Stated skein algebras and their representations

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1 Introduction

Skein algebras have been introduced by Przytycki and Turaev at the end of the 80's as a tool to study the SU(2) Witten-Reshetikhin-Turaev topological quantum field theories. Skein algebras appear in TQFTs through their finite dimensional representations. Such representations exist if and only if the parameter A is a root of unity. A generalization named stated skein algebras were introduced by Bonahon-Wong [BW11] and Lê [Le18]. The goal of the present note is to state what is known and to dress a series of open problems towards the resolution of the following

Problem 1.1. Classify all finite dimensional weight representations of stated skein algebras when A is a root of unity of odd order.

As we shall see, this problem is deeply connected to the study of the Poisson geometry of relative character varieties, more precisely to the computation of their symplectic leaves. Here we choose the order of A to be odd and restrict to weight representations for simplicity. Actually, even for the bigon, one of the simplest marked surface, Problem 1.1 is undecidable so we will reformulate later a more reasonable version in Problem 6.2.

2 Definition of stated skein algebras

Definition 2.1. A *marked surface* $\mathbf{\Sigma} = (\Sigma, \mathcal{A})$ is a compact oriented surface Σ (possibly with boundary) with a finite set $\mathcal{A} = \{a_i\}_i$ of orientation-preserving immersions $a_i : [0,1] \hookrightarrow \partial\Sigma$, *named boundary arcs*, whose restrictions to $(0,1)$ are embeddings and whose interiors are pairwise disjoint.

An *embedding* $f : (\Sigma, \mathcal{A}) \to (\Sigma', \mathcal{A}')$ of marked surfaces is a orientation-preserving proper embedding $f : \Sigma \to \Sigma'$ so that for each boundary arc $a \in \mathcal{A}$ there exists $a' \in \mathcal{A}$ such that $f \circ a$ is the restriction of a' to some subinterval of $[0,1]$. When several boundary arcs $a_1, \ldots, a_n$ in $\mathbf{\Sigma}$ are mapped to the same boundary arc b of $\mathbf{\Sigma}'$ we include in the definition of f the datum of a total ordering of $\{a_1, \ldots, a_n\}$. Marked surfaces with embeddings form a category MS with symmetric monoidal structure given by disjoint union.

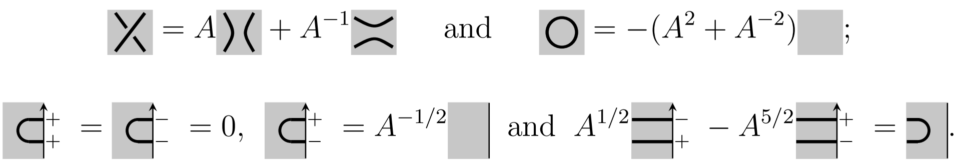
By abuse of notations, we will often denote by the same letter the embedding a_i and its image $a_i((0,1)) \subset \partial\Sigma$ and both call them boundary arcs. We will also abusively identify $\mathcal{A}$ with the disjoint union $\bigsqcup_i a_i((0,1)) \subset \partial\Sigma$ of open intervals. The main interest in considering marked surfaces is that they have a natural gluing operation. Let $\mathbf{\Sigma} = (\Sigma, \mathcal{A})$ be a marked surface and $a, b \in \mathcal{A}$ two boundary arcs. Set $\Sigma_{a\#b} := \Sigma \big/ a(t) \sim b(1-t)$ and $\mathcal{A}_{a\#b} := \mathcal{A} \setminus a \cup b$. The marked surface $\mathbf{\Sigma}_{a\#b} = (\Sigma_{a\#b}, \mathcal{A}_{a\#b})$ is said obtained from $\mathbf{\Sigma}$ by gluing a and b. We say that $\mathbf{\Sigma} = (\mathbf{\Sigma}, \mathcal{A})$ is *unmarked* if $\mathcal{A} = \emptyset$. A *puncture* is a connected component of $\partial\Sigma \setminus \mathcal{A}$.

Notations 2.2. Let us name some marked surfaces. Let $\Sigma_{g,n}$ be an oriented connected surface of genus g with n boundary components. We will sometimes write $\Sigma_g = \Sigma_{g,0}$.

1. The n^{th}*-punctured monogon* $\mathbf{m}_n = (\Sigma_{0,n+1}, \{a\})$ is $\Sigma_{0,n+1}$ with one boundary arc in one of its boundary component.
2. The n^{th}*-punctured bigon* $\mathbb{D}_n = (\Sigma_{0,n+1}, \{a, b\})$ is $\Sigma_{0,n+1}$ with two boundary arcs in the same boundary component. We call $\mathbb{B} := \mathbb{D}_0$ simply the *bigon*. We also write $\mathbb{D}_1^+ := (\Sigma_{0,2}, \{a, b\})$ the annulus with one boundary arc in each boundary component.
3. The triangle $\mathbb{T} = (D^2, \{a, b, c\})$ is a disc with three boundary arcs on its boundary.
4. We denote by $\mathbf{\Sigma}^0_{g,n} = (\Sigma_{g,n+1}, \{a\})$ the surface $\Sigma_{g,n+1}$ with a single boundary arc in one of its only boundary component.

A *tangle* is a compact framed, properly embedded 1-dimensional manifold $T \subset \Sigma \times (0,1)$ such that for every point of $\partial T \subset \mathcal{A} \times (0,1)$ the framing is parallel to the $(0,1)$ factor and points to the direction of 1. The *height* of $(v, h) \in \Sigma \times (0,1)$ is h. If a is a boundary arc and T a tangle, we impose that no two points in $\partial_a T := \partial T \cap a \times (0,1)$ have the same heights, hence the set $\partial_a T$ is totally ordered by the heights. Two tangles are isotopic if they are isotopic through the class of tangles that preserve the boundary height orders. By convention, the empty set is a tangle only isotopic to itself. A *state* is a map $s : \partial T \to \{\pm\}$ and a *stated tangle* is a pair (T, s).

Definition 2.3. Let k be a (unital associative) commutative ring and let $A^{1/2} \in k^\times$ be an invertible element. The *stated skein algebra* $\mathcal{S}_A(\mathbf{\Sigma})$ is the free k-module generated by isotopy classes of stated tangles in $\Sigma \times (0,1)$ modulo the following skein relations



The product of two classes of stated tangles $[T_1, s_1]$ and $[T_2, s_2]$ is defined by isotoping T_1 and T_2 in $\Sigma \times (1/2, 1)$ and $\Sigma \times (0, 1/2)$ respectively and then setting $[T_1, s_1] \cdot [T_2, s_2] = [T_1 \cup T_2, s_1 \cup s_2]$. Now consider an embedding $f : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ of marked surfaces and define a proper embedding $\widetilde{f} : \Sigma_1 \times (0,1) \to \Sigma_2 \times (0,1)$ such that: (1) $\widetilde{f}(x,t) = (f(x), \varphi(x,t))$ for φ a smooth map and (2) if a_1, a_2 are two boundary arcs of $\mathbf{\Sigma}_1$ mapped to the same boundary arc b of $\mathbf{\Sigma}_2$ and the ordering of f is $a_1 < a_2$, then for all $x_1 \in a_1, x_2 \in a_2, t_1, t_2 \in (0,1)$ one has $\varphi(x_1, t_1) < \varphi(x_2, t_2)$. We define $f_* : \mathcal{S}_A(\mathbf{\Sigma}_1) \to \mathcal{S}_A(\mathbf{\Sigma}_2)$ by $f_*([T, s]) := [\widetilde{f}(T), s \circ \widetilde{f}^{-1}]$. The assignment $\mathbf{\Sigma} \to \mathcal{S}_A(\mathbf{\Sigma})$ defines a symmetric monoidal functor $\mathcal{S}_A : \mathrm{MS} \to \mathrm{Alg}_k$.

Let a, b be two distinct boundary arcs of $\mathbf{\Sigma}$, denote by $\pi : \Sigma \to \Sigma_{a\#b}$ the projection and $c := \pi(a) = \pi(b)$. Let (T_0, s_0) be a stated framed tangle of $\Sigma_{a\#b} \times (0,1)$ transversed to $c \times (0,1)$ and such that the heights of the points of $T_0 \cap c \times (0,1)$ are pairwise distinct and the framing of the points of $T_0 \cap c \times (0,1)$ is vertical towards 1. Let $T \subset \Sigma \times (0,1)$ be the framed tangle obtained by cutting T_0 along c. Any two states $s_a : \partial_a T \to \{-,+\}$ and $s_b : \partial_b T \to \{-,+\}$ give rise to a state (s_a, s_0, s_b) on T. Both the sets $\partial_a T$ and $\partial_b T$ are in canonical bijection with the set $T_0 \cap c$ by the map π. Hence the two sets of states s_a and s_b are both in canonical bijection with the set $\mathrm{St}(c) := \{s : c \cap T_0 \to \{-,+\}\}$.

