Note on $(\tilde{\rho}, \tilde{D})$-separable polynomials in skew polynomial rings

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Abstract

The notion of $(\tilde{\rho}, \tilde{D})$-separable polynomials in skew polynomial rings was introduced by S. Ikehata. In this paper, we shall give a new characterization of $(\tilde{\rho}, \tilde{D})$-separable polynomials in skew polynomial rings which shows the difference between separable systems and $(\tilde{\rho}, \tilde{D})$-separable systems.

1 Introduction and Preliminaries

Let A/B be a ring extension with common identity. A/B is said to be *separable* if the A-A-homomorphism of $A\otimes_B A$ onto A defined by $z\otimes w \mapsto zw$ $(z, w \in A)$ splits. It is well known that A/B is separable if and only if there exists $\sum_i z_i \otimes w_i \in (A\otimes_B A)^A$ such that $\sum_i z_i w_i = 1$, where $(A\otimes_B A)^A = \{\theta \in A\otimes_B A \mid u\theta = \theta u \ (\forall\, u \in A)\}$. Then we say that $\{z_i, w_i\}$ is a *separable system* of A/B.

Throughout this paper, let B be an associative ring with identity element 1, ρ an automorphism of B, and D a ρ-derivation (that is, D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\beta + \rho(\alpha)D(\beta)$ for any $\alpha, \beta \in B$). By $B[X;\rho, D]$ we denote the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. Moreover, by $B[X;\rho, D]_{(0)}$, we denote the set of all monic polynomials f in $B[X;\rho,D]$ such that $fB[X;\rho,D] = B[X;\rho,D]f$. For each polynomial $f \in B[X;\rho,D]_{(0)}$, the residue ring $B[X;\rho,D]/fB[X;\rho,D]$ is a free ring extension of B.

From now on, let $B^\rho = \{b \in B \mid \rho(b) = b\}$, $f = \sum_{i=0}^m X^i a_i \in B[X;\rho,D]_{(0)} \cap B^\rho[X]$ $(m \geq 1, a_m = 1)$, $A = B[X;\rho,D]/fB[X;\rho,D]$, and $x = X + fB[X;\rho,D]$. Then A is a free ring extension of B with a free B-basis $\{1, x, x^2, \cdots, x^{m-1}\}$. Since $f \in B^\rho[X]$, there is a ring automorphism $\tilde{\rho}$ of A which is naturally induced by ρ, that is, $\tilde{\rho}$ is defined by

$$\tilde{\rho}\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j \rho(c_j) \quad (c_j \in B).$$

Similarly, there is a $\tilde{\rho}$-derivation $\tilde{D}$ of A which is naturally induced by D, that is, $\tilde{D}$ is defined by

$$\tilde{D}\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j D(c_j) \quad (c_j \in B).$$

Now we consider the following A-A-homomorphisms:

$$\begin{cases} \mu : A \otimes_B A \to A, & \mu(z \otimes w) = zw \\ \xi : A \otimes_B A \to A \otimes_B A, & \xi(z \otimes w) = \tilde{D}(z) \otimes \tilde{\rho}(w) + z \otimes \tilde{D}(w) \qquad (z, w \in A) \\ \eta : A \otimes_B A \to A \otimes_B A, & \eta(z \otimes w) = \tilde{\rho}(z) \otimes \tilde{\rho}(w) - z \otimes w \end{cases}$$

We say that f is a *separable polynomial* in $B[X;\rho,D]$ if A is a separable extension of B. By the definition, f is separable in $B[X;\rho,D]$ if and only if there exists an A-A-homomorphism $\nu : A \to A \otimes_B A$ such that $\mu\nu = 1_A$ (the identity map of A). Moreover, f is called a $(\tilde{\rho},\tilde{D})$*-separable polynomial* in $B[X;\rho,D]$ if there exists an A-A-homomorphism $\nu : A \to A \otimes_B A$ such that $\mu\nu = 1_A$, $\xi\nu = \nu\tilde{D}$, and $\eta\nu = \nu(\tilde{\rho} - 1_A)$. The notion of $(\tilde{\rho},\tilde{D})$-separable polynomials was introduced by S. Ikehata in [3]. Obviously, a $(\tilde{\rho},\tilde{D})$-separable polynomial is separable. We put here $B[X;\rho] = B[X;\rho,0]$ and $B[X;D] = B[X;1_B,D]$. If $D = 0$ then a $(\tilde{\rho},\tilde{0})$-separable polynomial in $B[X;\rho]$ is called $\tilde{\rho}$*-separable*. Similarly, if $\rho = 1_B$ then a $(\tilde{1_B},\tilde{D})$-separable polynomial in $B[X;D]$ is called $\tilde{D}$*-separable*.

