Weak Convergence Theorem for Infinite Families of Nonlinear Mappings in Banach Spaces

芝浦工業大学
北條真弓 (Mayumi Hojo)
Shibaura Institute of Technology, Oomiya, Saitama 337-8570, Japan

Abstract

In this article, we prove a weak convergence theorem of Mann's type iteration for infinite families of extended generalized hybrid mappings in a Banach space satisfying Opial's condition. This theorem solves a problem posed by Hojo and Takahashi [8]. Using this result, we get well-known and new weak convergence theorems in a Banach space. In particular, we obtain a weak convergence theorem of Mann's type iteration for finite families of extended generalized hybrid mappings in a Banach space.

2010 *Mathematics Subject Classification*: 47H10; 47H05
Keywords and phrases: Banach space, extended generalized hybrid mapping, fixed point, weak convergence theorem, Opial's condition

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 2010, Kocourek, Takahashi and Yao [12] defined a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings: Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called $(\alpha,\ \beta)$-*generalized hybrid*. We also know the following: For $\lambda \in \mathbb{R}$, a mapping $U : C \to H$ is called λ-*hybrid* [1] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings in a Hilbert space. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [13, 14] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [19] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [10]. We also know that λ-hybrid mappings in a Hilbert space are contained in the class of generalized hybrid mappings; see [9]. Hojo and Takahashi [7] extended the concept of generalized hybrid mappings in a Hilbert space to that in a Banach space as follows: Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called *extended generalized hybrid* [7] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \leq 0 \tag{1.3}$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid. Hojo and Takahashi [8] proved the following weak convergence theorem for finding a common fixed point of two extended generalized hybrid mappings in a Banach space by using Mann's type iteration [15]; see also [20].

Theorem 1.1 ([8])**.** *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha', \beta', \gamma', \delta' \in \mathbb{R}$. Let S and T be $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma,' \delta')$-extended generalized hybrid mappings of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$ and $\beta' \leq 0$ and $\gamma' \leq 0$, respectively. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < a \leq \alpha_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \gamma_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in F(S) \cap F(T)$, where $F(S) \cap F(T)$ is the set of common fixed points of S and T.

In this article, we prove a weak convergence theorem of Mann's type iteration for infinite families of extended generalized hybrid mappings in a Banach space satisfying Opial's condition. This theorem solves a problem posed by Hojo and Takahashi [8]. Using this result, we get well-known and new weak convergence theorems in a Banach space. In particular, we obtain a weak convergence theorem of Mann's type iteration for finite families of extended generalized hybrid mappings in a Banach space.

2 Preliminaries

Throughout this article, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$

for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [11]. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. The following result is in [18].

Lemma 2.1 ([18]). *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j \rangle,$$

where $j \in Jy$.

Let E be a Banach space and let $A \subset E \times E$. Then, A is accretive if for $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$, where J is the duality mapping of E. An accretive operator $A \subset E \times E$ is called m-accretive if $R(I + rA) = E$ for all $r > 0$, where I is the identity operator and $R(I + rA)$ is the range of $I + rA$. An accretive operator $A \subset E \times E$ is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where $\overline{D(A)}$ is the closure of the domain $D(A)$ of A. An m-accretive operator satisfies the range condition. If C is a nonempty, closed and convex subset of a Banach space and T is a nonexpansive mapping of C into itself, then $A = I - T$ is an accretive operator and $C = D(A) \subset R(I + rA)$ for all $r > 0$; see [18, Theorem 4.6.4].

Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \to E$ is said to be firmly nonexpansive [3] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j \rangle,$$

for all $x, y \in C$, where $j \in J(Tx - Ty)$; see also [2, 5]. It is known that the resolvent of an accretive operator satisfying the range condition in a Banach space is a firmly nonexpansive mapping of the closure of the domain into itself. In fact, let $C = \overline{D(A)}$ and $r > 0$. Define the resolvent J_r of A as follows:

$$J_r x = \{z \in D(A) : x \in z + rAz\}$$

for all $x \in \overline{D(A)}$. It is known that such $J_r x$ is a singleton; see [18]. We have that for $x_1, x_2 \in \overline{D(A)}$, $x_1 = z_1 + ry_1$, $y_1 \in Az_1$ and $x_2 = z_2 + ry_2$, $y_2 \in Az_2$. Since A is accretive, we have that $\langle y_1 - y_2, j \rangle \geq 0$, where $j \in J(z_1 - z_2)$. So, we have

$$\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j \rangle \geq 0.$$

Furthermore, we have that

$$\begin{aligned} &\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j \rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - z_1 - (x_2 - z_2), j \rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - x_2, j \rangle \geq \|z_1 - z_2\|^2. \end{aligned}$$