Definition 2.4. Let $\theta_{a\#b} : \mathcal{S}_A(\mathbf{\Sigma}_{a\#b}) \to \mathcal{S}_A(\mathbf{\Sigma})$ be the linear map given, for any (T_0, s_0) as above, by:

$$\theta_{a\#b}\left([T_0, s_0]\right) := \sum_{s \in \mathrm{St}(c)} [T, (s, s_0, s)].$$

The gluing map $\theta_{a\#b}$ is injective ([Le18, Theorem 3.1]) and coassociative in the sense that if a, b, c, d are four boundary arcs then $\theta_{a\#b} \circ \theta_{c\#d} = \theta_{c\#d} \circ \theta_{a\#b}$. While considering two copies $\mathbb{B} \cup \mathbb{B}'$ of the bigon and gluing b to a', we get another bigon. So we have a coassociative coproduct

$$\Delta := \theta_{b\#a'} : \mathcal{S}_A(\mathbb{B})^{\otimes 2} \to \mathcal{S}_A(\mathbb{B}).$$

The algebra $\mathcal{S}_A(\mathbb{B})$ with this coproduct has a structure of Hopf algebra that will be denoted by $\mathcal{O}_q[\mathrm{SL}_2]$. Now consider a marked surface $\mathbf{\Sigma}$ with two boundary arc c and d. Since $(\mathbb{B} \cup \mathbf{\Sigma})_{b\#c} \cong \mathbf{\Sigma}$, we have a left comodule map

$$\Delta^L_c := \theta_{b\#c} : \mathcal{S}_A(\mathbf{\Sigma}) \to \mathcal{O}_q[\mathrm{SL}_2] \otimes \mathcal{S}_A(\mathbf{\Sigma}).$$

Similarly, since $(\mathbf{\Sigma} \cup \mathbb{B})_{d\#a} \cong \mathbf{\Sigma}$, we have a right comodule map $\Delta^R_d := \theta_{d\#a} : \mathcal{S}_A(\mathbf{\Sigma}) \to \mathcal{S}_A(\mathbf{\Sigma}) \otimes \mathcal{O}_q[\mathrm{SL}_2]$. The main property of stated skein algebras is the

Theorem 2.5. *(K.-Quesney [KQ19a, Theorem 1.1], Costantino-Lê [CL19, Theorem 4.7]) The following sequence is exact:*

$$0 \to \mathcal{S}_A(\mathbf{\Sigma}_{c\#d}) \xrightarrow{\theta_{c\#d}} \mathcal{S}_A(\mathbf{\Sigma}) \xrightarrow{\Delta^L_c - \sigma \circ \Delta^R_d} \mathcal{O}_q[\mathrm{SL}_2] \otimes \mathcal{S}_A(\mathbf{\Sigma}),$$

where $\sigma(x \otimes y) := y \otimes x$.

Let $\mathbf{\Sigma}$ a marked surface and p a boundary puncture between two consecutive boundary arcs a and b on the same boundary component ∂ of $\partial\Sigma$. The orientation of Σ induces an orientation of ∂ so a cyclic ordering of the elements of $\partial \cap \mathcal{A}$ we suppose that a is followed by b in this ordering. We denote by $\alpha(p)$ an arc with one endpoint $v_a \in a$ and one endpoint $v_b \in b$ such that $\alpha(p)$ can be isotoped inside ∂. Let $\alpha(p)_{ij} \in \mathcal{S}_A(\mathbf{\Sigma})$ be the class of the stated arc $(\alpha(p), s)$ where $s(v_b) = i$ and $s(v_a) = j$.

Definition 2.6. We call *bad arc* associated to p the element $\alpha(p)_{-+} \in \mathcal{S}_A(\mathbf{\Sigma})$. The *reduced stated skein algebra* $\mathcal{S}^{red}_A(\mathbf{\Sigma})$ is the quotient of $\mathcal{S}_A(\mathbf{\Sigma})$ by the ideal generated by all bad arcs.

3 Some properties of stated skein algebras

Let us list some properties of stated skein algebras which are useful towards the resolution of Problem 1.1. From now on, we suppose that $A \in \mathbb{C}$ is a root of unity of odd order $N \geq 1$.

Theorem 3.1. *(Bonahon-Wong [BW11] for unmarked surfaces, K.-Quesney [KQ19a] for marked surfaces) There is an embedding*

$$Ch_A^{\mathbf{\Sigma}} : \mathcal{S}_{+1}(\mathbf{\Sigma}) \hookrightarrow \mathcal{Z}\left(\mathcal{S}_A(\mathbf{\Sigma})\right)$$

named Chebyshev-Frobenius morphism, sending the (commutative) algebra at $+1$ into the center of the skein algebra at $A^{1/2}$. Moreover, $Ch_A^{\mathbf{\Sigma}}$ is characterized by the facts that if γ is a closed curve, then $Ch_A^{\mathbf{\Sigma}}(\gamma) = T_N(\gamma)$, where $T_N(X)$ is the N^{th} Chebyshev polynomial of first type, and if α_{ij} is a stated arc, then $Ch_A^{\mathbf{\Sigma}}(\alpha_{ij}) = \alpha_{ij}^{(N)}$ is the class of N parallel copies of α_{ij} pushed along the framing direction.

If R is a ring with center $Z(R)$ such that R is a $Z(R)$-module of finite rank r, then R is a Polynomial Identity (PI) ring. If moreover R is prime, then writing $S := Z(R) \setminus \{0\}$, a theorem of Posner shows that the localization $S^{-1}R$ is a central simple algebra with center $S^{-1}Z(R)$, so is a matrix algebra is some algebraic closure of $S^{-1}Z(R)$. In particular, the rank r is a perfect square and we call *PI-dimension* of R its square root. Computing the PI-dimensions of stated skein algebras is an important step towards the classification of its representations. The orientation of Σ induces an orientation $\mathfrak{o}^+$ of the boundary arcs of $\mathcal{A}$.

Definition 3.2. 1. For p an inner puncture (i.e. an unmarked connected component of $\partial\Sigma$), we denote by $\gamma_p \in \mathcal{S}_A(\mathbf{\Sigma})$ the class of a peripheral curve encircling p once.

2. For $\partial \in \pi_0(\partial\Sigma)$ a boundary component which intersects $\mathcal{A}$ non trivially, denote by $p_1, \ldots, p_n$ the boundary punctures in ∂ cyclically ordered by $\mathfrak{o}^+$ and define the elements in $\mathcal{S}_A^{red}(\mathbf{\Sigma})$:

$$\alpha_\partial := \alpha(p_1)_{++} \ldots \alpha(p_n)_{++}, \quad \text{and} \quad \alpha_\partial^{-1} := \alpha(p_1)_{--} \ldots \alpha_(p_n)_{--}.$$

   We easily see that in $\mathcal{S}_A^{red}(\mathbf{\Sigma})$, we have $\alpha_\partial \alpha_\partial^{-1} = 1$.

3. For $\partial \in \pi_0(\partial\Sigma)$ a boundary component whose intersection with $\mathcal{A}$ is $2n$, for $n \geq 1$, denote by $p_1, \ldots, p_{2n}$ the boundary punctures in ∂ cyclically ordered by $\mathfrak{o}^+$. For $k \in \{1, \ldots, N-1\}$, write the product of bad arcs:

$$\beta_\partial^{(N,k)} := \alpha(p_1)_{-+}^k \alpha(p_2)_{-+}^{N-k} \ldots \alpha(p_{2n-1})_{-+}^k \alpha(p_{2n})_{-+}^{N-k} \in \mathcal{S}_A(\mathbf{\Sigma}).$$

   We will call *even* such a boundary component ∂.

Theorem 3.3. *1. (Bonahon-Wong [BW11] Przytycki-Sikora [PS19], Lê [Le18]) $\mathcal{S}_A(\mathbf{\Sigma})$ is a domain.*

2. (Bullock [Bul99] for unmarked surfaces, K. [Kor20]for marked surfaces) $\mathcal{S}_A(\mathbf{\Sigma})$ is finitely generated. When the marking is non trivial and the surface connexe, we even have explicit finite presentations of $\mathcal{S}_A(\mathbf{\Sigma})$.

3. *(Frohman-Lê-Kania-Bartoszynska [FKL19b]) If* $\mathbf{\Sigma}$ *is unmarked, then* (i) *the center of* $\mathcal{S}_A(\mathbf{\Sigma})$ *is generated by the image of the Chebyshev-Frobenius morphism together with the eventual peripheral curves* γ_p *for p an inner puncture.* (ii) $\mathcal{S}_A(\mathbf{\Sigma})$ *is finitely generated over the image of the Chebyshev-Frobenius morphism (so over its center) and* (iii) *for* $\mathbf{\Sigma} = (\Sigma_{g,n}, \emptyset)$ *the PI-dimension of* $\mathcal{S}_A(\mathbf{\Sigma})$ *is* N^{3g-3+n}.
4. *(K. [Kor21]) For any marked surface then* (i) *the center of* $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ *is generated by the image of the Chebyshev-Frobenius morphism together with the peripheral curves* γ_p *associated to inner punctures and the elements* $\alpha_\partial^{\pm 1}$ *associated to boundary components* $\partial \in \pi_0(\partial\Sigma)$. (ii) *both* $\mathcal{S}_A(\mathbf{\Sigma})$ *and* $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ *are finitely generated over the image of the Chebyshev-Frobenius morphisms (so over their center).* (iii) *For* $\mathbf{\Sigma} = (\Sigma_{g,n}, \mathcal{A})$, *the PI-dimension of* $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ *is* $N^{3g-3+n+|\mathcal{A}|}$.
5. *(Lê-Yu [LYa]: To appear) For any marked surfaces then* (i) *the center of* $\mathcal{S}_A(\mathbf{\Sigma})$ *is generated by the image of the Chebyshev-Frobenius morphism together with the peripheral curves* γ_p *and the elements* $\beta_\partial^{(N,k)}$ *associated to even boundary components and integers* $1 \leq k \leq N-1$. (ii) *For* $\mathbf{\Sigma} = (\Sigma_{g,n}, \mathcal{A})$, *the PI-dimension of* $\mathcal{S}_A(\mathbf{\Sigma})$ *is* $N^{3g-3+n_{even}+\frac{3}{2}(|\mathcal{A}|+n_{odd})}$, *where* n_{odd}, n_{even} *are the number of boundary components with an odd and even number of boundary arcs respectively*

Let $\mathcal{A}$ be a prime complex algebra which is finitely generated over its center $\mathcal{Z}$ and let D its PI-dimension. Write $\mathcal{X} := \mathrm{Specm}(\mathcal{Z})$ and for $x \in \mathcal{X}$ corresponding to a maximal ideal $\mathfrak{m}_x \subset \mathcal{Z}$, consider the finite dimensional algebra

$$\mathcal{A}_x := \mathcal{A}/_{\mathfrak{m}_x\mathcal{A}}.$$

Definition 3.4. The *Azumaya locus* of $\mathcal{A}$ is the subset

$$\mathcal{AL}(\mathcal{A}) = \{x \in \mathcal{X} | \mathcal{A}_x \cong Mat_D(\mathbb{C})\},$$

where $Mat_D(\mathbb{C})$ is the algebra of $D \times D$ matrices.

