NOTE ON SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR FRACTIONAL ORDER BEAM EQUATIONS

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1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ the set of all real numbers. In [19], we consider the boundary value problem for fractional order differential equation

$$\begin{cases} D^{\beta}_{0+}D^{\alpha}_{0+}u(t) - f(t,u(t),D^{\alpha}_{0+}u(t)) = 0, t \in [0,1] \\ u(0) = A, u(1) = B, D^{\alpha}_{0+}u(0) = C, D^{\alpha}_{0+}u(1) = D, \end{cases} \tag{1.1}$$

where D^{α}_{0+} is the Riemann-Liouville derivative of order α with respect to t, $1 < \alpha, \beta \leq 2$, A, B, C, D are constants, and f is a continuous function of $[0,1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. In this paper we propose the following differential equation (1.2) of order α, $3 < \alpha \leq 4$ with the two point boundary condition involving the form (1.1). For simplicity, we consider the cases of $A = B = C = D = 0$.

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t,u(t),D^{\alpha-3}_{0+}u(t),D_{0+}D^{\alpha-3}_{0+}u(t),D_{0+}D_{0+}D^{\alpha-3}_{0+}u(t)), \\ \quad 0 < t < 1, \\ u(0) = u(1) = 0, D_{0+}D^{\alpha-3}_{0+}u(0) = D_{0+}D^{\alpha-3}_{0+}u(1) = 0, \end{cases} \tag{1.2}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative and f is a function of $[0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Let $\alpha > 0$. The Riemann-Liouville fractional integral of order α of u, denoted $I^{\alpha}_{0+}u$, is defined by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided the right-hand side exists. The Riemann-Liouville fractional derivative of order α of a function u of $(0,\infty)$ into $\mathbb{R}$ is given by

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $n = [\alpha]+1$ ($[\alpha]$ denotes the integer part of α) and $\Gamma(\alpha)$ denotes the gamma function; see [11, 18]. Note that for $\alpha > \beta > 0$, we have

$$D^{\alpha}_{0+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.$$

2010 *Mathematics Subject Classification.* Primary 34A08, 34B15, 47H10.
Key words and phrases. Fixed point theorem, fractional order, differential equations, boundary value problem.

A function $u \in C[0,1]$ is called a solution of problem (1.2) if $D^{\alpha}_{0+}u \in C[0,1]$, $D^{\alpha-3}_{0+}u \in L^1[0,1]$, $D^{\alpha-2}_{0+}u \in L^1[0,1]$, $D^{\alpha-1}u \in L^1[0,1]$, u satisfies the boundary conditions and equality in (1.2) a.e. on [0, 1].

Many researchers have considered the differential equation (1.2) with $\alpha = 4$; see [1, 2, 9, 10, 14, 15, 21, 22, 23, 25, 26]. Equation (1.2) with $\alpha = 4$ can be used to model the deformations of an elastic beam; see [21, 22] and the references therein. The boundary conditions in (1.2) with respect to normal derivative ensures that both endpoints are simply supported. Meanwhile, fractional differential equations have been of interest recently; see [3, 5, 6, 7, 8, 11, 12, 17, 18, 19, 24]. In particular, for higher order boundary problems, see [17, 19, 20, 24]. However, to the best of our knowledge, there are no results for the boundary value problem represented by (1.2) for $3 < \alpha \leq 4$, which we consider in the present paper. We use the several methods to prove the existence and uniqueness of solutions. Moreover we consider the properties of Green function given by (1.2).

2. Lemmas

For a continuous mapping h of $[0,1]$ into $\mathbb{R}$, we consider the following fractional differential boundary problems defined by

$$\begin{cases} D^{\alpha}_{0+}u(t) = h(t), \quad 0 < t < 1, \\ u(0) = u(1) = 0, D_{0+}D^{\alpha-3}_{0+}u(0) = D_{0+}D^{\alpha-3}_{0+}u(1) = 0, \end{cases} \tag{2.1}$$

where $3 < \alpha \leq 4$. In this section, we show the unique solution to the boundary value problem represented by (2.1). A mapping u of $[0,1]$ into $\mathbb{R}$ is a solution of that boundary value problem if u is continuous on $[0,1]$ and u satisfies (2.1). The following lemma can be found in [6]; see [11] also. We denoted by $C(0,1)$ the set of all continuous mappings of $(0,1)$ into $\mathbb{R}$ and by $L(0,1)$ the set of all Lebesgue integrable mappings of $[0,1]$ into $\mathbb{R}$.