From $z_1 = J_r x_1$ and $z_2 = J_r x_2$, we have that J_r is a firmly nonexpansive mapping of C into itself; see also [3], [4] and [21]. Let E be a Banach space and let C be a nonempty subset of

E. A mapping $T : C \to E$ is called extended generalized hybrid if it satisfies (1.3), that is, there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid. We can also show that, in a Banach space, an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping is nonexpansive for $\alpha = 1$, $\beta = \gamma = 0$ and $\delta = -1$, nonspreading for $\alpha = 2$, $\beta = \gamma = -1$ and $\delta = 0$, and hybrid for $\alpha = 3$, $\beta = \gamma = -1$ and $\delta = -1$. Nonexpansive mappings, nonspreading mappings and hybrid mappings in a Banach space are deduced from firmly nonexpansive mappings as follows: Let T be a firmly nonexpansive mapping of C into E. Then we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle.$$

From Theorem 2.1 we have that

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \langle x - y, j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\ &\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\ &\Longleftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2. \end{aligned} \tag{2.1}$$

Futhermore, we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \langle x - y, j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x - Tx, j\rangle + 2\langle Ty - y, j\rangle \\ &\Longrightarrow 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\ &\Longleftrightarrow 0 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\ &\Longleftrightarrow 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2. \end{aligned} \tag{2.2}$$

Therefore, using (2.1) and (2.2), we have that

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \langle x - y, j\rangle \\ &\Longrightarrow 3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|x - y\|^2. \end{aligned}$$

Hojo and Takahashi [7] proved the following result.

Lemma 2.2 ([7])**.** *Let E be a Banach space, let C be a nonempty, closed and convex subset of E. Then an extended generalized hybrid mapping which has a fixed point is quasi-nonexpansive.*

The following result was proved by Xu [22].

Lemma 2.3 ([22])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\mu x + (1 - \mu)y\|^2 \leq \mu\|x\|^2 + (1 - \mu)\|y\|^2 - \mu(1 - \mu)g(\|x - y\|)$$

for all $x, y \in B_r$ and μ with $0 \leq \mu \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a Banach space. Then, E satisfies Opial's condition [16] if for any $\{x_n\}$ of E such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Let E be a Banach space. Let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be a mapping. Then, $p \in C$ is called an *asymptotic fixed point* of T [17] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $T : C \to E$ is said to be *demiclosed* if $\hat{F}(T) = F(T)$. We know the following result from Hojo and Takahashi [7].

Lemma 2.4 ([7]). *Let E be a Banach space satisfying Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into E which satisfies $\beta \leq 0$ and $\gamma \leq 0$. Then $\hat{F}(T) = F(T)$, i.e., T is demiclosed.*

If E is a Banach space satisfying Opial's condition, then nonexpansive mappings, nonspreading mappngs and hybrid mappings are demiclosed; see [7].

3 Weak Convergence Theorems

In this section, we first prove a weak convergence theorem of Mann's type iteration [15] for an infinite family of extended generalized hybrid mappings in a Banach space satisfying Opial's condition; see also Hojo[6].

Theorem 3.1. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R}$ for all $j \in \mathbb{N}$ and let $\{T_j\}$ be a sequence of $(\alpha_j, \beta_j, \gamma_j, \delta_j)$-extended generalized hybrid mappings of C into itself such that $\beta_j \leq 0$ and $\gamma_j \leq 0$ for all $j \in \mathbb{N}$. Suppose that $\cap_{j=1}^{\infty} F(T_j) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{j=1}^{\infty} \xi_j T_j x_n, \quad \forall n \in \mathbb{N},$$

where $a, b \in \mathbb{R}$ and $\{\xi_j\}, \{\alpha_n\} \subset (0, 1)$ satisfy the following:

(1) $\sum_{j=1}^{\infty} \xi_j = 1$;
(2) $0 < a \leq \alpha_n \leq b < 1, \quad \forall n \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \cap_{j=1}^{\infty} F(T_j)$.