In [3], S. Ikehata studied $(\tilde{\rho},\tilde{D})$-separable polynomials in $B[X;\rho,D]$. Moreover, in [11], X. Lou gave a characterization of $\tilde{\rho}$-separable polynomials in $B[X;\rho]$ by making use of the trace map. In this paper, we shall study $(\tilde{\rho},\tilde{D})$-separable polynomials in $B[X;\rho,D]$ in the case $\rho D = D\rho$. In section 2, we shall show a equivalent condition for $(\tilde{\rho},\tilde{D})$-separable polynomials in $B[X;\rho,D]$. In section 3, we shall give a new characterization of $(\tilde{\rho},\tilde{D})$-separable polynomial in $B[X;\rho,D]$. It shows the difference between separable systems and $(\tilde{\rho},\tilde{D})$-separable systems.

2 Equivalent condition for $(\tilde{\rho},\tilde{D})$-separability

From this section, assume that $\rho D = D\rho$, and let $B^\rho = \{b \in B \mid \rho(b) = b\}$, $B^{\rho,D} = \{b \in B \mid \rho(b) = b,\ D(b) = 0\}$, $C(B^{\rho,D})$ the center of $B^{\rho,D}$, m a positive integer, and f a monic polynomial in $B[X;\rho,D] \cap B^\rho[X]$ of the form $f = \sum_{i=0}^m X^i a_i$ ($a_m = 1$). As was shown in [10, Lemma 1.6 and Corollary 1.7], $f \in C(B^{\rho,D})[X]$ and

$$a_i \rho^m(\alpha) = \sum_{j=i}^{m} \binom{j}{i} \rho^i D^{j-i}(\alpha) a_j \quad (\forall\, \alpha \in B,\ 0 \le i \le m-1).$$

We shall use the following conventions:

- $A = B[X;\rho,D]/fB[X;\rho,D]$
- $x = X + fB[X;\rho,D]$
- $\tilde{\rho}$ is an automorphism of A defined by

$$\tilde{\rho}\left(\sum_{i=0}^{m-1} x^i c_i\right) = \sum_{i=0}^{m-1} x^i \rho(c_i) \quad (c_i \in B).$$

- $\tilde{D}$ is a $\tilde{\rho}$-derivation of A define by

$$\tilde{D}\left(\sum_{i=0}^{m-1} x^i c_i\right) = \sum_{i=0}^{m-1} x^i D(c_i) \quad (c_i \in B).$$

- $A^{\tilde{\rho},\tilde{D}} = \{z \in A \,|\, \tilde{\rho}(z) = z,\ \tilde{D}(z) = 0\}$.
- $C(A^{\tilde{\rho},\tilde{D}})$ is the center of $A^{\tilde{\rho},\tilde{D}}$.
- $V_{m-1} = \{z \in A \,|\, \rho^{m-1}(\alpha)z = z\alpha \ (\forall\, \alpha \in B)\}$.
- $T(A,B) = A \otimes_B A$.
- $T(A,B)^A = \{\theta \in T(A,B) \,|\, u\theta = \theta u \ (\forall\, u \in A)\}$.
- $T(C(A^{\tilde{\rho},\tilde{D}}), C(B^{\rho,D})) = C(A^{\tilde{\rho},\tilde{D}}) \otimes_{C(B^{\rho,D})} C(A^{\tilde{\rho},\tilde{D}})$.
- $T(C(A^{\tilde{\rho},\tilde{D}}), C(B^{\rho,D}))^{C(A^{\tilde{\rho},\tilde{D}})} = \{\theta \in T(C(A^{\tilde{\rho},\tilde{D}}), C(B^{\rho,D})) \,|\, u\theta = \theta u \ (\forall\, u \in C(A^{\tilde{\rho},\tilde{D}})\}$.
- $\pi_i : A \to B$ is the map defined by

$$\pi_i\left(\sum_{j=0}^{m-1} x^j c_j\right) = c_i \quad (c_i \in B, 0 \le i \le m-1).$$

