Toward the Construction of Toric Imprimitive Reflection Groups

Haruhisa Nakajima*
Department of Mathematics
College of Arts and Sciences
J. F. Oberlin University

Abstract

For an n-dimensional faithful imprimitive irreducible representation $G \to GL(V)$ over the complex number filed $\boldsymbol{C}$, the ring of invariants $\boldsymbol{C}[V]^G$ of G on polynomial function on V is a polynomial ring over $\boldsymbol{C}$ if and only if G is conjugate to $G(m, p, n)$ called a finite imprimitive unitary reflection groups. We try to generalize such finite groups to affine algebraic groups with the aid of cofree covers of algebraic tori related to Gorensteinness.

MSC: primary 13A50, 14R20, 20G05; secondary 14L30, 14M25

Keywords: pseudo-reflection; algebraic torus; reductive algebraic group; imprimitive representation; monomial group; cofree action

1 Introduction

All rings are assumed to be commutative. For a ring R, an element $\sigma \in GL_n(R)$ is said to monomial, if σ is a monomial matrix in $GL_n(R)$ and, moreover, a subgroup G of $GL_n(R)$ is said to be monomial, if each element of G is monomial. Clearly the group $GL_n(R)$ acts naturally on the n-dimensional polynomial ring $S = R[X_1, \ldots, X_n]$ as R-algebra automorphisms. An element $\sigma \in GL_n(K)$ with $\mathrm{rank}(\sigma - 1) \leqq 1)$ for a field K is called a pseudo-reflection in $GL_n(K)$. Some classical results on pseudo-reflections are referred to Bourbaki [1]. In the case where G is finite and monomial, we have

*Partially supported by Research Institute for Mathematical Sciences, Kyoto university.

Proposition 1.1 **([4])** *Suppose that R is an integral domain with its quotient field F of arbitrary characteristic. Let G be a finite monomial subgroup of $GL_n(R)$. Then the subring $R[X_1, \ldots, X_n]^G$ of invariants of G in S is a polynomial ring over R if and only if G is generated by pseudo-reflections in $GL_n(K)$.*

We say a subgroup G of $GL_n(K)$ is imprimitive, if K^n is represented as $K^n = \bigoplus_{i=1}^m V_i$ ($m \geqq 2$) of subspaces of K^n and G acts naturally on $\{V_1, \ldots, V_m\}$ as permutations.

Such finite groups in Proposition 1.1 are classified as a product of $G(m, p, n)$'s defined as follows :

Let $\Pi_n \cong S_n$ be the group of all $n \times n$ permutation matrices, and let $A(m, p, n)$ with $m, p \in \boldsymbol{N}$ such that $p|m$ be the group of diagonal $n\times n$ matrices whose diagonal elements are powers of some fixed primitive mth root of $1 \in K$, and whose products are m/pth roots of $1 \in K$. Then as Π_n normalizes $A(m, p, n)$, define $G(m, p, n)$ to be the semidirect product $A(m, p, n) \cdot \Pi_n$. If G is finite subgroup generated by pseudo-reflections and irreducible imprimitive in $GL_n(K)$, then G is conjugate to one of $G(m, p, n)$'s (e.g., [2, 7]).

The purpose of this paper is to study on

Problem 1.2 *Can we generalize finite imprimitive reflection groups $G(m, p, n)$ to infinite algebraic groups related with invariant rings?*

2 Preliminaries

2.1 *Some notations*

Let $\mathcal{Q}(A)$ denote the total quotient ring of a ring A and

$$\mathrm{Ht}_1(A) := \{\mathfrak{P} \in \mathrm{Spec}(A) \mid \mathrm{ht}(\mathfrak{P}) = 1\}.$$

For an integral domain A and a subring B of A such that $B = \mathcal{Q}(B) \cap A$ and $\mathcal{Q}(B) \subseteq \mathcal{Q}(A)$, we denote by

$$\mathrm{Ht}_1(A, B) := \{\mathfrak{P} \in \mathrm{Ht}_1(A) \mid \mathfrak{P} \cap B \in \mathrm{Ht}_1(B)\}.$$

Consider an action of a group G on a ring R as automorphisms. For a prime ideal $\mathfrak{P}$ of R, let

$$\mathcal{I}_G(\mathfrak{P}) = \{\sigma \in G \mid \sigma(x) - x \in \mathfrak{P} \ (x \in R)\}$$

which is referred to as the inertia group of $\mathfrak{P}$ under this action. Let $Z^1(G, \mathrm{U}(R))$ be the group of 1cocycles of G on the unit group $\mathrm{U}(R)$ of R. For a 1-cocycle χ,

$$R_\chi := \{x \in R \mid \sigma(x) = \chi(\sigma)x \ (\sigma \in G)\}$$

which is a module over the invariant subring R^G of G in R.

