Fixed point property and convergence theorems for hybrid-type mappings

Sachiko Atsushiba
Department of Mathematics, Graduate School of Education,
University of Yamanashi, 4-4-37,
Takeda Kofu, Yamanashi 400-8510, Japan

Abstract

In this paper, we establish the existence of absolute fixed points of normally 2-generalized hybrid mappings in a Hilbert space by using the idea of attractive points. We also prove some fixed point theorems.

1 Introduction

Throughout this paper, we denote a real Hilbert space by H, and its inner product and norm by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. For a mapping $T : C \to H$, we denote by $F(T)$ the set of fixed points of T and by $A(T)$ the set of attractive points [16] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;

(ii) $A(T) = \{z \in H : \|Tx - z\| \le \|x - z\|, \ \forall x \in C\}$.

Kohsaka and Takahashi [7], and Takahashi [15] introduced the following nonlinear mappings. A mapping $T : C \to H$ is said to be nonspreading [7] if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is said to be hybrid [15] if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

E-mail address: asachiko@yamanashi.ac.jp.

for all $x,y \in C$. They proved fixed point theorems for such mappings (see also [5, 8, 17]). In general, nonspreading and hybrid mappings are not continuous mappings. Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced the class of λ-hybrid mappings in a Hilbert space. This class contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Kocourek, Takahashi and Yao [6] introduced a broader class of nonlinear mappings than the class of λ-hybrid mappings in a Hilbert space. A mapping $T : C \to E$ is said to be generalized hybrid [6] if there are real numbers α, β such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x,y \in C$. Maruyama, Takahashi and Yao [10] introduced a broad class of nonlinear mappings called 2-generalized hybrid which contains generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H. A mapping $T : C \to C$ is said to be 2-generalized hybrid [10] if there exist real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 &+ \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ &\leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2\end{aligned}$$

for all $x,y \in C$. Kondo and Takahashi [9] introduced the following class of nonlinear mappings which covers 2-generalized hybrid mappings in a Hilbert space. A mapping $T : C \to C$ is said to be normally 2-generalized hybrid [9] if there exist real numbers $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0,$$

$$\alpha_2 + \alpha_1 + \alpha_0 > 0$$

and

$$\begin{aligned}\alpha_2\|T^2x - Ty\|^2 &+ \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ &+ \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0\end{aligned}$$

for all $x,y \in C$. We call such a mapping T an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-normally 2-generalized hybrid mapping. We know that the class of $(1-\alpha, -(1-\beta), \alpha, -\beta, 0, 0)$-normally 2-generalized hybrid mappings is the class of generalized hybrid mappings. Kondo and Takahashi [9] proved attractive point theorems, fixed point theorems and convergence theorems for the mappings.

On the other hand, Djafari Rouhani [11] introduced the concept of absolute fixed points for nonexpansive mappings. He studied an extension of nonexpansive mappings and established the existence of absolute fixed points of nonexpansive mappings. He established the existence of absolute fixed points of hybrid mappings and some fixed point theorems (see [12]). He also established the existence of absolute fixed points of generalized hybrid mappings and some fixed point theorems (see [13]).

In this paper, motivated by Kondo and Takahashi [9] and Djafari Rouhani [11, 12, 13], we establish the existence of absolute fixed points of normally 2-generalized hybrid mappings in a Hilbert space. We also prove some fixed point theorems.

2 Preliminaries and Lemmas

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. We know from [14] the following basic equality and inequality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Furthermore, we obtain that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$$

for all $x, y, z, w \in H$. In fact, we have

$$\begin{aligned} 2\langle x-y, z-w\rangle &= 2\langle x,z\rangle - 2\langle x,w\rangle - 2\langle y,z\rangle + 2\langle y,w\rangle \\ &= (-\|x\|^2 + 2\langle x,z\rangle - \|z\|^2) + (\|x\|^2 - 2\langle x,w\rangle + \|w\|^2) \\ &\quad + (\|y\|^2 - 2\langle y,z\rangle + \|z\|^2) + (-\|y\|^2 + 2\langle y,w\rangle - \|w\|^2) \\ &= \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2. \end{aligned}$$

