x - Ty\| \leq \|x - y\|$ for any $y \in C$. Let

$$A(T) = \{x \in H \mid \|x - Ty\| \leq \|x - y\| \text{ for any } y \in C\}.$$

A mapping T from C into H is said to be generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. Such a mapping is said to be (α, β)-generalized hybrid. The class of all generalized hybrid mappings is a new class of nonlinear mappings including nonexpansive mappings, nonspreading mappings [21] and hybrid mappings [23].

Motivated these mappings, in [16] Kawasaki and Takahashi introduced a new very wider class of mappings, called widely more generalized hybrid mappings, than the class of all generalized hybrid mappings. A mapping T from C into H is widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. This class includes the class of all generalized hybrid mappings and also the class of all k-pseudo-contractions [3] for $k \in [0, 1]$.

There are some studies on Banach space related to these results. In [25] Takahashi, Wong and Yao introduced the generalized nonspreading mapping and the skew-generalized nonspreading mapping in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is said to be generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y)\\ &\quad \le \varepsilon(\phi(Ty, Tx) - \phi(Ty, x)) + \zeta(\phi(y, Tx) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$, where J is the duality mapping on E and $\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading. A mapping T from C into E is said to be skew-generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y)\\ &\quad \le \varepsilon(\phi(Ty, Tx) - \phi(y, Tx)) + \zeta(\phi(Ty, x) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-skew-generalized nonspreading. These classes include the class of generalized hybrid mappings in a Hilbert space, however, it does not include the class of widely more generalized hybrid mappings. Moreover they introduced some extensions of attractive point and proved some attractive point theorems. $x \in E$ is an attractive point of T if $\phi(x, Ty) \le \phi(x, y)$ for any $y \in C$; $x \in E$ is a skew-attractive point of T if $\phi(Ty, x) \le \phi(y, x)$ for any $y \in C$. Let

$$\begin{aligned}A(T) &= \{x \in E \mid \phi(x, Ty) \le \phi(x, y) \text{ for any } y \in C\};\\ B(T) &= \{x \in E \mid \phi(Ty, x) \le \phi(y, x) \text{ for any } y \in C\}.\end{aligned}$$

Let C be a nonempty subset of a smooth Banach space E. A mapping T from C into E is said to be generalized nonexpansive [4] if the set of all fixed points of T is nonempty and $\phi(Tx, y) \le \phi(x, y)$ for any $x \in C$ and for any fixed point y of T. Let C be a nonempty subset of E of a Banach space E. A mapping R from E onto C is said to be sunny if $R(Rx + t(x - Rx)) = Rx$ for any $x \in E$ and for any $t \in [0, \infty)$. A mapping R from E onto C is called a retraction or a projection if $Rx = x$ for any $x \in C$.

On the other hand, in [1] Atsushiba, Iemoto, Kubota and Takeuchi introduced a concept of acute point as an extension of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H and $k \in [0, 1]$. $x \in H$ is called a k-acute point of T if $\|x - Ty\|^2 \le \|x - y\|^2 + k\|y - Ty\|^2$ for any $y \in C$. Let

$$\mathscr{A}_k(T) = \{x \in H \mid \|x - Ty\|^2 \le \|x - y\|^2 + k\|y - Ty\|^2 \text{ for any } y \in C\}.$$

Moreover, using a concept of acute point, they proved convergence theorems without convexity of C.

Motivated these results, in this paper we introduce some extensions of acute point and prove some acute point theorems and some extensions of weak convergence theorems.

2 Acute point and skew-acute point

Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell \in \mathbb{R}$. $x \in E$ is called a (k, ℓ)-acute point of T if

$$\phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y) \tag{2.1}$$

for any $y \in C$. $x \in E$ is called a (k, ℓ)-skew-acute point of T if

$$\phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y) \tag{2.2}$$

for any $y \in C$. Let

$$\begin{aligned}
&\mathscr{A}_{k,\ell}(T) \\
&= \{x \in E \mid \phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}; \\
&\mathscr{B}_{k,\ell}(T) \\
&= \{x \in E \mid \phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}.
\end{aligned}$$

It is obvious that

$$\mathscr{A}_{k_1,\ell_1}(T) \subset \mathscr{A}_{k_2,\ell_2}(T), \ \mathscr{B}_{k_1,\ell_1}(T) \subset \mathscr{B}_{k_2,\ell_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ with $k_1 \leq k_2$ and $\ell_1 \leq \ell_2$.

3 Acute point and skew-acute point theorems

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is called a generalized pseudocontraction [11] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1\phi(Tx, Ty) + \alpha_2\phi(Ty, Tx) + \beta_1\phi(x, Ty) + \beta_2\phi(Ty, x) \\
&\quad +\gamma_1\phi(Tx, y) + \gamma_2\phi(y, Tx) + \delta_1\phi(x, y) + \delta_2\phi(y, x) \\
&\quad +\varepsilon_1\phi(Tx, x) + \varepsilon_2\phi(x, Tx) + \zeta_1\phi(y, Ty) + \zeta_2\phi(Ty, y) \\
&\leq 0
\end{aligned} \tag{3.1}$$

for any $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction.

Theorem 3.1. *Let E be a reflexive and smooth Banach space, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction* *from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0,1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0;\\
&(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0.
\end{aligned}$$

Then there exists a $\left(-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}\right)$*-acute point.*

Theorem 3.2. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0,1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \geq 0;\\
&(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1) > 0.
\end{aligned}$$

Then there exists a $\left(-\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}, -\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}\right)$*-skew acute point.*

4 Weak convergence theorems

Theorem 4.1. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty subset of E, let $\{x_n\}$ be a sequence in C and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into E. Suppose that there exists $\lambda \in [0,1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0;\\
&(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0.
\end{aligned}$$

If $\{x_n\}$ is weakly convergent to q and $\{x_n - Tx_n\}$ is strongly convergent to 0, then

$$q \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

In particular, any fixed point of T belongs to

$$\mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

Theorem 4.2. *Let E be a uniformly convex Banach space with a uniformly Fréchet differentiable norm, let C be a nonempty convex subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0;\\
&(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0,
\end{aligned}$$

and suppose that

$$\mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T) \subset B(T) \neq \emptyset.$$

Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$ and let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n \in (0,1)$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$$

for any $n \in \mathbb{N}$ is weakly convergent to an element

$$q \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T),$$

where $q = \lim_{n\to\infty} Rx_n$.

Additionally, if C is closed and one of the following holds:

(1) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) > 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 \geq 0$*;*

(2) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) \geq 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 > 0$*,*

then q is a fixed point of T.

Theorem 4.3. *Let E be a uniformly convex Banach space with a uniformly Fréchet differentiable norm, let C be a nonempty subset of E satisfying $J(C)$ is convex and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \geq 0;\\
&(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1) > 0,
\end{aligned}$$

suppose that

$$\mathscr{B}_{-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)},-\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}}(T) \subset A(T) \neq \emptyset$$

and suppose that J^{-1} is weakly sequentially continuous. Let R^* be the sunny generalized nonexpansive retraction of E^* onto $J(A(T))$ and let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n \in (0, 1)$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n)$$

for any $n \in \mathbb{N}$ is weakly convergent to an element

$$q \in \mathscr{B}_{-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)},-\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}}(T),$$

where $q = \lim_{n\to\infty} J^{-1}R^*Jx_n$.

Additionally, if $J(C)$ is closed and one of the following holds:

(1) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1) > 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1 \geq 0$*;*

(2) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1) \geq 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1 > 0$*,*

then q is a fixed point of T.

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