Lemma 2.1. *Let $\alpha > 0$. If $u(t) \in C(0,1) \cap L(0,1)$ satisfying $D^{\alpha}_{0+}u(t) \in C(0,1) \cap L(0,1)$, then there exist constants $C_1, C_2, \ldots, C_n \in \mathbb{R}$ such that*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where $n = [\alpha] + 1$ and $I^{\alpha}_{0+}u$ is the Riemann-Liouville fractional integral of order α of a function u defined by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

Using Lemma 2.1, we obtain the following.

Lemma 2.2. *Let h be a continuous mapping of $[0,1]$ into $\mathbb{R}$. Let $3 < \alpha \leq 4$. Then the unique solution of the boundary value problem represented by (2.1) is*

$$u(t) = \int_0^1 G(t,s)h(s)ds,$$

where

(2.2)

$$G(t,s)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\big((t-s)^{\alpha-1}-(1-s)t^{\alpha-1}-(1-s)^{\alpha-1}t^{\alpha-3}+(1-s)t^{\alpha-3}\big)\\ (0\le s\le t\le 1),\\ -\dfrac{1}{\Gamma(\alpha)}\big((1-s)t^{\alpha-1}+(1-s)^{\alpha-1}t^{\alpha-3}-(1-s)t^{\alpha-3}\big)\\ (0\le t\le s\le 1).\end{cases}$$

Remark 2.3. If $\alpha=4$, then

$$G(t,s)=\begin{cases}\frac{1}{6}\big(s(1-t)(2t-t^2-s^2)\big)\ (0\le s\le t\le 1),\\ \frac{1}{6}\big(t(1-s)(2s-s^2-t^2)\big)\ (0\le t\le s\le 1).\end{cases}$$

Lemma 2.4. *Let $3<\alpha\le 4$. The function $G(t,s)$ in Lemma 2.2 satisfies the following conditions:*

(i) $m\le G(t,s)\le M$, *where*

$$m=\begin{cases}s(t(1-t)(1-t)-(t-s)(t-s))\ \textit{if}\ (s\le t),\\ 0\ \textit{if}\ (t\le s),\end{cases}$$

$$M=\begin{cases}(t-s)^2+(1-s)(2t-t^2-s^2)\ \textit{if}\ (s\le t),\\ (1-t)(2s-t^2-s^2)\ \textit{if}\ (t\le s).\end{cases}$$

(ii) *If s and t satisfy $0\le t\le\frac{3-\sqrt{5}}{2}$, or $1\ge t\ge\frac{3-\sqrt{5}}{2}$ and $t-\sqrt{t}(1-t)\le s\le t$, then $G(t,s)\ge 0$.*

We consider the following.

$$\int_0^t G(t,s)h(s)ds=[-G(t,s)v(s)]_0^t+\int_0^t G_1(t,s)v(s)ds,$$

and

$$\int_t^1 G(t,s)h(s)ds=[-G(t,s)v(s)]_t^1+\int_t^1 G_1(t,s)v(s)ds,$$

where $v(s)=\int_t^1 h(s)ds$ and $G_1(t,s)=\frac{\partial G}{\partial s}(t,s)$

(2.3)

$$G_1(t,s)=\begin{cases}\frac{1}{\Gamma(\alpha)}\big(-(\alpha-1)(t-s)^{\alpha-2}+t^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}t^{\alpha-3}-t^{\alpha-3}\big)\\ \text{if } 0\le s\le t\le 1\\ \frac{1}{\Gamma(\alpha)}t^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}t^{\alpha-3}-t^{\alpha-3}\\ \text{if } 0\le t\le s\le 1.\end{cases}$$

Moreover

$$\int_0^1 G(t,s)h(s)ds=\int_0^1 G_1(t,s)v(s)ds.$$

We also have

$$\int_0^1 D_{0+}^{\alpha-3} G(t,s)h(s)ds = \int_0^t G_2(t,s)v(s)ds,$$

$$\int_0^1 D_{0+} D_{0+}^{\alpha-3} G(t,s)h(s)ds = \int_0^1 G_3(t,s)v(s)ds,$$

where

$$G_2(t,s) = \begin{cases} \frac{1}{\Gamma(2)}((t-s)-(1-s)t^{\alpha-3}) & (0 \le s \le t \le 1), \\ -\frac{1}{\Gamma(2)}(1-s)t^{\alpha-3} & (0 \le t \le s \le 1), \end{cases} \tag{2.4}$$

and

$$G_3(t,s) = \frac{\partial G_2}{\partial s}(t,s) = \begin{cases} t^{\alpha-3} - 1 \ (0 \le s \le t \le 1) \\ t^{\alpha-3} \ (0 \le t \le s \le 1). \end{cases} \tag{2.5}$$