We obtain that, for all $x, y, w \in H$,

$$\langle (x-y)+(x-w), y-w\rangle = \|x-w\|^2 - \|x-y\|^2.$$

In fact, we have

$$\begin{aligned} &\langle (x-y)+(x-w), y-w\rangle \\ &\quad = \langle (x-y)+(x-w), (y-x)+(x-w)\rangle \\ &\quad = \|x-w\|^2 - \|x-y\|^2 + \langle x-y, x-w\rangle + \langle x-w, y-x\rangle \\ &\quad = \|x-w\|^2 - \|x-y\|^2. \end{aligned}$$

Let C be a nonempty closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping P_C is called the metric projection of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$; see [14] for more details.

The following result is well-known (see [14]).

Theorem 2.1. *Let C be a nonempty, bounded, closed and convex subset of H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

Let $\{x_n\}$ be a sequence in H. We use the following notations:

$$F_1 = \{q \in H : \text{the sequence} \{\|x_n - q\|\} \text{ is nonincreasing}\};$$
$$F_\ell = \{q \in H : \lim_{n\to\infty} \|x_n - q\| \text{ exists}\}.$$

It is clear that $F_1 \subset F_\ell$.

Lemma 2.2 ([2])**.** *Let $\{x_n\}$ be a sequence in H. Then, F_1 and F_ℓ are closed and convex subsets of H.*

Let C be a nonempty subset C of a Hilbert space H and let T be a mapping of a nonempty subset from C to H. We denote by $A(T)$ the set of attractive points of T, i.e.,

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}.$$

The concept of attractive points was introduced by Takahashi and Takeuchi [16].

Let $x \in C$. If $\{x_n\} = \{T^n x\}$, then $A(T) \subset F_1$ (see also [12, 13]).

Lemma 2.3 ([16])**.** *Let C be a nonempty closed and convex subset of H. Let T be a mapping of C into itself and let P_C be the metric projection of H onto C. Assume that $A(T) \neq \emptyset$. Then, if $u \in A(T)$, then $P_C u \in F(T)$. Thus, if $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$.*

Lemma 2.4 ([16])**.** *Let C be a nonempty subset of H and let T be a mapping of C to H. Then, $A(T)$ is closed and convex subset of H.*

Lemma 2.5 ([16])**.** *Let C be a nonempty subset of H and let T be a mapping of C to H. Then,*

$$A(T) \cap C \subset F(T).$$

3 Absolute fixed points

In this section, using the concepts of attractive points, we establish the existence of absolute fixed points of normally 2-generalized hybrid mappings (see also [3, 11, 12, 13]).

Kondo and Takahashi [9] proved the following result.

Theorem 3.1 ([9])**.** *Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. Then, the following are equivalent.*

(i) for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C,

(ii) there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C,

(iii) $A(T) \neq \emptyset$;

The concept of absolute fixed points was introduced by Djafari Rouhani [11] (see also [3, 12, 13]). Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. A point $p \in H$ is said to be an absolute fixed point of T if there exists a normally 2-generalized hybrid extension S of T from $C \cup \{p\}$ to $C \cup \{p\}$ such that $Sp = p$, and if p is a fixed point of every normally 2-generalized hybrid extension of T to the union of C and a subset of H containing p.

Kondo and Takahashi [9] proved the following nonlinear ergodic theorem of Baillon's type ([4]) (see also [2, 3, 4]).

Theorem 3.2 ([9])**.** *Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. Assume that $\{T^n z\}$ is bounded for some $z \in D$. Let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Then, for each $x \in C$, $\frac{1}{n}\sum_{k=1}^{n-1} T^k x$ converges weakly to $u \in A(T)$, where $u = \lim_{n\to\infty} P_{A(T)} T^n x$.*

By Theorem 3.2, we have the following.

Proposition 3.3 ([3])**.** *Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. Assume that $\{T^n z\}$ is bounded for some $z \in C$. Let $x \in C$. Let*

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} T^k x.$$

Then, for each $y \in C$ and $n \in \mathbb{Z}^+$,

$$\|T^{n+1}y - u\| \leq \|T^n y - u\|$$

holds. Thus, $u \in F_1$.