- $\tau : A \to B$ is the map defined by

$$\tau\,(z) = \sum_{i=0}^{m-1} \pi_i(x^i z) \quad (z \in A).$$

Clearly, π_i $(0 \le i \le m-1)$ and τ are $B^{\rho,D}$-B-homomorphisms. Moreover, we define polynomials $Y_i \in B[X;\rho,D]$ $(0 \le i \le m-1)$ as follows:

$$\begin{aligned}
Y_0 &= X^{m-1} + X^{m-2}a_{m-1} + \cdots + Xa_2 + a_1,\\
Y_1 &= X^{m-2} + X^{m-3}a_{m-1} + \cdots + Xa_3 + a_2,\\
&\cdots\cdots\\
Y_i &= X^{m-i-1} + X^{m-i-2}a_{m-1} + \cdots + Xa_{i+2} + a_{i+1} \left(= \sum_{k=i}^{m-1} X^{k-i}a_{k+1}\right),\\
&\cdots\cdots\\
Y_{m-2} &= X + a_{m-1},\\
Y_{m-1} &= 1.
\end{aligned}$$

The polynomials Y_i were introduced by Y. Miyashita to characterize separable polynomials in $B[X;\rho,D]$ (cf. [5]). We set $y_i = Y_i + fB[X;\rho,D] \in A$ $(0 \le i \le m-1)$.

Remark 1. (1) Since $f \in C(B^{\rho,D})$, we see that $\tau(x^k)$ is in $C(B^{\rho,D})$ for any non-negative integer k. Moreover, Y_i $(0 \leq i \leq m-1)$ is in $C(B^{\rho,D})[X]$ and y_i $(0 \leq i \leq m-1)$ is in $C(A^{\tilde{\rho},\tilde{D}})$.

(2) It is easy to see that

$$\begin{aligned} C(A^{\tilde{\rho},\tilde{D}}) &= \left(C(B^{\rho,D})[X] + fB[X;\rho,D]\right)/fB[X;\rho,D] \\ &\cong C(B^{\rho,D})[X]/fC(B^{\rho,D})[X]. \end{aligned}$$

In particular, a free B-basis $\{1, x, x^2, \cdots, x^{m-1}\}$ of A can be regarded as a free $C(B^{\rho,D})$-basis of $C(A^{\tilde{\rho},\tilde{D}})$, and the restriction map $\tau|_{C(A^{\tilde{\rho},\tilde{D}})}$ is a trace map from $C(A^{\tilde{\rho},\tilde{D}})$ to $C(B^{\rho,D})$.

(3) As was shown in [10, Lemma 2.1], it is already known that

$$T(A,B)^A = \left\{ \sum_{i=0}^{m-1} y_i v \otimes x^i \,\middle|\, v \in V_{m-1} \right\}.$$

In particular, every separable system of A/B is of the form $\{y_i v, x^i\}$ for some $v \in V_{m-1}$. Similarly, we can see that

$$T(C(A^{\tilde{\rho},\tilde{D}}), C(B^{\rho,D}))^{C(B^{\rho,D})} = \left\{ \sum_{i=0}^{m-1} y_i v \otimes x^i \,\middle|\, v \in C(A^{\tilde{\rho},\tilde{D}}) \right\}.$$

(4) Note that $\sum_{i=0}^{m-1} Y_i X^i = \sum_{i=0}^{m-1} X^i Y_i = f'$, where f' is the derivative of f.

The following is a equivalent condition for $(\tilde{\rho}, \tilde{D})$-separability in $B[X;\rho,D]$.

Lemma 2.1. *The following are equivalent.*

(1) f *is* $(\tilde{\rho}, \tilde{D})$*-separable in* $B[X;\rho,D]$.

(2) *There exists* $v \in V_{m-1} \cap A^{\tilde{\rho},\tilde{D}}$ *such that* $\sum_{i=0}^{m-1} y_i v x^i = 1$.

(3) f' *is invertible in* $B[X;\rho,D]$ *modulo* $fB[X;\rho,D]$, *where* f' *is the derivative of* f.