Remark 2.5. The function $\int_0^1 G(\cdot, s)ds$ is continuous on $[0,1]$. In fact, we have

$$\int_0^1 G(t,s)ds = \frac{1}{\alpha\Gamma(\alpha)} \left(t^\alpha - \frac{\alpha}{2} t^{\alpha-1} + \left(1 - \frac{\alpha}{2}\right) t^{\alpha-3} \right) \text{ for all } 0 \le t \le 1.$$

3. Main result

Next we use the method of order reduction to transform (1.2) to a nonlinear integral equation. To do this, let

$$T_1 v(t) = I_{0+}^{\alpha-3} T_2 v(t) = \int_0^1 G_1(t,s)v(s)ds,$$

$$T_2 v(t) = \int_0^1 G_2(t,s)v(s)ds, \ T_3 v(t) = \int_0^1 G_3(t,s)v(s)ds,$$

where $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ are given by (2.3), (2.4) and (2.5). From the above formulas, it follows that

$$D_{0+} D_{0+} D_{0+}^{\alpha-3} T_1 v(t) = D_{0+} D_{0+} T_2 v(t) = D_{0+} T_3 v(t) = -v(t).$$

Note that since

$$T_1 v(t) = \int_0^1 G_1(t,s)v(s)ds = \int_0^1 G(t,s)f(s)ds,$$

we have

$$T_1 v(0) = T_1 v(1) = 0.$$

Moreover by definition,

$$T_2 v(0) = \int_0^1 G_2(0,s)ds = 0, T_2 v(1) = \int_0^1 G_2(1,s)ds = 0.$$

Boundary value problem (1.1) can be converted into a ternminal value problem

$$D_{0+} v(t) = -f(t, T_1 v(t), T_2 v(t), T_3 v(t), -v(t)), \int_0^1 v(s)ds = 0.$$

From the above formulas, it follows that

$$D_{0+}v(t) = f(t, T_1v(t), T_2v(t), T_3v(t), -v(t))$$

where

$$D_{0+}T_3v(t) = -v(t),\ D_{0+}T_2v(t) = T_3v(t),\ D_{0+}^{\alpha-3}T_1v(t) = T_2v(t).$$

Then we have the following lemma.

Lemma 3.1. *Let $3 < \alpha \leq 4$. The boundary value problem (1.2) is equivalent to the following integral equations forms;*

$$\begin{cases} v(t) = \int_t^1 f(s, T_1v(s), T_2v(s), T_3v(s), -v(s))ds, \\ T_1v(t) = \int_0^1 G_1(t,s)v(s)ds, \\ T_2v(t) = \int_0^1 G_2(t,s)v(s)ds, \\ T_3v(t) = \int_0^1 G_3(t,s)v(s)ds, \end{cases}$$

where $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ are given by (2.3), (2.4) and (2.5).

Lemma 3.2. *$G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ satisfy the followings.*

(i) $-8 \leq 3s+3s^3-3t-4t^2-1 \leq G_1(t,s) \leq 3t^2-2s^2+3-4s \leq 6$, *if* $0 \leq s \leq t \leq 1$
$-5 \leq 3t^3 - 4s^2 + 2t - 1 \leq G_1(t,s) \leq t^2 + 3 - 4t \leq 3$, *if* $0 \leq t \leq s \leq 1$.

(ii) $-1 \leq G_2(t,s) \leq 0$, $-1 \leq G_3(t,s) \leq 1$.

Next we define an operator A from $C[0,1]$ into $C[0,1]$ by

$$Av(t) = \int_t^1 f(s, T_1v(s), T_2v(s), T_3v(s), -v(s))ds$$

,

where $v \in C[0,1]$. Then the solution of boundary value problem (1.2) is a fixed point of mapping A. Also let

$$(T_1v)(t) = \int_0^1 G_1(t,s)v(s)ds,\ (T_2v)(t) = \int_0^1 G_2(t,s)v(s)ds,\ (T_3v)(t) = \int_0^1 v(s)ds.$$

Then the existence of solution of the boundary value problem (1.2) is equivalent to the existence of fixed point of A on $C[0,1]$. Take $u_0(t) = 1 - t$.