Note that any irreducible representation $\rho : \mathcal{A} \to \mathrm{End}(V)$ sends central elements to scalar operators so admits a (unique) $x \in \mathcal{X}$ such that $\mathfrak{m}_x\mathcal{A} \subset \ker(\rho)$. If $x \in \mathcal{AL}$, then V is D dimensional. By a theorem of Posner, if x does not belong to the Azumaya locus, then $\mathcal{A}_x$ has PI-dimension strictly smaller than $\mathcal{A}$, therefore any irreducible representation $\rho : \mathcal{A} \to \mathrm{End}(V)$ inducing x has dimension $\dim(V) < D$. So the Azumaya locus admits the following alternative definition:

$$\mathcal{AL}(\mathcal{A}) = \{x \in \mathcal{X} | x \text{ is induced by an irrep of maximal dimension } D\}.$$

Therefore, if $\rho : \mathcal{A} \to \mathrm{End}(V)$ is a D-dimensional central representation inducing $x \in \mathcal{X}$, then ρ is irreducible if and only if $x \in \mathcal{AL}$. When $\mathcal{A}$ is prime, finitely generated and has finite rank over its center, then $\mathcal{AL}(\mathcal{A})$ contains a Zariski open dense subset, therefore we have the

Theorem 3.5. *([FKL19b] for unmarked surfaces, [Kor21] for marked surfaces) The Azumaya loci of* $\mathcal{S}_A(\mathbf{\Sigma})$ *and* $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ *contain open dense subsets.*

Notations 3.6. Let $\mathcal{Z}$ denote the center of $\mathcal{S}_A(\mathbf{\Sigma})$ and write $\widehat{\mathcal{X}}(\mathbf{\Sigma}) := \mathrm{Specm}(\mathcal{Z})$ and $\mathcal{X}(\mathbf{\Sigma}) := \mathrm{Specm}(\mathcal{S}_{+1}(\mathbf{\Sigma}))$. The Chebyshev-Frobenius morphism induces a surjective finite morphism $\pi : \widehat{\mathcal{X}}(\mathbf{\Sigma}) \to \mathcal{X}(\mathbf{\Sigma})$. We also define a branched covering $\pi' : \widehat{\mathcal{X}}^{red}(\mathbf{\Sigma}) \to \mathcal{X}^{red}(\mathbf{\Sigma})$ associated to reduced stated skein algebras in the same manner.

Definition 3.7. The *fully Azumaya locus* is the subset $\mathcal{FAL}(\mathbf{\Sigma}) \subset \mathcal{X}(\mathbf{\Sigma})$ of elements x such that all elements of $\pi^{-1}(x)$ belong to the Azumaya locus of $\mathcal{S}_A(\mathbf{\Sigma})$. The fully Azumaya locus $\mathcal{FAL}^{red}(\mathbf{\Sigma}) \subset \mathcal{X}^{red}(\mathbf{\Sigma})$ is defined similarly.

Since finite morphisms send closed sets to closed sets, Corollary 3.5 implies that the fully Azumaya loci are dense.

Theorem 3.8. *(Brown-Gordon [BG01, Corollary 2.7]) Let $\mathcal{A}$ be an affine prime $\mathbb{C}$-algebra finitely generated over its center $\mathcal{Z}$ and denote by D its PI-dimension. Let $R \subset \mathcal{Z}$ be a subalgebra such that $\mathcal{Z}$ if finitely generated as a R-module. Let $M \in \mathcal{AL}(\mathcal{A})$ and $\mathfrak{m} := M \cap R$. Then*

$$\mathcal{A} /_{\mathfrak{m}\mathcal{A}} \cong Mat_D \left(\mathcal{Z} /_{\mathfrak{m}\mathcal{Z}} \right).$$

For $x \in \mathcal{X}(\mathbf{\Sigma})$, set

$$\mathcal{S}_A(\mathbf{\Sigma})_x := \mathcal{S}_A(\mathbf{\Sigma}) \Big/ Ch_A(\mathfrak{m}_x)\mathcal{S}_A(\mathbf{\Sigma}) .$$

In order to solve Problem 1.1, we need to classify all indecomposable modules of the finite dimensional algebras $\mathcal{S}_A(\mathbf{\Sigma})_x$.

Let $\mathcal{Z}$ be the center of $\mathcal{S}_A(\mathbf{\Sigma})$ and write

$$Z(x) := \mathcal{Z} \Big/ Ch_A(\mathfrak{m}_x)\mathcal{Z} .$$

For $x \in \mathcal{X}^{red}(\mathbf{\Sigma})$, we define $\mathcal{S}_A^{red}(\mathbf{\Sigma})_x$ and $Z^{red}(x)$ similarly.

Corollary 3.9. *If $x \in \mathcal{FAL}(\mathbf{\Sigma})$ and $D := PI{-}Dim(\mathcal{S}_A(\mathbf{\Sigma}))$, then $\mathcal{S}_A(\mathbf{\Sigma})_x \cong Mat_D(Z(x))$. Similarly, if $x \in \mathcal{FAL}^{red}(\mathbf{\Sigma})$ and $D' := PI{-}Dim(\mathcal{S}_A^{red}(\mathbf{\Sigma}))$, then $\mathcal{S}_A^{red}(\mathbf{\Sigma})_x \cong Mat_{D'}(Z^{red}(x))$.*

Since the algebras $Z(x)$ are easy to compute explicitly using Theorem 3.3, by putting Corollary 3.5 and Theorem 3.9 together, we have solved Problem 1.1 generically, i.e. for every classical shadows lying in the fully Azumaya locus. We need thus to solve the:

Problem 3.10. Compute the (fully) Azumaya loci of $\mathcal{S}_A(\mathbf{\Sigma})$ and $\mathcal{S}_A^{red}(\mathbf{\Sigma})$.

Here is what is known concerning Problem 3.10:

Theorem 3.11.
1. *(Brown-Goodearl :) The Azumaya locus of $\mathcal{S}_A(\mathbb{B}) \cong \mathcal{O}_q[\mathrm{SL}_2]$ is equal to the smooth locus. Therefore (Brown-Gordon [BG02a]), its fully Azumaya locus is the set of non-diagonal matrices of $\mathrm{SL}_2 = \mathcal{X}(\mathbb{B})$.*
2. *(Takenov [Tak, Theorem 15, Theorem 17]:) Takenov described explicitly two open dense subsets $\mathcal{O}_{1,1}$ and $\mathcal{O}_{0,4}$ which are included into the Azumaya loci of $\mathcal{S}_A(\Sigma_{1,1})$ and $\mathcal{S}_A(\Sigma_{0,4})$ respectively.*
3. *(K. [Kor19a]:) The fully Azumaya locus of $\mathcal{S}_A^{red}(\mathbb{D}_1)$ is equal to the set of elements $(g_+, g_-) \in B_+ \times B_- = \mathcal{X}^{red}(\mathbb{D}_1)$ such that $g_+ g_-^{-1} \neq \pm \mathbb{1}_2$.*

4. *(Bonahon-Wong [BW16b]:) For a closed surface, the central representations of* $\mathcal{X}_{\mathrm{SL}_2}(\Sigma_g) \cong \mathcal{X}(\Sigma_g, \emptyset)$ *are not in the Azumaya locus.*
5. *(Ganev-Jordan-Safranov [GJSa]:) For a closed surface, the smooth locus of* $\mathcal{X}_{\mathrm{SL}_2}(\Sigma_g)$ *is included in the Azumaya locus. Moreover, for* $\mathbf{\Sigma}_{g,0}^0$*, the Azumaya locus is equal to the dense bad arcs leaf* $\mathcal{X}^0(\mathbf{\Sigma}_{g,0}^0)$.
6. *(Consequence of Alekseev-Malkin [AM94]:) The open dense subset* D_{00} *is included in the fully Azumaya locus of* $D_q^+(\mathrm{SL}_2) = \mathcal{S}_A(\mathbb{D}_1^+)$.

The open dense subsets $\mathcal{X}^0(\underline{\mathbf{\Sigma}}_{g,n})$ and D_{00} will be defined in the next section. The last two items of Theorem 3.11 are consequences of a more general theorem review in the next section: by Brown-Gordon theory of Poisson orders, if $\mathcal{X}(\mathbf{\Sigma})$ contains a symplectic leaf which is dense, then it is included in the fully Azumaya locus. Let us make an obvious remark: the quotient map $\mathcal{S}_{+1}(\mathbf{\Sigma}) \to \mathcal{S}_{+1}^{red}(\mathbf{\Sigma})$ induces an inclusion $\mathcal{X}^{red}(\mathbf{\Sigma}) \subset \mathcal{X}(\mathbf{\Sigma})$.

Lemma 3.12. *For a connected marked surface* $\mathbf{\Sigma}$ *with non-trivial marking, then the leaf* $\mathcal{X}^{red}(\mathbf{\Sigma}) \subset \mathcal{X}(\mathbf{\Sigma})$ *does not intersect the fully Azumaya locus of* $\mathcal{S}_A(\mathbf{\Sigma})$.