(4) f *is separable in* $C(B^{\rho,D})[X]$ *(i.e. a commutative ring* $C(A^{\tilde{\rho},\tilde{D}})$ *is separable over* $C(B^{\rho,D})$*).*

(5) *There exists* $v \in C(A^{\tilde{\rho},\tilde{D}})$ *such that* $\sum_{i=0}^{m-1} y_i v x^i = 1$ *and* $\sum_{i=0}^{m-1} y_i v \tau(x^i u) = u$ *for any* $u \in C(A^{\tilde{\rho},\tilde{D}})$.

Proof. We have already known that (1), (2), (3), and (4) are equivalent by [3, Theorem 2.1]. We shall show that (4) is equal to (5).

(4) $\Longrightarrow$ (5) Assume that f is separable in $C(B^{\rho,D})[X]$, that is, a commutative ring $C(A^{\tilde{\rho},\tilde{D}})$ is (finitely generated projective and) separable over $C(B^{\rho,D})$. Note that τ is a trace map from $C(A^{\tilde{\rho},\tilde{D}})$ to $C(B^{\rho,D})$ by Remark 1 (2). Let u be arbitrary element in $C(A^{\tilde{\rho},\tilde{D}})$. Then, by [1, chapter III, Theorem 2.1], there exists a finite set $\{z_i, w_i\}$ ($z_i, w_i \in C(A^{\tilde{\rho},\tilde{D}})$) such that $\sum_i z_i w_i = 1$ and $\sum_i z_i \tau(w_i u) = u$. Concerning $\sum_i z_i \otimes w_i \in T(C(A^{\tilde{\rho},\tilde{D}}), C(B^{\rho,D}))$, we have

$$\begin{aligned}\sum_i z_i \otimes w_i u &= \sum_i z_i \otimes \sum_j z_j \tau(w_j w_i u)\\ &= \sum_j \sum_i z_i \tau(w_i w_j u) \otimes z_j\\ &= \sum_j u w_j \otimes z_j.\end{aligned}$$

Thus $\sum_i z_i \otimes w_i = \sum_j w_j \otimes z_j \in T(C(A^{\tilde{\rho},\tilde{D}}), C(B^{\rho,D}))^{C(A^{\tilde{\rho},\tilde{D}})}$. So, by Remark 1 (3), there exists $v \in C(A^{\tilde{\rho},\tilde{D}})$ such that $\sum_i z_i \otimes w_i = \sum_{i=0}^{m-1} y_i v \otimes x^i$. Let μ and $\widehat{\tau}$ be additive endomorphisms from $T(C(A^{\tilde{\rho},\tilde{D}}), C(B^{\rho,D}))$ to $C(A^{\tilde{\rho},\tilde{D}})$ defined by $\mu(z \otimes w) = zw$ and $\widehat{\tau}(z \otimes w) = z\tau(w)$ ($z, w \in C(A^{\tilde{\rho},\tilde{D}})$), respectively. We obtain then

$$\begin{aligned}1 &= \sum_i z_i w_i = \mu\left(\sum_i z_i \otimes w_i\right) = \mu\left(\sum_{i=0}^{m-1} y_i v \otimes x^i\right) = \sum_{i=0}^{m-1} y_i v x^i,\\ u &= \sum_i z_i \widehat{\tau}(w_i u) = \widehat{\tau}\left(\sum_i z_i \otimes w_i u\right) = \widehat{\tau}\left(\sum_{i=0}^{m-1} y_i v \otimes x^i u\right) = \sum_{i=0}^{m-1} y_i v \tau(x^i u).\end{aligned}$$

(5) $\Longrightarrow$ (4) It is obvious by [1, chapter III, Theorem 2.1]. □

3 Characterization of $(\tilde{\rho}, \tilde{D})$-separability

The conventions and notations employed in the preceding section will be used in this section. First we shall state the following.

