Uniform Exponential Growth in Twisted Polynomial Algebras

Christopher Briggs
Department of Mathematics
Embry-Riddle Aeronautical University
Christopher.Briggs@erau.edu

Abstract: We prove that the growth of a twisted polynomial algebra is either uniformly exponential or polynomially bounded.

1 Introduction

1.1 Growth in groups

Let $T = \{x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}\}$ be a generating set of group G. For $g \in G$, write $g = y_1 \ldots y_k$, where each $y_i \in T$. The *length* of g (with respect to T) is the minimal k for which such an expression of g is possible. The *length function* $\lambda_{G,T} : G \to \mathbb{Z}_{\geq 0}$ maps each $g \in G$ to its length. The *growth function* of G (with respect to T) is $\gamma_{G,T}(n) = \{g \in G \mid \lambda_{G,T}(g) \leq n\}$. If T is understood, we write λ_G and γ_G. The asymptotic nature (polynomially bounded, exponential, neither) of γ_G is independent of the chosen generating set T, and so we speak of groups of polynomially bounded, exponential, and intermediate growth respectively.

Growth in groups was introduced independently by Schwarz in 1955 and Milnor in 1968 [8, 9]. In 1968, Wolf proved that a virtually nilpotent group has polynomial growth, and that a virtually polycyclic group which is not virtually nilpotent has exponential growth [11]. In 1981, Gromov proved that a finitely generated group of polynomial growth is virtually nilpotent [6], thereby characterizing polynomially bounded groups as precisely the finitely generated virtually nilpotent groups.

In 1968, Milnor asked whether there exist groups of intermediate growth. The question was answered affirmatively by Grigorchuk in 1983 [5]. In 1981, Gromov defined a group G to have *uniform exponential growth* if $\inf_T \left(\lim_n \gamma_{G,T}(n)^{1/n}\right) > 1$, where the infimum is over all finite generating sets T of G. In the same year, Gromov asked: does a group of exponential growth necessarily have uniform exponential growth [7]? The question was answered negatively by Wilson in 2004 [10]. However, the question has a positive answer for several classes of groups. In 2002, Alperin proved that a virtually polycyclic group has either polynomial or uniform exponential growth [1]. In 2005, Eskin, Moses, and Oh proved a linear group over a field of characteristic zero has polynomial or uniform exponential growth [4]. In 2008, Breuillard and Gelander proved that a linear group over any field has polynomial or uniform exponential growth [2].

1.2 Growth in algebras

Write $A = K[T]$ to denote that A is generated (as an algebra) by the set T over the field K. The growth function of A with respect to T is $\gamma_{A,T}(n) = \dim_K \left(\sum_{i=0}^{n} KT^i\right)$. Growth type is independent of the generating set, so, as with groups, we define polynomial, exponential, and intermediate growth for algebras. Also as with groups, a commutative algebra has polynomial growth, and the free algebra on at least two letters has exponential growth.

We say an algebra A has *uniform exponential growth* if

$$\inf_T \left(\lim_n \gamma_{A,T}(n)^{1/n} \right) > 1$$

where the infimum is over all finite generating sets T of A. Examples of algebras of uniform exponential growth include Golod-Shafarevich algebras, group algebras of Golod-Shafarevich groups, and any algebra graded by $\mathbb{N}$ with exponential growth [3]. An example of an algebra of nonuniform exponential growth is the group algebra (over any field) of Wilson's group of nonuniform exponential growth.

Results on algebras of uniform exponential growth apply to groups as follows: if G is a group and KG is its group algebra over field K, then G has uniform exponential growth if KG does. It is not known to the author whether the converse is true; that question motivates the present study.

2 Preliminary results

For the remainder, we will use the following notation: X is a finite, nonempty alphabet; S is the free abelian semigroup generated by X; σ is an endomorphism of S; K is a field; and $R = K^\sigma[X, t]$ is the twisted polynomial algebra generated by $\{X, \sigma\}$ over K. Specifically, $R = \operatorname{span}_K\{wt^n | w \in S, t \geq 0\}$, with multiplication $tx = x^\sigma t$ for each $x \in X$. Another view is that R is the K-span of the (nonabelian) semigroup $\langle S, t\rangle$ generated by $S \cup \{t\}$, where $tx = x^\sigma t$ for each $x \in X$.