Proof. This follows from the fact, proved in Theorem 3.3, that $PI - Dim(\mathcal{S}_A^{red}(\mathbf{\Sigma})) < PI - Dim(\mathcal{S}_A(\mathbf{\Sigma}))$. □

4 Relative character varieties and Poisson orders

4.1 Deformation quantization and Poisson orders

Let A_q be an associative unital $\mathbb{C}[q^{\pm 1}]$-algebra which is free and flat as a $\mathbb{C}[q^{\pm 1}]$-module. Consider the $\mathbb{C}$ algebra $A_{+1} = A_q \otimes_{q=1} \mathbb{C}$ and the $\mathbb{C}[[\hbar]]$ algebra $A_\hbar = A_q \otimes_{q=\exp(\hbar)} \mathbb{C}[[\hbar]]$. We suppose that A_{+1} is commutative. Fix $\mathcal{B}$ a basis of A_q, so by flatness, $\mathcal{B}$ can be also considered as a basis of $A_\hbar$ and A_{+1}. The basis $\mathcal{B}$ induces an isomorphism of $\mathbb{C}[[\hbar]]$-modules $\Psi^{\mathcal{B}} : A_{+1} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]] \xrightarrow{\cong} A_\hbar$ sending $b \in \mathcal{B}$ to itself. Denote by $\star$ the pull-back in $A_{+1} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$ of the product in $A_\hbar$. Define a Poisson bracket $\{\cdot, \cdot\}$ on A_{+1} by the formula:

$$x \star y - y \star x = \hbar \{x, y\} \pmod{\hbar^2}. \tag{1}$$

We say that the algebra $(A_{+1} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]], \star)$ is a *deformation quantization* of the Poisson algebra $(A_{+1}, \{\cdot, \cdot\})$. If $\mathcal{B}'$ is another basis, then $\Psi^{\mathcal{B}'} \circ \Psi^{\mathcal{B}}$ is an algebra isomorphism $(A_{+1} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]], \star_{\mathcal{B}}) \cong (A_{+1} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]], \star_{\mathcal{B}'})$ whose reduction modulo $\hbar$ is the identity: such an isomorphism is called a *gauge equivalence* and it is clear that two gauge equivalent star products induces the same Poisson bracket, in particular $\{\cdot, \cdot\}$ is independent on the choice of $\mathcal{B}$. Note that when A_{+1} is reduced and finitely generated, $\mathrm{Specm}(A_{+1})$ is a Poisson variety.

Remark 4.1. If $\Psi_A : A_q^1 \to A_q^2$ is an algebra morphism, it induces some morphisms $\Psi_\hbar : A_\hbar^1 \to A_\hbar^2$ and $\Psi_{+1} : A_{+1}^1 \to A_{+1}^2$. Since Ψ_{+1} is the reduction modulo $\hbar$ of $\Psi_\hbar$, it follows from the definition of the Poisson bracket that Ψ_{+1} is a Poisson morphism.

Example 4.2. Let E be a free finitely generated $\mathbb{Z}$ module and $(\cdot, \cdot) : E \times E \to \mathbb{Z}$ a skew-symmetric map and write $\mathbb{E} = (E, (\cdot, \cdot))$. The *quantum torus* $\mathbb{T}_q(\mathbb{E})$ is the algebra with underlying vector space the group algebra $k[E] = \text{Span}\{Z_e, e \in E\}$ and product given by $Z_a \cdot Z_b := q^{(a,b)} Z_{a+b}$. Note that given $e = (e_1, \ldots, e_n)$ a basis of E, the quantum torus $\mathbb{T}_q(\mathbb{E})$ is isomorphic to the complex algebra generated by invertible elements $Z_{e_i}^{\pm 1}$ with relations $Z_{e_i} Z_{e_j} = q^{2(e_i,e_j)_E} Z_{e_j} Z_{e_i}$. Setting $q = \exp(\hbar)$, we see that

$$Z_a \star Z_b - Z_b \star Z_a = (q^{(a,b)} - q^{-(a,b)}) Z_{a+b} = \hbar 2(a,b) Z_{a+b} \pmod{\hbar^2},$$

giving the Poisson bracket $\{Z_a, Z_b\} = 2(a,b) Z_{a+b}$ of $\mathcal{X}(\mathbb{E}) := \text{Specm}(\mathbb{C}[E]) \cong (\mathbb{C}^*)^n$ so $\mathbb{T}_q(\mathbb{E})$ is a deformation quantization of the torus $\mathcal{X}(\mathbb{E})$.

Definition 4.3. A *Poisson order* is a 4-uple $(\mathcal{A}, \mathcal{X}, \phi, D)$ where:

1. $\mathcal{A}$ is an (associative, unital) affine $\mathbb{C}$-algebra finitely generated over its center $\mathcal{Z}$;
2. $\mathcal{X}$ is a Poisson affine $\mathbb{C}$ -variety;
3. $\phi : \mathcal{O}[\mathcal{X}] \hookrightarrow \mathcal{Z}$ a finite injective morphism of algebras;
4. $D : \mathcal{O}[\mathcal{X}] \to Der(\mathcal{A}) : z \mapsto D_z$ a linear map such that for all $f, g \in \mathcal{O}[\mathcal{X}]$, we have

$$D_f(\phi(g)) = \phi(\{f, g\})$$

.

Here is our main source of examples. Let $\mathcal{A}_q$ a free flat affine $\mathbb{C}(q^{\pm 1})$-algebra like before, $N \geq 1$ and, writing $t := N(q^N - 1)$, the $\mathbb{C}(q^{\pm 1}) \big/ (q^N - 1)$ algebra $\mathcal{A}_N := \mathcal{A} / t$ and $\pi : \mathcal{A}_q \to \mathcal{A}_N$ the quotient map. By fixing a basis $\mathcal{B}$ of $\mathcal{A}_q$ by flatness we can define a linear embedding $\hat{\cdot} : \mathcal{A}_N \to \mathcal{A}_q$ sending a basis element $b \in \mathcal{B}$ seen as element in $\mathcal{A}_N$ to the same element $\hat{b}$ seen as an element in $\mathcal{A}_q$. Note that $\hat{\cdot}$ is a left inverse for π. Suppose that the algebra $\mathcal{A}_{+1} = \mathcal{A}_q \otimes_{q=1} \mathbb{C}$ is commutative and suppose there exists a central embedding $\phi : \mathcal{A}_{+1} \hookrightarrow \mathcal{A}_N$ into the center of $\mathcal{A}_N$. Write $\mathcal{X} := \text{Specm}(\mathcal{A}_{+1})$ and define $D : \mathcal{A}_{+1} \to Der(\mathcal{A}_N)$ by the formula

$$D_x y := \pi \left(\frac{[\widehat{\phi(x)}, \hat{y}]}{N(q^N - 1)} \right).$$

Clearly D_x is a derivation, is independent on the choice of the basis $\mathcal{B}$ and preserves $\phi(\mathcal{A}_{+1})$, so it defines a Poisson bracket $\{\cdot, \cdot\}_N$ on $\mathcal{A}_{+1}$ by

$$D_x \phi(y) = \phi(\{x, y\}_N). \tag{2}$$

So, writing $\mathcal{X} = \text{Specm}(\mathcal{A}_{+1})$, then $(\mathcal{A}_N, \mathcal{X}, \phi, D)$ is a Poisson order for this bracket. Note that if ζ_N is an N-th root of unity and $\mathcal{A}_{\zeta_N} = \mathcal{A}_q \otimes_{q=\zeta_N} \mathbb{C}$, we get a Poisson order $(\mathcal{A}_{\zeta_N}, \mathcal{X}, \phi, D)$ as well simply by tensoring by $\mathbb{C}$.

What is not clear is how to compare the above bracket $\{\cdot, \cdot\}_N$ with the one coming from deformation quantization defined by Equation (1).

Example 4.4. Consider the quantum torus $\mathbb{T}_q(\mathbb{E})$ of Example 4.2 and define the *Frobenius morphism* $Fr_N : \mathbb{T}_{+1}(\mathbb{E}) \hookrightarrow \mathcal{Z}(\mathbb{T}_{\zeta_N}(\mathbb{E}))$ by $Fr_N(Z_e) := Z_e^N$. The preceding discussion defines a Poisson order $(\mathbb{T}_{\zeta_N}(\mathbb{E}), \mathcal{X}(\mathbb{E}), Fr_N, D)$. The Poisson bracket $\{\cdot,\cdot\}_N$ defined by Equation (2) is computed as follows:

$$\pi\left(\frac{[Fr_N(Z_a), Fr_N(Z_b)]}{N(q^N-1)}\right) = \pi\left(\frac{[Z_a^N, Z_b^N]}{N(q^N-1)}\right)$$
$$= \pi\left(\frac{q^{N^2(a,b)} - q^{-N^2(a,b)}}{N(q^N-1)}\right) Z_{a+b}^N$$
$$= \pi\left(q^{-N(a,b)}\frac{1}{N}\sum_{i=1}^{2N(a,b)} q^{Ni}\right) Fr_N(Z_{a+b}) = Fr_N(2(a,b)Z_{a+b}).$$

So $\{Z_a, Z_b\}_N = 2(a,b)Z_{a+b}$ and the Poisson bracket $\{\cdot,\cdot\}_N$ coincides with the bracket coming from deformation quantization.

Using the Chebyshev-Frobenius morphism, by the preceding discussion we have Poisson orders $(\mathcal{S}_A(\mathbf{\Sigma}), \mathcal{X}(\mathbf{\Sigma}), Ch_A, D)$ and $(\mathcal{S}_A^{red}(\mathbf{\Sigma}), \mathcal{X}^{red}(\mathbf{\Sigma}), Ch_A, D)$ where we set $q := A^2$. To prove that the two brackets coming from Equations (1) and (2) coincide, we can use the quantum trace. A marked surface is *triangulable* if it can be obtained from a disjoint union of triangles by gluing some pairs of edges. The data of the triangles plus the pairs of glued edges is called a triangulation.

Theorem 4.5. *1. (Bonahon-Wong [BW11]) For $(\mathbf{\Sigma}, \Delta)$ a triangulated marked surface, there exists a quantum torus $\mathbb{T}_q(\mathbf{\Sigma}, \Delta)$, named the balanced Chekhov-Fock algebra, and an embedding $Tr^\Delta : \mathcal{S}_A^{red}(\mathbf{\Sigma}) \hookrightarrow \mathbb{T}_q(\mathbf{\Sigma}, \Delta)$, named the quantum trace, such that Tr^Δ intertwines the actions of the Chebyshev-Frobenius morphism and the Frobenius morphism.*

2. (Lê-Yu [LYb]) Let $\mathbf{\Sigma}^*$ be the marked surface obtained from $\mathbf{\Sigma}$ by gluing a triangle to each boundary arc. Then there exists an embedding $\iota : \mathcal{S}_A(\mathbf{\Sigma}) \hookrightarrow \mathcal{S}_A^{red}(\mathbf{\Sigma}^*)$ commuting with the Chebyshev-Frobenius morphism. In particular, by composition

$$\mathcal{S}_A(\mathbf{\Sigma}) \xrightarrow{\iota} \mathcal{S}_A^{red}(\mathbf{\Sigma}^*) \xrightarrow{Tr^\Delta} \mathbb{T}_q(\mathbf{\Sigma}^*, \Delta),$$

we get an embedding of $\mathcal{S}_A(\mathbf{\Sigma})$ into a quantum torus that intertwines the actions of the Chebyshev-Frobenius morphism and the Frobenius morphism.