$$\int_t^1 (T_1u_0)(t)dt \leq \frac{1}{\Gamma(\alpha+2)}u_0(t),\ \int_t^1 (T_2u_0)(t)dt \leq \left(\frac{1}{3(\alpha-2)} - \frac{1}{8}\right)u_0(t).$$

$$\int_t^1 (T_3u_0)(t)dt \leq 0.0276515u_0.(\alpha = 3.5,$$

$t = 1/6(8 - 8/(-109 + 27\sqrt{17})^{1/3} + (-109 + 27\sqrt{17})^{1/3}) \sim 0.5474636625659386)$

Moreover if $\alpha = 4$,

$$\int_t^1 (T_3u_0)(t)dt \leq \frac{3}{32}u_0 = 0.09375u_0,\ (t = 1/4).$$

If $\alpha = 3$, $\int_t^1 (T_3u_0)(t)dt \leq \frac{1}{6}u_0(t) = 0.166667u_0(t)$ $(t = 0)$.

$$C_1 = \frac{1}{\Gamma(\alpha+2)},\ C_2 = \frac{1}{3(a-2)} - \frac{1}{8},\ C_3 = 0.0276515,\ (\alpha = 3.5). \tag{3.1}$$

If $\alpha = 4$, then we have

$$C_1 = \frac{1}{120},\, C_2 = \frac{1}{24}.$$

However if $\alpha = 4$, calculate directly, then we can take

$$C_1 = \frac{291}{30720} > \frac{1}{120},\, C_2 = \frac{1}{9\sqrt{6}} = 0.04536090 > \frac{1}{24} = 0.0416667,$$
$$C_3 = \frac{3}{32} = 0.09375 > 0.0276515.$$

Now we have the following theorem, which is the version of [27, Theorem 1].

Theorem 3.3. *Suppose that there exist four nonnegative constants M_i ($i = 1, 2, 3, 4$) with $M_1C_1 + M_2C_2 + M_3C_3 + M_4 < 1$ such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \le \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.2) has a unique solution.

Next we consider the Banach contraction principal. Then, there exist constants

$$D_1 = \frac{4}{\Gamma(\alpha)}\left(\frac{\alpha-1}{\alpha(\alpha-2)}\right),\, D_2 = \frac{1}{6} + \frac{1}{2(\alpha-2)},\, D_3 = \frac{1}{2} - \frac{3-\alpha}{(\alpha-1)(\alpha-2)}.$$

If $\alpha = 4$, then

$$D_1 = \frac{1}{4},\, D_2 = \frac{5}{12},\, D_3 = \frac{2}{3}.$$

Note that for $\alpha = 4$, if we calculate directly, result is same. In this case we also have the theorem, which is the version of [27, Theorem 3].

Theorem 3.4. *Suppose that there exist four nonnegative constants M_i ($i = 1, 2, 3, 4$) with $M_1D_1 + M_2D_2 + M_3D_3 + M_4 < 1$ such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \le \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.2) has a unique solution.

We also have the theorem, which is the version of [27, Theorem 3]. In this section, we consider the existence and uniqueness of solutions of the boundary value problem represented by (1.2). It seems that there are few uniqueness results if the norm of related linear operator is greater than 1. In fact, Theorems 3.3, conclude that $r(T)$ is less than 1, where $r(T)$ is the spectral radius of linear operator T. Note that $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. By Theorem 3.3, since for any $v \in C[0.1]$, $T^n v(t) \le NM^n u_0(t)$, we have $\|T^n\|^{\frac{1}{n}} \le M < 1$, thus we have $r(T) < 1$.

Theorem 3.5. *Suppose that there exist four nonnegative constants M_i ($i = 1, 2, 3, 4$) such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \le \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R},$$

and $r(T) < 1$. Then the boundary value problem (1.2) has a unique solution.

Finally we consider the method in [28]. In order to do this, we give several lemmas. First put $H_1(t,s) = D_{0+}^{\alpha-3}G(t,s)$. Thus we have

(3.2)
$$H_1(t,s) = \begin{cases} \frac{1}{2}(t-s)^2 - \frac{1}{2}(1-s)t^2 + \frac{1}{(\alpha-1)(\alpha-2)}(-(1-s)^{\alpha-1} + (1-s))) \text{ if } 0 \le s \le t \le 1, \\ -\frac{1}{2}(1-s)t^2 + \frac{1}{(\alpha-1)(\alpha-2)}(-(1-s)^{\alpha-1} + (1-s)) \text{ if } 0 \le t \le s \le 1. \end{cases}$$

Also put $H_2(t,s) = D_{0+}D_{0+}^{\alpha-3}G(t,s)$.