The main result is that such an algebra must have either polynomially bounded or uniform exponential growth: intermediate and nonuniform exponential growth are excluded. The idea behind the proof of this theorem is to find an element $x \in X$ which experiences rapid growth (in degree) under applications of σ. If this is the case, then $\{x, t\}$ freely generate a free subsemigroup of $\langle S, t\rangle$, and so any set of the form $\{t^{\epsilon_0} x t^{\epsilon_1} x \dots x t^{\epsilon_n} \mid \epsilon_i \in \{0, 1\}\}$ is linearly independent over K. These words have length at most $2n + 1$ in the generating set $X \cup \{t\}$, and there are at least 2^{n+1} of them, which demonstrates exponential growth.

There are complications in the details. To prove uniform exponential growth, we cannot begin with a generating set of our choosing. Rather, we must begin with an arbitrary finite generating set of the algebra. The element x we seek may not appear as an element of our generating set, but instead as a summand of an element. The presence of other summands makes it difficult to determine the dimension of the span of a set of products of generating set elements. The solution is to separate the elements of X which undergo exponential growth (in degree) under applications of σ from those elements of X which do not.

For $w \in S$ we denote by $\deg(w)$ the degree of w expressed as a word in X, and for $y \in X$ we take $\deg_y(w)$ to be the exponent of y in the same expression. We say that $x \in X$ is *active* if there is some $k > 0$ such that $\deg_x\left(x^{\sigma^k}\right) > 1$. We focus on active elements of X when searching for free generators of a free subsemigroup of $\langle S, t\rangle$.

Proposition 2.1. *Let $x \in X$. If x is active, or if there is some active $y \in X$ and some $k > 0$ such that $deg_y\left(x^{\sigma^k}\right) > 0$, then the sequence $\{deg\left(x^{\sigma^n}\right)\}_{n\in\mathbb{N}}$ is exponential. Otherwise, the sequence is polynomially bounded.*

The following lemma implies that, if $x \in X$ is active, then any word of which x is a factor experiences exponential growth under applications of σ.

Lemma 2.2. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences. Suppose $\{a_n\}$ is exponential, and $\{b_n\}$ is either exponential or polynomially bounded. Then there is some k such that, for each n, $c_n \geq 2c_{n-1}$ where $c_n = a_{kn} + b_{kn}$.*

One more lemma assists the proof of the main result. It is used in the case of polynomially bounded growth. Here, $|\cdot|_1$ denotes the L_1 norm.

Lemma 2.3. *If the square matrix $A_{r\times r}$ with entries from $\mathbb{C}$ has all eigenvalues of magnitude at most 1, then for each $\mathbf{v} \in \mathbb{C}^r$, the sequence $\{|A^n\mathbf{v}|_1\}_{n\in\mathbb{N}}$ is polynomially bounded.*

3 Main Result

Theorem 3.1. *Let K be a field and X be a finite, nonempty alphabet. Let S be the free abelian semigroup generated by X, and σ be an endomorphism of S. If every eigenvalue of σ has magnitude 0 or 1, then the twisted polynomial algebra $R = K^{\sigma}[X, t]$ has polynomially bounded growth. Otherwise, R has uniformly exponential growth.*

References

[1] R. Alperin, *Uniform exponential growth of polycyclic groups*, Geom. Dedicata **92** (2002) 105–113

[2] E. Breuillard and T. Gelander, *Uniform independence for linear groups*, Invent. Math. **173** (2008) no. 2, 225–263

[3] C. Briggs, *Examples of Uniform Exponential Growth in Algebras*, J. Algebra Appl., **16** (2017) no. 12, 1750241 1–7

[4] A. Eskin, S Mozes and H. Oh, *On uniform exponential growth for linear groups*, Invent. Math. **160** (2005) no. 1, 1–30

[5] R. I. Grigorchuk, *On Milnor's problem of group growth*, Soviet. Math. Dokl. **28** (1983) 23–26

[6] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981) 53–73

[7] M. Gromov, *Structures métriques pour les variétés riemanniennes*, CEDIC, Paris, 1981, MR 85e:53051

[8] J. Milnor, *Growth in finitely generated solvable groups*, J. Diff. Geom. **2** (1968) 447–449

[9] A. S. Schwarz, *A volume invariant of coverings*, Dokl. Ak. Nauk USSR **105** (1955) 32–34

[10] J. Wilson, *On exponential growth and uniformly exponential growth for groups*, Proc. Amer. Math. Soc. **60** (1976) 22–24

[11] J. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Diff. Geom. **2** (1968) 434–446