Lemma 4.6. *The Poisson brackets of $\mathcal{S}_{+1}(\mathbf{\Sigma})$ and $\mathcal{S}_{+1}^{red}(\mathbf{\Sigma})$ coming from Equations* (1) *and* (2) *are equal.*

Proof. When $\mathbf{\Sigma}$ is triangulable, this follows from Theorem 4.5 and the computations in 4.2 and 4.4. For the bigon, this was proved by De Concini-Lyubashenko in [DCL94]. If $\mathbf{\Sigma} = (\Sigma_{g,0}, \emptyset)$ is a closed surface, consider the triangulable surface $\mathbf{\Sigma}' = (\Sigma_{g,1}, \emptyset)$ obtained by removing an open disc. By functoriality, the embedding $\mathbf{\Sigma}' \to \mathbf{\Sigma}$ induces an algebra morphism $\phi : \mathcal{S}_A(\mathbf{\Sigma}') \to \mathcal{S}_A(\mathbf{\Sigma})$ which is clearly surjective. Let $\mathcal{I} \subset \mathcal{S}_A(\mathbf{\Sigma})$ be the kernel of ϕ so that ϕ induces an isomorphism $\mathcal{S}_A(\mathbf{\Sigma}) \cong \mathcal{S}_A(\mathbf{\Sigma}')/_{\mathcal{I}}$. By [KQ19a, Proposition 2.18], the ideal $\mathcal{I}$ is generated by the elements $[\gamma] - [\gamma']$, where γ, γ' are two closed curves in $\Sigma_{g,1}$ such that $\iota(\gamma)$ and $\iota(\gamma')$ are isotopic in $\Sigma_{g,0}$, so ϕ intertwines the Chebyshev-Frobenius morphisms. By Remark 4.1, the morphism $\phi_{+1} : \mathcal{S}_{+1}(\mathbf{\Sigma}') \to \mathcal{S}_{+1}(\mathbf{\Sigma})$ preserves the Poisson brackets coming from Equation (1) and (2) and we deduce that they are equal in $\mathcal{S}_{+1}(\mathbf{\Sigma})$ from the fact that they are equal in $\mathcal{S}_{+1}(\mathbf{\Sigma}')$.

□

Definition 4.7. Let G be an affine Lie group. A Poisson order $(\mathcal{A}, \mathcal{X}, \phi, D)$ is said G-*equivariant* if G acts on $\mathcal{A}$ by automorphism such that its action preserves $\phi(\mathcal{O}[\mathcal{X}]) \subset \mathcal{A}$ and such that it is D equivariant in the sense that for every $g \in G$, $z \in \mathcal{O}[\mathcal{X}]$ and $a \in \mathcal{A}$, one has

$$D_{g\cdot z}(a) = gD_z(g^{-1}a).$$

We now endow the previous Poisson orders with a structure of $(\mathbb{C}^*)^{\mathcal{A}}$-equivariant Poisson order. Let $\varphi : \mathcal{O}_q[\mathrm{SL}_2] \to \mathbb{C}[X^{\pm 1}]$ be the surjective morphism defined by

$$\varphi(\alpha_{+-}) = \varphi(\alpha_{-+}) = 0, \quad \varphi(\alpha_{++}) = X, \quad \varphi(\alpha_{--}) = X^{-1}.$$

The morphism φ is clearly a morphism of Hopf algebras and the induced morphism on $\mathcal{O}[\mathrm{SL}_2] \xrightarrow{Ch_A} \mathcal{O}_q[\mathrm{SL}_2]$ is the diagonal embedding $\mathbb{C}^* \to \mathrm{SL}_2$ sending z to $\begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}$. Note that, while identifying $\mathbb{C}[X^{\pm 1}]$ with the reduced algebra $\mathcal{S}_A^{red}(\mathbb{B})$, then φ is just the quotient map $\mathcal{S}_A(\mathbb{B}) \to \mathcal{S}_A^{red}(\mathbb{B})$. Define an algebraic action of $(\mathbb{C}^*)^{\mathcal{A}}$ on $\mathcal{S}_A(\mathbf{\Sigma})$ by the co-action

$$\Delta^{diag} : \mathcal{S}_A(\mathbf{\Sigma}) \xrightarrow{\Delta^L} \left(\mathcal{O}_q[\mathrm{SL}_2]^{\otimes \mathcal{A}}\right) \otimes \mathcal{S}_A(\mathbf{\Sigma}) \xrightarrow{(\varphi^{\otimes \mathcal{A}})\otimes id} \mathbb{C}[X^{\pm 1}] \otimes \mathcal{S}_A(\mathbf{\Sigma}),$$

where Δ^L is the comodule map obtained by composing the Δ^L_a for all a (i.e. by gluing a bigon to each boundary arc). The above action induces by quotient a similar action on $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ and both action preserve the image of the Chebyshev-Frobenius morphism. The equivariance of D for this action is an immediate consequence of the definition of D.

Definition 4.8. Let $\mathcal{X}$ be a complex Poisson affine variety.

1. Define a first partition $X = X^0 \bigsqcup \ldots \bigsqcup X^n$ where X^0 is the smooth locus of X and for $i = 0, \ldots, n-1$, X^{i+1} is the smooth locus of $X \setminus X^i$. Each X^i is a smooth complex affine variety that can be seen as an analytic Poisson variety. Define an equivalence relation $\sim$ on X^i by writing $x \sim y$ if there exists a finite sequence $x = p_0, p_1, \ldots, p_k = y$ and functions $h_0, \ldots, h_{k-1} \in \mathcal{O}[X^i]$ such that p_{i+1} is obtained from p_i by deriving along the Hamiltonian flow of h_i. Write $X^i = \bigsqcup_j X^{i,j}$ the orbits of this relation. Note that the $X^{i,j}$ are analytic subvarieties: they are the biggest connected smooth symplectic subvarieties of X^i. The elements $X^{i,j}$ of the (analytic) partition $X = \bigsqcup_{i,j} X^{i,j}$ are called the *symplectic leaves* of X.

2. An ideal $\mathcal{I} \subset \mathcal{O}[\mathcal{X}]$ is a *Poisson ideal* if $\{\mathcal{I}, \mathcal{O}[\mathcal{X}]\} \subset \mathcal{I}$. Since the sum of two Poisson ideals is a Poisson ideal, every maximal ideal $\mathfrak{m} \subset \mathcal{O}[\mathcal{X}]$ contains a unique maximal Poisson ideal $P(\mathfrak{m}) \subset \mathfrak{m}$. Define an equivalence relation $\sim$ on $\mathcal{X} = \mathrm{Specm}(\mathcal{O}[\mathcal{X}])$ by $\mathfrak{m} \sim \mathfrak{m}'$ if $P(\mathfrak{m}) = P(\mathfrak{m}')$. The equivalence class of $\mathfrak{m}$ is named its *symplectic core* and denoted by $C(\mathfrak{m})$. The partition $\mathcal{X} = \bigsqcup_{\mathfrak{m} \in \mathcal{X}/_{\sim}} C(\mathfrak{m})$ is called the *symplectic cores partition* of $\mathcal{X}$.

3. If moreover an algebraic Lie group G acts on $\mathcal{X}$ we call *equivariant symplectic leaves* and *equivariant symplectic cores* the G orbits of the symplectic leaves and symplectic cores respectively.

For $x \in \mathcal{X}$ with symplectic leaf $\mathcal{F}(x)$ and symplectic core $C(x)$, $C(x)$ is the smallest algebraic set containing the analytic set $\mathcal{F}(x)$. For $(\mathcal{A}, \mathcal{X}, \phi, D)$ a Poisson order and $x \in \mathcal{X}$, write $\mathcal{A}_x := \mathcal{A} \big/ \phi(\mathfrak{m}_x)\mathcal{A}$. The main theorem of the theory of Poisson orders is

Theorem 4.9. *(Brown-Gordon [BG03])*

1. *[BG03, Proposition 4.3]) For $(\mathcal{A}, \mathcal{X}, \phi, D)$ a G-equivariant Poisson order, if $x, y \in \mathcal{X}$ belong to the same equivariant symplectic core then $\mathcal{A}_x \cong \mathcal{A}_y$.*

2. *[BG03, Proposition 3.6]) The (equivariant) symplectic leaves partition is a refinement of the (equivariant) symplectic cores partition.*

By putting Theorem 3.5 and Theorem 4.9 together, we obtain the

Corollary 4.10. *If $\mathcal{F} \subset \mathcal{X}(\mathbf{\Sigma})$ is a dense equivariant symplectic leaf (or a dense equivariant symplectic core) then $\mathcal{F}$ is included in the fully Azumaya locus.*

Problem 4.11. Compute the equivariant symplectic leaves of $\mathcal{X}(\mathbf{\Sigma})$ and $\mathcal{X}^{red}(\mathbf{\Sigma})$.

As we shall review, the problem was solved:

1. for closed surfaces by Goldman [Gol84];

2. for unmarked non-closed surfaces, independently by Fock-Rosly [FR99] and Guruprasad-Huebschmann-Jeffrey-Weitsman [GHJW97];

3. for the once-punctured monogon $\mathbf{m}_1$ and $\mathbb{D}_1^+$ by Alekseev-Malkin [AM94];

4. for the bigon, independently by Alekseev-Malkin [AM94] and Hodges-Levasseur [Hod93];

5. Ganev-Jordan-Safranov [GJSa] found an explicit open dense symplectic leaf in $\mathcal{X}(\mathbf{\Sigma}_{g,0}^0)$, for $g \geq 1$.