Using Theorem 3.1, we obtain the following weak convergence theorem for a finite family of extended generalized hybrid mappings in a Banach space satisfying Opial's condition; see Hojo and Takahashi [7] for two extended generalized hybrid mappings.

Theorem 3.2 ([7]). *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R}$ for all $j \in \{1, 2, \ldots, M\}$ and let $\{T_j\}_{j=1}^{M}$ be a finite family of $(\alpha_j, \beta_j, \gamma_j, \delta_j)$-extended generalized hybrid mappings of C into itself such that $\beta_j \leq 0$ and $\gamma_j \leq 0$ for all $j \in \{1, 2, \ldots, M\}$. Suppose that $\cap_{j=1}^{M} F(T_j) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{j=1}^{M} \xi_j T_j x_n, \quad \forall n \in \mathbb{N},$$

where $a, b \in \mathbb{R}$ and $\{\xi_j\}, \{\alpha_n\} \subset (0,1)$ satisfy the following:

(1) $\sum_{j=1}^{M} \xi_j = 1$;
(2) $0 < a \leq \alpha_n \leq b < 1, \quad \forall n \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \cap_{j=1}^{M} F(T_j)$.

Using Theorem 3.2, we obtain the following result.

Theorem 3.3. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R}$ for all $j \in \{1, 2, \ldots, M\}$ and let $\{T_j\}_{j=1}^{M}$ be a finite family of $(\alpha_j, \beta_j, \gamma_j, \delta_j)$-extended generalized hybrid mappings of C into itself such that $\beta_j \leq 0$ and $\gamma_j \leq 0$ for all $j \in \{1, 2, \ldots, M\}$. Suppose that $\cap_{j=1}^{M} F(T_j) \neq \emptyset$. Let λ be a real number with $0 < \lambda < 1$. Define a mapping $U : C \to C$ by*

$$U = \lambda I + (1-\lambda) \sum_{j=1}^{M} \xi_j T_j,$$

where $\{\xi_j\} \subset (0,1)$ satisfies $\sum_{j=1}^{M} \xi_j = 1$. Then for any $x \in C$, $U^n x$ converges weakly to an element $z \in \cap_{j=1}^{M} F(T_j)$.

Using Theorem 3.2, we also obtain the following result [7].

Theorem 3.4 ([7])**.** *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$ and define a sequence $\{x_n\}$ of C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \quad \forall n \in \mathbb{N}.$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Using Theorems 3.1 and 3.2, we can also prove the following weak convergence theorems for families of nonexpansive mappings and nonspreading mappings in a Banach space.

Theorem 3.5. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\{T_j\}$ be a sequence of nonexpansive mappings of C into itself. Let $\{\xi_j\}$ be a family of real numbers in $(0,1)$ such that $\sum_{j=1}^{\infty} \xi_j = 1$. Suppose that*

$$\Omega := \cap_{j=1}^{\infty} F(T_j) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) \sum_{j=1}^{\infty} \xi_j T_j x_n, \quad \forall n \in \mathbb{N},$$

where $a, b \in \mathbb{R}$ and $\{\lambda_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n \leq b < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \Omega$.

Theorem 3.6. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\{T_j\}_{j=1}^M$ be a sequence of nonspreading mappings of C into itself. Let $\{\xi_j\}$ be a family of real numbers in $(0,1)$ such that $\sum_{j=1}^M \xi_j = 1$. Suppose that*

$$\Omega := \cap_{j=1}^M F(T_j) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) \sum_{j=1}^M \xi_j T_j x_n, \quad \forall n \in \mathbb{N},$$

where $a, b \in \mathbb{R}$ and $\{\lambda_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n \leq b < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \Omega$.

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