2.2 *Pseudo-reflection groups of actions*

Here algebraic groups are affine (linear) and defined over a fixed algebraically closed field K of an arbitrary characteristic p. In general K-algebras R are not necessarily affine, i.e., finite generated as algebras over K.

We say an action (R, L) of an affine algebraic group L on R is regular, when L acts rationally on the K-algebra R as K-algebra automorphisms.

For an affine algebraic group L, L^0 denotes the identity (connected) component of L.

Definition 2.1 (Pseudo-reflection groups, [6]) *Suppose that R is a Krull K-domain with (R, L) a K-regular action of an algebraic group L. Define the subgroup*

$$\mathfrak{R}(R, L) := \left\langle \bigcup_{\mathfrak{P} \in \mathrm{Ht}_1(R, R^L)} \mathcal{I}_L(\mathfrak{P}) \right\rangle$$

of L which is called the pseudo-reflection group of the action (R, L).

Theorem 2.2 ([6]) *Suppose that R is a Krull K-domain with (R, L) a regular action of L. If L is reductive as an algebraic group, then the pseudo-reflection group $\mathfrak{R}(R, L)$ of the action (R, L) is finite on R.*

For an affine variety X, we denote by $K[X]$ the K-algebra consisting of polynomial functions on X. For an algebraic group L, we say (X, L) is a regular action if L acts on $K[X]$ as a rational L-module which induces K-algebra automorphisms. Clearly (X, L) is an regular action if and only if so is $(K[X], L)$. Thus we define the pseudo-reflection group $\mathfrak{R}(X, L)$ of the action (X, L) to be the group $\mathfrak{R}(K[X], L)$ as above.

Corollary 2.3 *Suppose that (X, L) is a regular action of a reductive algebraic group L on an affine normal variety X. Then the pseudo-reflection group $\mathfrak{R}(X, L)$ of the action (X, L) is finite on X.*

2.3 *Stable actions, representations*

Let $\mathfrak{X}(L)$ denotes the module of rational characters of L. Suppose that L is reductive. Then $K[X]^L$ is finitely generated as a K-algebra. For a finite dimensional vector space V, a morphism $\rho : L \to GL(V)$ of algebraic groups is a (rational) representation of L and then (V, L) is regarded as a regular action of L on an affine space V.

We say that (X, L) is stable if X contains a non-empty open set consisting of closed L-orbits in X.

Proposition 2.4 *Suppose that the identity component L^0 of L is an algebraic torus. Then there is an affine variety X_{st} with a stable regular action such that a canonical dominant G-invariant morphism $X \to X_{\mathrm{st}}$ induces $K[X]^L = K[X_{\mathrm{st}}]^L$. Here*

$$K[X_{\mathrm{st}}] = K[\{K[X]_\chi \mid K[X]_\chi K[X]_{-\chi} \neq \{0\}\} \ (\chi \in \mathfrak{X}(L^0))].$$

In the case where $X = V$ is the representation space of L, V_{st} is regarded as a rational L-submodule of V, i.e., $L \to GL(V)$ induces $L \to GL(V_{\mathrm{st}})$.

2.4 *Paralleled linear hulls*

We now suppose that L^0 is an algebraic torus and $L = Z_L(L^0)$. For a finite dimensional (rational) representation (V, L), a pair (W, L_w) is defined to be a paralleled linear hull of (V, L), if W is a L-submodule of V_{st} such that L is diagonal on V_{st}/W, $w \in V_{\mathrm{st}}$ is a non-zero vector satisfying $W \cap \langle Lw \rangle_K == \{0\}$ and the L_w-invariant morphism

$$(\bullet + w) : W \ni x \mapsto x + w \in V_{\mathrm{st}}$$

induces the canonical isomorphism

$$K[W]^{L_w} \stackrel{\sim}{\to} K[V_{\mathrm{st}}]^L (= K[V]^L)$$

If a W is minimal with this property, we say (W, L_w) is minimal paralleled linear hull of (V, L).

Theorem 2.5 *For any (V, L) as above, there exists a minimal paralleled linear hull (W, L_w) and it has the following properties :*

(1) $KW]^{L_w} \to K[W]$ *is no-blowing-up of codimension one and L_w acts transitively on the set* $\mathrm{Ht}_1(K[W], K[W]^{L_w})$.

(2) $\mathrm{Cl}(K[V]^L) \cong \mathfrak{X}(L/\mathfrak{R}(W, L_w))$

Remark 2.6 We note that Theorem 2.5 can be generalized for a Krull K-domain R with a regular action of L, which is useful in studying on a reduced expression of actions.

3 Toric imprimitive reflection groups

3.1 *Imprimitive actions on sub-toric quotients*

Hereafter for convenience sake suppose that the base filed K is of characteristic zero. We will define certain imprimitive algebraic group with toric identitiy component as follows.