Let us first state a trivial, but useful result towards the resolution of this problem. Consider an algebra A_q as before and let $x \in \mathcal{A}_q$ be such that $xA_q = A_q x$, i.e. the left and right (and bilateral) ideals generated by x coincide. Let $I_q = (x) \subset A_q$ this ideal and $I = I_q \otimes_{q=1} \mathbb{C} \subset A_{+1}$. Since we have $[I_q, A_q] \subset I_q$, it follows from the definition of the Poisson bracket that $\{I, A_{+1}\} \subset I$, i.e. that I is a Poisson ideal of A_{+1}. Partition the set $X = \mathrm{Specm}(A_{+1})$ into $X = X^0 \bigsqcup X^1$ where X^0 is the open subset of $x \in X$ such that $\chi_x(I) = 0$ and X^1 its closed complement. Clearly each set X^i is a disjoint union of symplectic leaves, i.e. the partition into symplectic leaves is a refinement of the partition $X = X^0 \bigsqcup X^1$.

Lemma 4.12. *(Lê-Yu [LYb, Lemma 4.4.(a)]) Let $p \in \mathcal{P}^\partial$ be a boundary puncture and $\alpha(p)_{-+} \in \mathcal{S}_A(\mathbf{\Sigma})$ its associated bad arc. For any $[D, s] \in \mathcal{B}^{\mathfrak{o}^+}$, there exists $n \in \mathbb{Z}$ such that $\alpha(p)_{-+}[D, s] = A^{n/2}[D, s]\alpha(p)_{-+}$. In particular $\alpha(p)_{-+}\mathcal{S}_A(\mathbf{\Sigma}) = \mathcal{S}_A(\mathbf{\Sigma})\alpha(p)_{-+}$.*

For $\varepsilon : \mathcal{P}^\partial \to \{0,1\}$, denote by $\mathcal{X}^{(\varepsilon)}(\mathbf{\Sigma}) \subset \mathcal{X}(\mathbf{\Sigma})$ the subset of these $x \in \mathcal{X}(\mathbf{\Sigma})$ such that $\chi_x(\alpha(p)_{-+}) = 0$ if $\varepsilon(p) = 0$ and $\chi_x(\alpha(p)_{-+}) \neq 0$ else.

Definition 4.13. We call the *bad arcs partition* the partition $\mathcal{X}(\mathbf{\Sigma}) = \bigsqcup_\varepsilon \mathcal{X}^{(\varepsilon)}(\mathbf{\Sigma})$.

Note that for $\varepsilon = 0$ (the map sending every p to 0), one has $\mathcal{X}^{(0)}(\mathbf{\Sigma}) = \mathcal{X}^{red}(\mathbf{\Sigma})$ by definition. By Lemma 4.12 and the preceding discussion, we obtain the

Lemma 4.14. *The partition into equivariant symplectic cores is a refinement of the bad arcs partition.*

Let us state a second obvious remark towards the resolution of Problem 4.11. Recall from Definition 3.2, that for each inner puncture p we defined a central element $\gamma_p \in \mathcal{S}_A(\Sigma)$ and for each boundary component ∂, we defined an invertible central element $\alpha_\partial \in \mathcal{S}_A^{red}(\Sigma)$. Let $\mathrm{Cas} \subset \mathcal{S}_{+1}(\mathbf{\Sigma})$ (resp. $\mathrm{Cas}^{red} \subset \mathcal{S}_{+1}^{red}(\mathbf{\Sigma})$) denote the subgroup generated by the elements γ_p (resp. by the elements $\gamma_p, \alpha_\partial^{\pm 1}$). Since these elements are central in the skein algebras with parameter $A = \exp(\hbar/2)$, the elements in Cas and Cas^{red} are Casimir elements, i.e. they are in the kernel of the Poisson bracket. Therefore, if we consider the following *Casimir partition*

$$\mathcal{X}(\mathbf{\Sigma}) = \bigsqcup_{\pi : \mathrm{Cas} \to \mathbb{C}} \mathcal{X}_{(\pi)}(\mathbf{\Sigma}), \quad \mathcal{X}^{red}(\mathbf{\Sigma}) = \bigsqcup_{\pi : \mathrm{Cas}^{red} \to \mathbb{C}} \mathcal{X}^{red}_{(\pi)}(\mathbf{\Sigma}),$$

where the π are characters over the Casimir groups and $\mathcal{X}_{(\pi)}(\mathbf{\Sigma})$ is the (algebraic) subset of elements x such that $\chi_x(c) = \pi(c)$, for all $c \in \mathrm{Cas}$ and similarly for the reduced version, then

Lemma 4.15. *The partition into symplectic cores is a refinement of the Casimir partition.*

Note that the group $(\mathbb{C}^*)^{\mathcal{A}}$ preserves the Casimir leaves $\mathcal{X}_{(\pi)}(\mathbf{\Sigma})$ but not the leaves $\mathcal{X}^{red}_{(\pi)}(\mathbf{\Sigma})$.

Lemma 4.16. *If $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ is commutative for A generic, then for every $x \in \mathcal{X}^{red}(\mathbf{\Sigma}) = \mathcal{X}^{(0)}(\mathbf{\Sigma})$, then the singleton $\{x\}$ is a symplectic core of $\mathcal{X}(\mathbf{\Sigma})$.*

Proof. Let $\mathcal{I}^{bad} \subset \mathcal{S}_A^{red}(\mathbf{\Sigma})$ be the ideal generated by the bad arcs. If $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ is commutative for $A = \exp(\hbar)$, then we have $[\mathcal{S}_\hbar(\mathbf{\Sigma}), \mathcal{S}_\hbar(\mathbf{\Sigma})] \subset \mathcal{I}^{bad}$, so by definition of the Poisson bracket we have $\{\mathcal{S}_{+1}(\mathbf{\Sigma}), \mathcal{S}_{+1}(\mathbf{\Sigma})\} \subset \mathcal{I}^{bad}$. Therefore the restriction of the Poisson bracket to $\mathcal{X}^{red}(\mathbf{\Sigma})$ vanishes. □

4.2 Relative character varieties

The Poisson variety $\mathcal{X}(\mathbf{\Sigma})$ has a geometric interpretation that we now sketch and refer to [Kor19b] for further details. Let $\mathbb{V} \subset \Sigma$ be a finite set such that (1) $\mathbb{V}$ intersects each boundary arc exactly once and (2) $\mathbb{V}$ intersects each connected component of Σ at least once. Let $\Pi_1(\Sigma, \mathbb{V})$ be the full subcategory of the fundamental groupoid $\Pi_1(\Sigma)$ generated by $\mathbb{V}$. The set of functors $\rho : \Pi_1(\Sigma, \mathbb{V}) \to \mathrm{SL}_2$, where we see SL_2 as a groupoid with one element, forms the closed points of an affine variety $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma})$. Define the discrete gauge group $\mathcal{G}$ as the algebraic group of maps $g : \mathbb{V} \to \mathrm{SL}_2$ such that $g(v) = \mathbb{1}_2$ if $v \in \mathcal{A}$. It acts on $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ by the formula

$$g \cdot \rho(\alpha) = g(v_2)\rho(\alpha)g(v_1)^{-1} \quad \text{for } \alpha : v_1 \to v_2 \in \Pi_1(\Sigma, \mathbb{V}), g \in \mathcal{G}, \rho \in \mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}).$$

The *relative character variety* is the GIT quotient:

$$\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}) := \mathcal{R}_{\mathrm{SL}_2}(\Sigma) /\!\!/ \mathcal{G}.$$

The relative character variety does not depend, up to canonical isomorphism, on $\mathbb{V}$. If $\mathfrak{o}$ is an orientation of the boundary arcs of $\mathbf{\Sigma}$, the relative character variety has a structure of Poisson affine variety, denoted $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})^{\mathfrak{o}}$, whose Poisson bracket can be describe by a generalized Goldman formula. Note that if Σ is connected and $\mathcal{A}$ non empty, we can choose $\mathbb{V} \subset \mathcal{A}$, in which case the group $\mathcal{G}$ is trivial. In this case, we easily see that $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}) \cong (\mathrm{SL}_2)^n$ for some $n \geq 1$. In particular, the relative character variety is smooth in that case.

Theorem 4.17. *([Bul97, Tur91] for unmarked surfaces, [KQ19a] for marked surfaces) The Poisson varieties $\mathcal{X}(\mathbf{\Sigma})$ and $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ are (non canonically) isomorphic.*

When $\mathbf{\Sigma} = (\Sigma_g, \emptyset)$ is closed, choose $\mathbb{V} = \{v\}$ a singleton and $\mathcal{X}_{\mathrm{SL}_2}(\Sigma_g) = \mathrm{Hom}(\pi_1(\Sigma_g, v) \to \mathrm{SL}_2) /\!\!/ \mathrm{SL}_2$. The action of SL_2 on $\mathrm{Hom}(\pi_1(\Sigma, v), \mathrm{SL}_2)$ by conjugacy is not free. We decompose the set of representations $\rho : \pi_1(\Sigma, v) \to \mathrm{SL}_2$ into three classes:

1. The *central representations* taking value in $\pm\mathbb{1}_2$ and for which the stabilizer is SL_2.
2. The *diagonal representations* which are conjugate to a non central representation valued in the subgroup $D \subset \mathrm{SL}_2$ of diagonal matrices and for which the stabilizer is the group of diagonal matrices.
3. The *irreducible representations* for which the stabilizer is $\pm\mathbb{1}_2$.