Theorem 3.4 ([3])**.** *Let C be a nonempty subset of H. Let T be an $(\alpha_0,\beta_0,\alpha_1,\beta_1,\alpha_2,\beta_2)$ -normally 2-generalized hybrid mapping of C into itself. Assume that $\{T^n z\}$ is bounded for some $z \in C$. Let $x \in C$. Let*

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} T^k x.$$

Let M be a nonempty subset of H such that $M \supset C \cup \{u\}$. Assume that S is a normally 2-generalized hybrid extension of T to M. Then, $Su = u$.

Adding that C is closed and convex, we can obtain the following fixed point theorem ([9]) by Theorem 3.4 (see also [9, 13, 16]).

Theorem 3.5 ([3])**.** *Let C be a nonempty closed and convex subset H and let T be a normally 2-generalized hybrid mapping of C into itself. Assume that $\{T^n z\}$ is bounded for some $z \in C$. Then, $F(T) \neq \emptyset$.*

We give a sufficient condition for a normally 2-generalized hybrid mapping of C into itself with a bounded orbit to have a normally 2-generalized hybrid extension to $C \cup \{u\}$, where $u = \text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} T^i x$.

Lemma 3.6 ([3])**.** *Let C be a nonempty subset H and let T be an $(\alpha_0,\beta_0,\alpha_1,\beta_1,\alpha_2,\beta_2)$-normally 2-generalized hybrid mapping of C into itself. Assume that $\{T^n z\}$ is bounded for some $z \in C$. Let $x \in C$. Let*

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} T^i x.$$

Then, the mapping $S : C \cup \{u\} \to C \cup \{u\}$ defined by $Sz = Tz$ for all $z \in C$, and $Su = u$ is an $(\alpha_0,\ \beta_0,\ \alpha_1,\ \beta_1,\ \alpha_2,\beta_2)$ -normally 2-generalized hybrid mapping of $C \cup \{u\}$ into itself, if either $\alpha_2+\beta_2 \geq 0$, $\alpha_1+\beta_1 \geq 0$ and $\sum_{n=0}^{2}(\alpha_n+\beta_n) = 0$, or $\alpha_2+\beta_2 < 0$, $\alpha_1+\beta_1 < 0$, $\sum_{n=0}^{2}(\alpha_n+\beta_n) = 0$ and the orbit $\{T^k y\}$ of every $y \in C$ lies on the sphere centered at y, with the radius $\|y-u\|$.

By Lemma 3.6, we establish the existence of absolute fixed points of normally 2-generalized hybrid mappings.

Theorem 3.7 ([3])**.** *Let C be a nonempty subset H and let T be an $(\alpha_0,\beta_0,\alpha_1,\beta_1,\alpha_2,\beta_2)$ -normally 2-generalized hybrid mapping of C into itself. Assume that $\{T^n z\}$ is bounded for some $z \in C$. Let $x \in C$. Let*

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} T^k x.$$

Then, u is an absolute fixed point of T if either $(\alpha_2+\beta_2) \geq 0$, $(\alpha_1+\beta_1) \geq 0$ and $\displaystyle\sum_{n=0}^{2}(\alpha_n+\beta_n) = 0$, or $\alpha_2+\beta_2 < 0$, $\alpha_1+\beta_1 < 0$, $\sum_{n=0}^{2}(\alpha_n+\beta_n) = 0$ and the orbit $\{T^k y\}$ of every $y \in C$ lies on the sphere centered at y, with the radius $\|y-c\|$.

4 Fixed point properties

In this section, motivated by [12, 13], we get some fixed point theorems for normally 2-generalized hybrid mappings defined on nonconvex domains in H.

Let C be a nonempty subset of H. We say that C is Chebyshev with respect to its convex closure, if, for any $y \in \overline{\text{co}}C$, there is a unique $q \in C$ such that

$$\|y-q\| = \inf\{\|y-z\| : z \in C\}$$

(see [12, 13]).

Theorem 4.1 ([3])**.** *Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. Then, T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$, and for any $y \in \overline{\text{co}}\{T^n x : n \in \mathbb{Z}^+\}$, there is a unique $p \in C$ such that $\|y-p\| = \inf\{\|y-z\| : z \in C\}$.*

As a directed consequence of Theorem 4.1, we obtain the following theorem (see [3]).