Definition 3.1 (A toric imprimitive triplet (G, H, U)) *A triplet (G, H, U), is said to be toric imprimitive, consists of*

(i) an algebraic group G whose identity component G^0 is an algebraic torus.

(ii) a maximal diagonalizable closed subgroup H of G such that G is centralize H, i.e., $G = Z_G(H)$

(iii) a finite dimensional representation U of G whose action (G, U) is stable.

(iv) U satisfies the following condition : U can be expressed as a direct sum

$$U = \bigoplus_{i=1}^{m} U_i$$

of rational H-submodules U_i satisfying

(iv)-1 (U_i, H) is a stable action with $\dim K[U_i]^H = 1$ and $\det_{U_i}(H) = 1$.

(iv)-2 $\{U_1, \dots, U_m\}$ is a system of imprimitivities of U under the action of G, i.e., G acts on $\{U_1, \dots, U_m\}$ as a permutation group). We say that U_i are the G-imprimitive H-isobaric component of U.

Definition 3.2 ((H, U)-toric transpositions) *We say for a toric imprimitive triplet (G, H, U) an element $\tau \in G$ is a (H, U)-toric transposition, if $\tau|_U \in H|U$ or the following conditions are satisfied : there exists a pair $1 \leqq i_1 \neq \ i_2 \leqq m$ of indices such that*

(1) $\sigma(U_{i1}) = U_{i2}, \sigma(U_{i2}) = U_{i1}$ and $\mathrm{ord}(\sigma|_{U_{i1} \oplus U_{i2}}) = 2$,

(2) $\sigma(U_i) = U_i$ and $\det_{U_i}(\sigma) = 1$ for $1 \leqq i \leqq m, i \neq i_1, i_2$.

Definition 3.3 (Toric imprimitive reflection groups) *We say that s toric imprimitive triplet (G, H, U) defines a (H, U)-toric imprimitive reflection group, if $G|_U$ is generated by (H, U)-toric transpositions.*

Example 3.4 *Consider the dual space* $U^\vee = \bigoplus_{i=1}^4 KX_i$ *of* U *on which* $H = K^\times$ *acts by* $H \ni \sigma = t = \mathrm{diag}[t, t^{-1}, t, t^{-1}]$ *on* $\{X_1, \dots, X_4\}$*. Put* $U_1 = KX_1 \oplus KX_2$*,* $U_2 = KX_3 \oplus KX_4$*. Let*

$$\tau = \begin{bmatrix} O & E_2 \\ E_2 & O \end{bmatrix} \in GL(U^\vee)$$

where E_n *denotes the unit matrix of degree* n*. Define* $G = \langle H, \tau \rangle$*. Then* (G, H, U) *and* (H, H, U) *are respectively* (H, U)*-toric imprimitive reflection groups*

3.2 *Cofree Covers*

Definition 3.5 *A finite dimensional representation* $L \to GL(V)$ *of a reductive group* G *is said to be coregular (resp. cofree), if* $K[V]^L$ *is a polynomial ring over* K *(resp. if* $K[V]$ *is a free* $K[V]^L$*-module).*

The following fact is well known.

Proposition 3.6 ([8]) *For a representation* (V, L) *of a reductive* L*, if* (V, L) *is cofree, then it is coregular.*

Hereafter suppose that $p = 0$ or $(p, |G/G^0|) = 1$.

We introduce cofree covers of representations as follows :

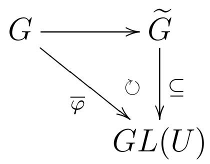
For an n-dimensional faithful rational representation $\varphi : G \to GL(W)$ such that (W, G) is a stable action; as $\mathfrak{R}(W, G)$ is a finite group, there is an n-dimensional natural representation $\overline{\varphi} : G \to GL(U)$ satisfying

$$K[W]^{\mathfrak{R}(W,G)} = K[U].$$

Definition 3.7 (Cofree cover) *A cofree cover* $(W, \widetilde{G})$ *of* (W, G) *is defined as follows : we ca choose a closed subgroup* $\widetilde{G}$ *of* $GL(U)$ *in such a way that*

(1) $(U, \widetilde{G})$ *is stable.*

(2) the following diagram is commutative



(3) $\widetilde{G}^0$ *is an algebraic torus with* $\widetilde{G} = \widetilde{G}^0 G$

(4) $\mathfrak{R}(U, \widetilde{G}) = \{1\}$

(5) $(W, \widetilde{G})$ *is cofree (see bellows).*

We say that $(W, \widetilde{G})$ is a minimal cofree cover of (W, G) if $\widetilde{G}$ is minimal in cofree covers of (W, G).