Denote by $\mathcal{X}^i_{\mathrm{SL}_2}(\Sigma_g)$ the set of classes of irreducible, diagonal and central representations when $i = 0, 1, 2$ respectively. It follows from the work of Goldman [Gol84] that the partition into symplectic leaves of $\mathcal{X}_{\mathrm{SL}_2}(\Sigma_g)$ is simple: both $\mathcal{X}^0_{\mathrm{SL}_2}(\Sigma_g)$ and $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_g)$ are symplectic leaves and for every central representation r the singleton $\{[r]\}$ is a symplectic leaf. When $g = 1$, then $\mathcal{X}^0_{\mathrm{SL}_2}(\Sigma_1)$ is empty and $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_1)$ is an open dense symplectic leaf, so it is included in the Azumaya locus. As we shall see, since no central representation belong to the Azumaya locus, then $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_1)$ is equal to the Azumaya locus. For $g \geq 2$, since the smooth locus $\mathcal{X}^0_{\mathrm{SL}_2}(\Sigma_g)$ is symplectic, it is included in the Azumaya locus as noticed by Ganev-Jordan-Safranov in [GJSa]. It remains the

Question 4.18. For $g \geq 2$, is the symplectic leaf $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_g)$ included in the Azumaya locus of $\mathcal{S}_A(\Sigma_g)$?

Note that one diagonal representation belong to the Azumaya locus if and only if all of them do. Lê and Yu conjectured that the answer is no.

Let us consider the marked surfaces $\mathbb{B}, \mathbf{m}_1, \mathbb{D}_1$ and $\mathbb{D}_1^+$ and write $\mathrm{SL}_2^D := \mathcal{X}(\mathbb{B})$, $\mathrm{SL}_2^{STS} := \mathcal{X}(\mathbf{m}_1)$, $D(\mathrm{SL}_2) := \mathcal{X}(\mathbb{D}_1)$ and $D_+(\mathrm{SL}_2) := \mathcal{X}(\mathbb{D}_1^+)$. Both SL_2^D and SL_2^{STS} are isomorphic to SL_2 but with two different Poisson structures. The Poisson structure of SL_2^D endows SL_2 with a structure of Poisson-Lie group and was defined by Drinfel'd, the Poisson structure of SL_2^{STS} were defined by Semenov-Tian-Shansky and appeared in the work of Alekseev-Malkin. Note that SL_2^D has four bad arcs leaves which correspond to the double Bruhat cells of SL_2 whereas SL_2^{STS} has two bad arcs leaves, say SL_2^0 and SL_2^1 which correspond to the simple Bruhat cells of SL_2. We denote by SL_2^0 the open dense cell. Note that both $\mathcal{S}_A^{red}(\mathbb{B})$ and $\mathcal{S}_A^{red}(\mathbf{m}_1)$ are commutative, so by Lemma 4.16, if $g \in \mathrm{SL}_2$ is diagonal, then the singleton $\{g\}$ is a symplectic leaf of SL_2^D and if $g \in \mathrm{SL}_2^1$ the singleton $\{g\}$ is a symplectic leaf of SL_2^{STS}.

Similarly, the two Poisson varieties $D(\mathrm{SL}_2)$ and $D_+(\mathrm{SL}_2)$ (named Heisenberg doubles) are isomorphic to $\mathrm{SL}_2 \times \mathrm{SL}_2$, though only $D(\mathrm{SL}_2)$ is a Poisson Lie group, and both were considered by Alekseev-Malkin. By extending the techniques of Semenov-Tian-Shansky, Alekseev-Malkin proved

Theorem 4.19. *1. (Alekseev-Malkin [AM94, Theorem 2]) The four bad arcs leaves of $D_+(\mathrm{SL}_2)$ are symplectic.*

2. (Hodges-Levasseur [Hod93, Theorem B.2.1], Alekseev-Malkin [AM94, Section 4]) The equivariant symplectic leaves of SL_2^D are the double Bruhat cells of SL_2 (which correspond to the bad arcs leaves of SL_2^D).

3. (Alekseev-Malkin [AM94, Section 4]) The symplectic leaves of SL_2^{STS} are

(a) The leaves $\mathrm{SL}_2^0 \cap C$, for C a conjugacy class;

(b) the singletons $\{g\}$ for $g \in \mathrm{SL}_2^1$.

Note that one bad arc leaf, say D_{00} of $D_+(\mathrm{SL}_2)$ is dense, so Corollary 4.10 implies that D_{00} is included in the Azumaya locus of $\mathcal{S}_A(\mathbb{D}_1^+)$. Define an embedding $\mathbf{m}_1 \to \mathbf{\Sigma}_{g,0}^0$ by sending the underlying annulus of $\mathbf{m}_1$ in a tubular neighborhood of the (unique) boundary component of $\mathbf{\Sigma}_{g,0}^0$ and let $\mu_q : \mathcal{S}_A(\mathbf{m}_1) \to \mathcal{S}_A(\mathbf{\Sigma}_{g,0}^0)$ be the induced morphism. Note that μ_q sends the unique bad arc of $\mathcal{S}_A(\mathbf{m}_1)$ to the unique bad arc of $\mathcal{S}_A(\mathbf{\Sigma}_{g,0}^0)$. Let $\mu : \mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}_{g,0}^0) \to \mathrm{SL}_2^{STS}$ be the Poisson morphism induced by μ_{+1}. The bad arcs decomposition write $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}_{g,0}^0) = \mathcal{X}_{\mathrm{SL}_2}^0(\mathbf{\Sigma}_{g,0}^0) \cup \mathcal{X}_{\mathrm{SL}_2}^1(\mathbf{\Sigma}_{g,0}^0)$, where $\mathcal{X}_{\mathrm{SL}_2}^i(\mathbf{\Sigma}_{g,0}^0) = \mu^{-1}(\mathrm{SL}_2^i)$.

Theorem 4.20. *(Ganev-Jordan-Safranov [GJSa, Theorem 2.14]) The open dense bad arc leaf $\mathcal{X}_{\mathrm{SL}_2}^0(\mathbf{\Sigma}_{g,0}^0) = \mu^{-1}(\mathrm{SL}_2^0)$ is symplectic.*

So $\mathcal{X}_{\mathrm{SL}_2}^0(\mathbf{\Sigma}_{g,0}^0)$ is included in the Azumaya locus and by Lemma 3.12 it is equal to the Azumaya locus.

5 Three families of representations

In addition to the general theorems cited above, there exist three families of representations of skein algebras which are powerful tools to solve Problem 1.1.

5.1 Representations coming from modular TQFTs

The TQFTs defined by Witten and Reshetikhin-Turaev produces representations of skein algebras for unmarked surfaces:

$$\rho^{WRT} : \mathcal{S}_A(\Sigma_g) \to \mathrm{End}(V_A(\Sigma_g)).$$

The dimension of $V_A(\Sigma_g)$ is computed using the Verlinde formula, in particular $Dim(V_A(\Sigma_g)) < PI - Dim(\mathcal{S}_A(\Sigma_g))$. The following theorem was formulated for A a root of unity of even order $2N$, but its proof extends word-by-word to the odd case.

Theorem 5.1. *1. (Gelca-Uribe [GU10, Theorem 6.7], see also [BW16b]) The representation ρ^{WRT} is irreducible.*

2. (Bonahon-Wong [BW16b]) The classical shadow of ρ^{WRT} is a central representation.

The identification $\mathcal{X}(\Sigma_g) \cong \mathcal{X}_{\mathrm{SL}_2}(\Sigma_g)$ in Theorem 4.17 depends on a choice of spin structure S and which central representation is the shadow of ρ^{WRT} depends on this non canonical choice. Let $\chi \in \mathrm{H}^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{X}^2_{\mathrm{SL}_2}(\Sigma_g)$ be the classical shadow of ρ^{WRT}. For $\chi' \in \mathrm{H}^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$, define a representation $\chi' \cdot \rho^{WRT}$ by $\chi' \cdot \rho^{WRT}(\gamma) = (-1)^{\chi'([\gamma])}\rho^{WRT}(\gamma)$. Then $\chi' \cdot \rho^{WRT}$ has classical shadow $\chi + \chi'$. In particular, every central representation in $\mathcal{X}^2_{\mathrm{SL}_2}(\Sigma_g)$ is the classical shadow of an irreducible representation whose dimension is strictly smaller than the PI-dimension of $\mathcal{S}_A(\Sigma_g)$ therefore:

Corollary 5.2. *The locus $\mathcal{X}^2_{\mathrm{SL}_2}(\Sigma_g)$ of central representations does not intersect the Azumaya locus of $\mathcal{S}_A(\Sigma_g)$.*

5.2 Representations coming from non semi-simple TQFTs

Blanchet, Costantino, Geer and Patureau-Mirand defined in [BCGP16] a new family of TQFTs named non semi-simple because their algebraic input is no longer a modular category but rather a so-called G-modular relative category (which is non semi-simple in general) as described by De Renzi in [DR]. The categories giving rise to representations of the Kauffman-bracket skein algebras are the categories of projective weight representations of the unrolled quantum group $\overline{U}_q^H \mathfrak{sl}_2$ that we consider here at odd roots of unity. For every cohomology class $\omega \in \mathrm{H}^1(\Sigma_g; \mathbb{C}/\mathbb{Z})$, these TQFTs define some representations

$$\rho^{BCGP} : \mathcal{S}_A(\Sigma_g) \to \mathrm{End}(V_A(\Sigma_g, \omega)).$$

When $\omega \in \mathrm{H}^1(\Sigma_g; \mathbb{C}/\mathbb{Z}) \setminus \mathrm{H}^1(\Sigma_g; \frac{1}{2}\mathbb{Z}/\mathbb{Z})$, these representations satisfy:

1. The dimension of $V_A(\Sigma_g, \omega)$ is equal to the PI-dimension of $\mathcal{S}_A(\Sigma_g)$.
2. The classical shadow of ρ^{BCGP} is the class of the diagonal representations $r_\omega : \pi_1(\Sigma_g, v) \to \mathrm{SL}_2$ defined by

$$r_\omega(\gamma) = (-1)^{w_S([\gamma])} \begin{pmatrix} \exp(2i\pi\omega([\gamma])) & 0 \\ 0 & \exp(-2i\pi\omega([\gamma])) \end{pmatrix},$$

where w_S is the Johnson quadratic form of the spin structure S used in the identification $\mathcal{X}(\Sigma_g) \cong \mathcal{X}_{\mathrm{SL}_2}(\Sigma_g)$.