Lemma 3.1. *$\sum_{i=0}^{m-1} y_i \tau(x^i)$ is in $C(A^{\tilde{\rho},\tilde{D}})$ and*

$$\rho^{1-m}(\alpha) \sum_{i=0}^{m-1} y_i \tau(x^i) = \sum_{i=0}^{m-1} y_i \tau(x^i) \alpha \quad (\forall\, \alpha \in B).$$

Proof. Since $y_i \in C(A^{\tilde{\rho},\tilde{D}})$ and $\tau(x^i) \in C(B^{\rho,D})$, it is obvious that $\sum_{i=0}^{m-1} y_i \tau(x^i) \in C(A^{\tilde{\rho},\tilde{D}})$. Let α be arbitrary element in B, and $\widehat{\tau} : A \otimes_B A \to A$ an A-B-homomorphism defined by $\widehat{\tau}(z \otimes w) = z\tau(w)$ ($z, w \in A$). As was shown [10], we have already known that

$$\alpha y_j = \sum_{i=j}^{m-1} y_i \binom{i}{j} (-1)^{i-j} \rho^{m-j-1} D^{i-j}(\alpha) \quad (0 \leq j \leq m-1).$$

Noting that $x^i\alpha = \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} \rho^{-j} D^{i-j}(\alpha) x^j$, we obtain

$$\begin{aligned}
\sum_{j=0}^{m-1} \rho^{1-m}(\alpha) y_j \tau(x^j) &= \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} y_i \binom{i}{j} (-1)^{i-j} \rho^{-j} D^{i-j}(\alpha) \tau(x^j) \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{i} y_i \binom{i}{j} (-1)^{i-j} \rho^{-j} D^{i-j}(\alpha) \tau(x^j) \\
&= \widehat{\tau} \left(\sum_{i=0}^{m-1} \sum_{j=0}^{i} y_i \binom{i}{j} (-1)^{i-j} \rho^{-j} D^{i-j}(\alpha) \otimes x^j \right) \\
&= \widehat{\tau} \left(\sum_{i=0}^{m-1} y_i \otimes \left(\sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} \rho^{-j} D^{i-j}(\alpha) x^j \right) \right) \\
&= \widehat{\tau} \left(\sum_{i=0}^{m-1} y_i \otimes x^i \alpha \right) \\
&= \sum_{i=0}^{m-1} y_i \tau(x^i) \alpha.
\end{aligned}$$

This completes the proof. □

Corollary 3.2.

$$\tilde{\rho}^{1-m}(z) \sum_{i=0}^{m-1} y_i \tau(x^i) = \sum_{i=0}^{m-1} y_i \tau(x^i) z \quad (\forall\, z \in A).$$

Proof. It is obvious by Lemma 3.1. □

So we shall state the following theorem which shows a new equivalent condition for $(\tilde{\rho}, \tilde{D})$-separability. It shows the difference between separable systems and $(\tilde{\rho}, \tilde{D})$-separable systems of A/B.

Theorem 3.3. *The following are equivalent.*

(1) *f is $(\tilde{\rho}, \tilde{D})$-separable in $B[X; \rho, D]$.*

(2) *f is separable in $B[X; \rho, D]$ with a separable system $\{y_i v, x^i\}$ of A/B such that $\sum_{i=0}^{m-1} y_i v \tau(x^i) = 1$, where $v \in V_{m-1}$.*

Proof. (1) $\Longrightarrow$ (2) Let f be $(\tilde{\rho}, \tilde{D})$-separable in $B[X; \rho, D]$. So, by Lemma 2.1 (5), there exists $v \in C(A^{\tilde{\rho},\tilde{D}})$ such that $\sum_{i=0}^{m-1} y_i v x^i = 1$ and $\sum_{i=0}^{m-1} y_i v\tau(x^i u) = u$ for any $u \in C(A^{\tilde{\rho},\tilde{D}})$. Clearly, $\sum_{i=0}^{m-1} y_i v\tau(x^i) = 1$. To show that $\{y_i v, x^i\}$ is a separable system of A/B, it suffices to prove that $\sum_{i=0}^{m-1} y_i v \otimes x^i \in (A \otimes_B A)^A$. Let $\tilde{f}' = f' + fB[X; \rho, D]$, where f' is the derivative of f. Noting that $\sum_{i=0}^{m-1} y_i x^i = \tilde{f}'$, we have

$$1 = \sum_{i=0}^{m-1} y_i v x^i = \tilde{f}' v = v \tilde{f}'.$$

Thus $\tilde{f}'$ is invertible in A (this is the assertion (3) of Lemma 2.1). For any $\alpha \in B$, it follows from the proof of [3, Lemma 1.2] that $\alpha f' = f' \rho^{m-1}(\alpha)$, and hence we obtain

$$\begin{aligned}\alpha = \alpha \cdot 1 = \alpha \tilde{f}' v &= \tilde{f}' \rho^{m-1}(\alpha) v, \\ &= 1 \cdot \alpha = \tilde{f}' v \alpha.\end{aligned}$$