3.3 *Representations having cofree covers*

Example 3.8 *Consider the dual space* $W^\vee = \bigoplus_{i=1}^{6} KX_i$ *of* W *on which* $G = K^\times$ *acts by* $G \ni \sigma = t = \mathrm{diag}[t, t^{-1}, t, t^{-1}, t, t^{-1}]$ *on* $\{X_1, \ldots, X_6\}$. *Let* $\widetilde{G} = (K^\times)^3$ *be the closed subgroup* $\{\mathrm{diag}[t_1, t_1^{-1}, t_2, t_2^{-1}, t_3, t_3^{-1}] | (t_1, t_2, t_3) \in \widetilde{G}\}$ *on* $\{X_1, \ldots, X_6\}$ *in* $GL(W^\vee)$. *Then* $(W, \widetilde{G})$ *is a minimal cofree cover of* (W, G).

Example 3.9 *Consider the dual space* $V^\vee = \bigoplus_{i=1}^{4} KX_i$ *of* V *on which* $L = K^\times$ *acts by* $L \ni \sigma = t = \mathrm{diag}[t, t^{-1}, t^{-1}, t^{-1}]$ *on* $\{X_1, \ldots, X_4\}$. *Then* (V, L) *is coregular but is not cofree. A minimal paralleled linear hull* (W, L_w) *is expressed as* $W^\vee = \bigoplus_{i=2}^{4} KX_i$ *and* $L_w = \{1\}$. *Then* (W, L_w) *is itself a minimal cofree cover of* (W, L_w).

3.4 *On Problem 1.2*

A partial answer to this problem is :

Theorem 3.10 *Let* L *be a linearly reductive algebraic group with a maximal diagonalizable closed subgroup* D *containing* L^0 *such that* $Z_L(D) = L$. *Let* $\rho : L \to GL(V)$ *be an* l*-dimensional rational representation of* L. *Let* (W, w) *be a minimal paralleled linear hull of* (V, L) *with* $G = L_w, H = D_w$. *Then the following conditions are equivalent :*

(1) (W, G) *has a minimal cofree cover.*

(2) $G|_U$ *is generated by* (H, U)*-toric transpositions.*

Remark 3.11 In this theorem, the existence of cofree covers is only related to minimal paralleled linear hulls. This situation can similarly be examined as Example 3.9.

To show this, we need a special case (cf. [5]) where $L = D$ as follows :

Theorem 3.12 *Under the circumstances as in Theorem 3.10, we suppose that* L *is diagonalizable. Then the following conditions are equivalent :*
(1) $K[V]^L$ *is a Gorenstein ring.*
(2) (W, G) *has a minimal cofree cover.*

The next well known result plays an important role in the proof of this :

Theorem 3.13 (G. Kempf, R. P. Stanley and V. Danilov [[3, 9]) *Suppose that*

$$\varphi : H \to GL(W)$$

is a finite dimensional representation of a diagonalizable algebraic group H whose action is stable. Then the canonical module $\omega_{K[W]^H}$ of the graded algebra $K[W]^H$ is isomorphic to the graded module

$$K[W]_{\det_W |H}(-\dim W)$$

of invariants relative to $\det_W |H$ in $K[W]$ of H shifted by the number $-\dim W$.

Regarding Problem 1.2, there seems to be a different answer from the one here based on subtoric quotients of representations. Further studies will be conducted.

References

[1] N. Bourbaki, Groupes et Algèbres de Lie : Chapitres 4 à 6 (Éléments de Mathématique) Springer-Verlag, Berlin-Heidelberg-New York, 2006.

[2] R. Corran, E.-K.Lee, S.-J. Lee, Braid groups of imprimitive complex reflection groups, J. Algebra 427 (2015), 387–425.

[3] V. I. Danilov, The geometry of toric varieties, Uspehi Mat. Nauk 33 (1978), 85–134.

[4] Haruhisa Nakajima, Invariants of finite groups generated by pseudo-reflections in positive characteristc, Tsukuba J. Math. 3 (1979), 109–122.

[5] Haruhisa Nakajima, Cofree embeddings of algebraic tori preserving canonical sheaves, Proc. Japan Acad., 82, Ser. A 82A (2006), 155–160.

[6] Haruhisa Nakajima, Reductivities and finiteness of pseudo-reflections of algebraic groups and homogeneous fiber bundles, J. Pure and Appl. Algebra 217 (2013), 1548-1562.

[7] E. W. Read, On the finite imprimitive unitary reflection groups, J. Algbera 45 (1977), 439–452.

[8] Gerald W. Shwarz, Rrepresentations of simple Lie groups with a free module of invariants, Invent. Math. 50 (1978), 1–12.

[9] Richard P. Stanley, Hilbert functions of graded algebras, Advances in Math. 28 (1978), 57–83.