Question 5.3. For $\omega \in \mathrm{H}^1(\Sigma_g;\mathbb{C}/\mathbb{Z})\setminus \mathrm{H}^1(\Sigma_g;\frac{1}{2}\mathbb{Z}/\mathbb{Z})$, is the representation ρ^{BCGP} irreducible ?

Clearly, solving Question 5.3 is equivalent to solving Question 4.18. At the author's knowledge, not much is known concerning the representation ρ^{BCGP} when $\omega \in \mathrm{H}^1(\Sigma_g;\frac{1}{2}\mathbb{Z}/\mathbb{Z})$. The above two properties are expected to hold in that case too, in which case Corollary 5.2 would imply that they are not irreducible.

Question 5.4. For $\omega \in \mathrm{H}^1(\Sigma_g;\frac{1}{2}\mathbb{Z}/\mathbb{Z})$, what is the dimension of $V_A(\Sigma_g,\omega)$? Is the representation ρ^{WRT} indecomposable ? Projective ? What is its classical shadow ?

5.3 Representations coming from quantum Teichmüller theory

Recall from Theorem 4.5 the quantum trace $Tr^{\Delta} : \mathcal{S}_A^{red}(\mathbf{\Sigma}) \hookrightarrow \mathbb{T}_q(\mathbf{\Sigma},\Delta)$. De Concini and Procesi proved that any quantum torus at roots of unity is Azumaya of constant rank, so the isomorphism classes of irreducible representations of $\mathbb{T}_q(\mathbf{\Sigma},\Delta)$ are in 1-to-1 correspondance with the characters over the center of $\mathbb{T}_q(\mathbf{\Sigma},\Delta)$. Let $r : \mathcal{Z}_\omega(\mathbf{\Sigma},\Delta) \to \mathrm{End}(V)$ an irreducible representation. Then the composition

$$\rho^{BW} : \mathcal{S}_A^{red}(\mathbf{\Sigma}) \xrightarrow{\mathrm{tr}^\Delta} \mathcal{Z}_\omega(\mathbf{\Sigma},\Delta) \xrightarrow{r} \mathrm{End}(V),$$

is called a *quantum Teichmüller representation.* This procedure permits to construct representations of reduced stated skein algebras for triangulable surfaces. The quantum trace embeds the center of $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ into the center of $\mathbb{T}_q(\mathbf{\Sigma},\Delta)$ so ρ^{BW} is a central representation. For each inner puncture p, one can define a central element $H_p \in \mathbb{T}_q(\mathbf{\Sigma},\Delta)$ such that $tr^\Delta(\gamma_p) = H_p + H_p^{-1}$ (see [BW17] for the definition of H_p). We can use this construction to produce representations of skein algebras of closed surfaces as follows. Let $\Sigma_{g,n}$ be obtained from Σ_g by removing n open discs. The inclusion $\Sigma_{g,n} \subset \Sigma_g$ is an embedding of marked surfaces and defines a morphism $\phi : \mathcal{S}_A(\Sigma_{g,n}) \to \mathcal{S}_A(\Sigma_g)$ which is clearly surjective. Let $\mathcal{I} \subset \mathcal{S}_A(\Sigma_{g,n})$ be the kernel of ϕ. Let Δ be triangulation of $(\Sigma_{g,n},\emptyset)$ which is combinatoric in the sense that every edges has its two endpoints distinct. Let $r : \mathcal{Z}_\omega(\mathbf{\Sigma},\Delta) \to End(V)$ be an irreducible representations sending each central element H_{p_i} to $r(H_{p_i}) = -q^{-1} id_V$ and consider the subspace

$$V^0 := \{v \in V | \quad r(x)v = 0, \forall x \in \mathcal{I}\}.$$

The representation $\rho^{BW} = r \circ \mathrm{tr}^\Delta : \mathcal{S}_A(\Sigma_{g,n}) \to \mathrm{End}(V)$ induces via ϕ a representation $\rho^{BW} : \mathcal{S}_A(\Sigma_g) \to \mathrm{End}(V^0)$ which was studied in [BW19]. Here is what is known concerning the representations ρ^{BW}.

Theorem 5.5. *1. (Bonahon-Wong [BW17, BW19] for unmarked surfaces, K.-Quesney [KQ19b, Kor21] for marked surfaces) The dimension of ρ^{BW} is equal to the PI-dimension of $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ except maybe when $\mathbf{\Sigma}$ is closed of genus $g \geq 2$ and the shadow is a central element in $\mathcal{X}^2_{\mathrm{SL}_2}(\Sigma_g)$ in which case it is only known that the dimension is $\leq N^{3g-3}$.*

2. ([BW17, BW19]) When $\mathbf{\Sigma}$ is closed, all elements of $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ are the classical shadow of a quantum Teichmüller representation ρ^{BW}. When $\mathbf{\Sigma}$ is not closed, the set of shadows of representations ρ^{BW} is dense.

Question 5.6.

1. When $\mathbf{\Sigma}$ is not closed, which elements of $\mathcal{X}^{red}(\mathbf{\Sigma})$ are the classical shadows of quantum Teichmüller representations ?
2. What is the dimension of ρ^{BW} when $\mathbf{\Sigma}$ is closed of genus $g \geq 2$ and the shadow is central ?
3. If two quantum Teichmüller representations induce the same character over the center of $\mathcal{S}_A(\mathbf{\Sigma})$, are they isomorphic ?
4. When are the representations ρ^{BW} simple ? indecomposable ? projective ?
5. If ρ^{BW} and ρ^{BCGP} have the same classical shadow, are they isomorphic ? If the shadow is central, is ρ^{WRT} a sub-representation of one of them ?

Question 5.6 was solved by the author in [Kor19a] for $\mathbb{D}_n$.

Proposition 5.7. *The following assertion are equivalent:*

1. *There exists a diagonal representation in $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_g)$ that belongs to the Azumaya locus of $\mathcal{S}_A(\Sigma_g)$;*
2. *All diagonal representations belong to the Azumaya locus of $\mathcal{S}_A(\Sigma_g)$;*
3. *There exists a class $\omega \in \mathrm{H}^1(\Sigma_g; \mathbb{C}/\mathbb{Z}) \setminus \mathrm{H}^1(\Sigma; \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ for which the representation ρ^{BCGP}_ω coming from non semi simple TQFTs is irreducible;*
4. *For all $\omega \in \mathrm{H}^1(\Sigma_g; \mathbb{C}/\mathbb{Z}) \setminus \mathrm{H}^1(\Sigma; \frac{1}{2}\mathbb{Z}/\mathbb{Z})$, then ρ^{BCGP}_ω is irreducible;*
5. *There exists a quantum Teichmüller representation ρ^{BW} with classical shadow in $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_g)$ which is irreducible;*
6. *All quantum Teichmüller representations ρ^{BW} with classical shadow in $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_g)$ are irreducible.*

Moreover, if these assertions are true, then any two representation ρ^{BCGP} and ρ^{BW} having the same diagonal classical shadow in $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_g)$ are isomorphic.

Proof. The equivalence between the first two assertions follows from the fact that $\mathcal{X}^1_{\mathrm{SL}_2}(\Sigma_g)$ is a symplectic leaf together with Theorem 4.9. The other equivalences follow from the fact that both families of representations ρ^{BCGP} and ρ^{BW} have dimension equal to the PI-dimension of $\mathcal{S}_A(\Sigma_g)$. □

6 Reformulation of the representations classification problem

Call semi-weight representation of $\mathcal{S}_A(\mathbf{\Sigma})$ a representation which is semi-simple as a module over $\mathcal{S}_{+1}(\mathbf{\Sigma})$ (through the Chebyshev-Frobenius morphism). An indecomposable semi-weight representation ρ sends the elements of $\mathcal{S}_{+1}(\mathbf{\Sigma})$ to scalars operators so admits a classical shadow $[r] \in \mathcal{X}(\mathbf{\Sigma})$ and is a representation of the finite dimensional algebra $\mathcal{S}_A(\mathbf{\Sigma})_{[r]}$. Drozd classified finite dimensional algebras in three families:

Definition 6.1. A finite dimensional $\mathbb{C}$ algebra $\mathcal{A}$ is

1. *of finite representation type* if it has a finite number of isomorphism classes of indecomposable finite dimensional modules;

2. *of tame representation type* if it is not of finite representation type and if for every $d \geq 0$, there exists a finite collection of $\mathcal{A} - \mathbb{C}[X]$ bimodules $M_1, \ldots, M_n$ such that any d-dimensional indecomposable $\mathcal{A}$ module is isomorphic to a module $M_i \otimes S$ for S a simple $\mathbb{C}[X]$-module.

3. *of wild representation type* if there exists a functor $F : \mathbb{C}\langle x, y\rangle - \mathrm{Mod} \to \mathcal{A} - \mathrm{Mod}$ that preserves indecomposability and isomorphism classes.

By Drozd's Trichotomy theorem ([Dro79]) a finite dimensional algebra belongs to exactly one of these families. Classifying the indecomposable representations of a wild algebra is an undecidable problem (the word problem for finitely presented groups can be embedded into that problem) so we need to reformulate our initial problem.

Problem 6.2. 1. Classify the equivariant symplectic leaves of $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ (4.11);

2. For each leaf, choose a representative $[r]$ and determine the representation type of $\mathcal{S}_A(\mathbf{\Sigma})_{[r]}$; in particular if $[r]$ belongs to the fully Azumaya locus (which happens for a dense leaf for instance), then Corollary 3.9 gives the answer;

3. If $\mathcal{S}_A(\mathbf{\Sigma})_{[r]}$ it is not wild, classify its indecomposable finite dimensional representations.

Of course, Problem 6.2 has an equivalent version for reduced stated skein algebras. This problem was solved by Brown-Gordon for the bigon in [BG02a], by the author for $\mathcal{S}_A^{red}(\mathbb{D}_1)$ in [Kor19a] and can be deduced for Σ_1 and $\Sigma_{1,1}$ from [HP01].

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