$$H_2(t,s) = \begin{cases} st - s = s(t-1) \text{ if } 0 \le s \le t \le 1, \\ s-1 \text{ if } 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

$$H_3(t,s) = \begin{cases} s \text{ if } 0 \le s \le t \le 1, \\ 0 \text{ if } 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

Lemma 3.6. *$H_1(t,s)$ satisfies the following.*

(i) $-\frac{5}{6}s(1-s) \le H_1(t,s) \le \frac{1}{2}\left((t-s)^2 + (1-s)(s(1-s) + t(1-t))\right)$ *if* $0 \le s \le t \le 1$,
$\frac{1}{6}(1-s)t(1-3t) \le H_1(t,s) \le \frac{1}{2}(1-s)(s(1-s) + t(1-t))$ *if* $0 \le t \le s \le 1$.

(ii) *If* $0 \le t \le 1 - \sqrt{1-s} \le s$, *then* $H_1(t,s) \ge 0$.

For $u \in C^\alpha[0,1]$, put $\varphi(t) = f(t, u(t), D_{0+}^{\alpha-3}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-1}u(t))$. Then the boundary value problem (1.2) becomes

$$\begin{cases} D_{0+}^{\alpha}u\varphi(t) = \varphi(t) \\ u(0) = u(1) = 0, D_{0+}^{\alpha-2}u(0) = D_{0+}^{\alpha-2}u(1) = 0, \end{cases}$$

where

$$u(t) = \int_0^1 G(t,s)\varphi(s)ds,\ D_{0+}^{\alpha-3}u(t) = \int_0^1 H_1(t,s)\varphi(s)ds,$$
$$D_{0+}^{\alpha-2}u(t) = \int_0^1 H_2(t,s)\varphi(s)ds,\ D_{0+}^{\alpha-1}u(t) = \int_0^1 H_3(t,s)\varphi(s)ds.$$

For φ, we have the equation $A\varphi = \varphi$, where A is a non-linear operator defined by

$$A\varphi(t) = f(t, u_\varphi(t), v_\varphi(t), w_\varphi(t), x_\varphi(t)),$$

with

$$v_\varphi(t) = D_{0+}^{\alpha-3}u(t),\ w_\varphi(t) = D_{0+}^{\alpha-2}u(t),\ x_\varphi(t) = D_{0+}^{\alpha-1}u(t).$$

Thus we have the following lemma.

Lemma 3.7. *Let $3 < \alpha \leq 4$. The boundary problem (1.2) is equivalent to the following integral equations forms;*

$$\begin{cases} \varphi(t) = f(t, u_\varphi(t), v_\varphi(t), w_\varphi(t), x_\varphi(t)), \\ v_\varphi(t) = \int_0^1 H_1(t,s)\varphi(s)ds, \\ w_\varphi(t) = \int_0^1 H_2(t,s)\varphi(s)ds, \\ x_\varphi(t) = \int_0^1 H_3(t,s)\varphi(s)ds, \end{cases}$$

where $H_1(t,s)$, $H_2(t,s)$ and $H_3(t,s)$ are given by (3.2), (3.3) and (3.4).

By (3.2) and Lemma 3.6, there exists E_1, E_2, E_3 and E_4 such that

$$E_1 = \sup_{t\in[0,1]} \int_0^1 |G(t,s)|ds, E_2 = \sup_{t\in[0,1]} \int_0^1 |H_1(t,s)|ds,$$

$$E_3 = \sup_{t\in[0,1]} \int_0^1 |H_2(t,s)|ds, E_4 = \sup_{t\in[0,1]} \int_0^1 |H_3(t,s)|ds.$$

In this case following theorem holds. It is a version of [28].

Theorem 3.8. *Suppose that there exist four nonnegative constants M_i $(i = 1,2,3,4)$ with $M_1E_1 + M_2E_2 + M_3E_3 + M_4E_4 < 1$ such that*

$$|f(t,x_1,x_2,x_3,x_4) - f(t,y_1,y_2,y_3,y_4)| \leq \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.2) has a unique solution.

For the case that $\alpha = 4$, we have the following; see Dang and Ngo [28].

Corollary 3.9. *Let f be a continuous function of $[0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Let g be a Lipschitz continuous function of $\mathbb{R}$ into itself with a nonnegative constant L. Assume that there exists nonnegative constants M_1, M_2, M_3 and M_4 with*

$$\frac{1}{120}M_1 + \frac{1}{6}M_2 + \frac{5}{12}M_3 + M_4 < 1$$

such that

$$|f(t,x_1,x_2,x_3,x_4) - f(t,y_1,y_2,y_3,y_4)| \leq \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem represented by (1.2) has a unique solution.

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