Since $\tilde{f}'$ is invertible, we have $\rho^{m-1}(\alpha)v = v\alpha$. Therefore $v \in V_{m-1}$, and hence $\sum_{i=0}^{m-1} y_i v \otimes x^i \in (A \otimes_B A)^A$ by Remark 1 (3).

(2) $\Longrightarrow$ (1) Assume that f is separable in $B[X; \rho, D]$ with a separable system $\{y_i v, x^i\}$ of A/B such that $\sum_{i=0}^{m-1} y_i v\tau(x^i) = 1$, where $v \in V_{m-1}$. If $v \in A^{\tilde{\rho},\tilde{D}}$ then f is $(\tilde{\rho}, \tilde{D})$-separable by Lemma 2.1 (2). Therefore we shall show that $v \in A^{\tilde{\rho},\tilde{D}}$. By Corollary 3.2, we see that

$$1 = \sum_{i=0}^{m-1} y_i v\tau(x^i) = \sum_{i=0}^{m-1} y_i \tau(x^i) v = \tilde{\rho}^{1-m}(v) \sum_{i=0}^{m-1} y_i \tau(x^i).$$

On the other hand, since $\sum_{i=0}^{m-1} y_i \tau(x^i) \in C(A^{\tilde{\rho},\tilde{D}})$, we have

$$\begin{aligned}1 = \tilde{\rho}(1) = \tilde{\rho}\left(\sum_{i=0}^{m-1} y_i \tau(x^i) v\right) &= \sum_{i=0}^{m-1} y_i \tau(x^i) \tilde{\rho}(v), \\ 0 = \tilde{D}(1) = \tilde{D}\left(\sum_{i=0}^{m-1} y_i \tau(x^i) v\right) &= \sum_{i=0}^{m-1} y_i v\tau(x^i) \tilde{D}(v).\end{aligned}$$

This implies that $\tilde{\rho}(v) = v$ and $\tilde{D}(v) = 0$, whence $v \in A^{\tilde{\rho},\tilde{D}}$. □

References

[1] F. DeMeyer, *Separable algebra over commutative rings*, Lecture Notes in Mathematics, Vol.181, Springer-Verlag, Berlin-Heidelberg-New York.

[2] K. Hirata and K. Sugano, *On semisimple extensions and separable extensions over noncommutative rings*, J. Math. Soc. Japan, **18** (1966), 360–373.

[3] S. Ikehata, *On separable polynomials and Frobenius polynomials in skew polynomial rings*, Math. J. Okayama Univ., **22** (1980), 115–129.

[4] S. Ikehata, *Azumaya algebras and skew polynomial rings*, Math, J. Okayama Univ., **23** (1981), 19–32.

[5] Y. Miyashita, *On a skew polynomial ring*, J. Math. Soc. Japan, **31** (1979), no.2, 317–330.

[6] T. Nagahara, *On separable polynomials over a commutative rings II*, Math. J. Okayama Univ., **15**, (1972), 149–162.

[7] T. Nagahara, *On separable polynomials of degree* 2 *in skew polynomial rings*, Math. J. Okayama Univ., **19** (1976), 65–95.

[8] K. Sugano, *Separable extensions and Frobenius extensions*, Osaka J. Math., **7** (1970), 29–40.

[9] S. Yamanaka and S. Ikehata, *An alternative proof of Miyasita's theorem in a skew polynomial rings*, Int. J. Algebra, **21** (2012), 1011–1023

[10] S. Yamanaka, *An alternative proof of Miyasita's theorem in a skew polynomial rings II*, Gulf J. Math., **5** (2017), 9–17.

[11] X.Lou, *On $\tilde{\rho}$-separability in skew polynomial rings*, Port. Math., Vol.51, Frac.2 (1994), 235–241.

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