Theorem 4.2. *Let C be a nonempty subset of H which is Chebyshev with respect to its convex closure. Let T be a normally 2-generalized hybrid mapping of C into itself. Then, T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

If T is a normally 2-generalized hybrid mapping of C into itself, then $\{T^n x\}$ is a normally 2-generalized hybrid sequence in the sense of [2]. Let $\{x_n\} = \{T^n x\}$. Then, we obtain the following weak convergence theorems for normally 2-generalized hybrid mappings by [2] (see also [12, 13]).

Theorem 4.3 ([3])**.** *Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. Suppose that T is weakly asymptotically regular, i.e.,*

$$T^{n+1} - T^n x \rightharpoonup 0$$

for each $x \in C$. Then, the following are equivalent.

(i) $F_1 \neq \emptyset$;

(ii) $F_\ell \neq \emptyset$;

(iii) $A(T) \neq \emptyset$;

(iv) $\{T^n x\}$ *is bounded in H for each $x \in C$.*

(v) $\{T^n z\}$ *is bounded in H for some $z \in C$.*

(vi) $\{T^n x\}$ *converges weakly to an element* $v \in H$.

Moreover, in this case $v = \lim_{n\to\infty} P_{F_1} x_n \in A(T)$, *where* P_{F_1} *is the metric projection of* H *onto* F_1.

Adding that C is closed and convex in Theorem 4.3, we can obtain the following theorem (see [2, 16]).

Theorem 4.4 ([3]). *Let* C *be a nonempty closed and convex subset* H *and let* T *be a normally* 2*-generalized hybrid mapping of* C *into itself. Suppose that* T *is weakly asymptotically regular, i.e.,*

$$T^{n+1} - T^n x \rightharpoonup 0$$

for each $x \in C$. *Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;

(ii) $F_\ell \neq \emptyset$;

(iii) $A(T) \neq \emptyset$;

(iv) $F(T) \neq \emptyset$;

(v) $\{T^n x\}$ *is bounded in* H *for each* $x \in C$.

(vi) $\{T^n z\}$ *is bounded in* H *for some* $z \in C$.

(vii) $\{T^n x\}$ *converges weakly to an element* $v \in H$.

Moreover, in this case $v = \lim_{n\to\infty} P_{F_1} T^n x \in A(T)$, *where* P_{F_1} *is the metric projection of* H *onto* F_1.

References

[1] K. Aoyama, S. Iemoto, F. Kohsaka, W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335-343.

[2] S. Atsushiba, *Nonlinear ergodic theorems for normally* 2*-generalized hybrid sequences*, J. Nonlinear Convex Anal. **21** (2019), 251—264.

[3] S. Atsushiba, *Fixed point property for normally 2-generalized hybrid mappings*, J. Nonl. Var. Anal. **3** (2019), 149–158.

[4] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sei. Paris Ser. A-B **280** (1975), 1511-1514.

[5] S. Iemoto, W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, Nonlinear Anal.* **71** (2009), 2082-2089.

[6] P. Kocourek, W. Takahashi, J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497-2511.

[7] F. Kohsaka, W. Takahashi *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166-177.

[8] F. Kohsaka, W. Takahashi *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008) 824-835.

[9] A. Kondo, W. Takahashi, *Attractive point and weak convergence theorems for normally N-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **3** (2017), 297-310.

[10] T. Maruyama, W. Takahashi, M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185-197.

[11] B. Djafari Rouhani, *On the fixed point property for nonexpansive mappings and semigroups*, Nonlinear Anal. **30** (1997), 389-396.

[12] B. Djafari Rouhani, *Ergodic theorems for hybrid sequences in a Hilbert space with applications*, J. Math. Anal. Appl. **409** (2014), 205-211.

[13] B. Djafari Rouhani, *Ergodic and fixed point theorems for sequences and nonlinear mappings in a Hilbert space*, Demonstratio Math. **51** (2018), 27-36.

[14] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[15] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79-88.

[16] W. Takahashi, Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399-406.

[17] W. Takahashi, J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457-472.