On the asymptotic behavior of families of nonlinear mappings and some weak convergence theorems

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Abstract

In this paper, we study the asymptotic behavior of orbits of nonexpansive semigroups in Banach spaces. We also establish a weak convergence theorem for two normally 2-generalized hybrid mappings and we give some convergence theorems.

1 Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [25] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;

(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

The behavior of the sequence of Picard iterates of T is one of the important problems in metric fixed point theory because this allows us to approximate a fixed point in the simplest way. Moreau [20] proved that if C is a closed subset of a Hilbert space and if $F(T)$ has nonempty interior, then for each $x \in C$, the sequence $\{T^n x\}$ converge strongly to a point in $F(T)$. Kirk and Sims [12] generalized this result to Banach spaces. Grzesik, Kaczor, Kuczumow and Reich [10] proved convergence of iterates of nonexpansive mappings: Let C be a bounded closed and convex subset of a uniformly convex Banach space E. Assume that C has nonempty interior and that it is locally uniformly rotund. Let T be a nonexpansive mapping of C into itself and let $x \in C$. If T has no fixed point in the interior of C, then

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there exists a unique point z_0 on the boundary ∂C of C such that each sequence $\{T^n x : n = 1,2,3,\dots\}$ converges strongly to z_0. They [10] also proved the convergence of orbits of one-parameter nonexpansive semigroups.

In this paper, we study the asymptotic behavior of orbits of nonexpansive semigroups with no common fixed points in the interior of their domains (see [2]). Motivated by [10], we give convergence theorems for abstract semigroups. We also establish a weak convergence theorem for two normally 2-generalized hybrid mappings and we give some convergence theorems.

2 Preliminaries and notations

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$. We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or it $\text{w-}\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, $\text{co}A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a\cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s+t$. Let $B(S)$ be the Banach space of all bounded real-valued functions defined on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. For each $s \in S$ and $g \in B(S)$, we can define an element $\ell_s g \in B(S)$ by $(\ell_s g)(t) = g(st)$ for all $t \in S$. We also denote by ℓ_s^* the conjugate operator of ℓ_s. Let $C(S)^*$ be the dual space of $C(S)$. A linear functional μ on $C(S)$ is called a mean on $C(S)$ if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(g(t))$ or $\int g(t)d\mu(t)$ instead of $\mu(g)$ for $\mu \in C(S)^*$ and $g \in C(S)$. A mean μ on $C(S)$ is called invariant if $\mu(\ell_s g) = \mu(g)$ for all $s \in S$ and $g \in C(S)$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(g) = g(s)$ for every $g \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean μ on S is also a mean on $C(S)$ containing constants.

The following definition which was introduced by Takahashi [23] is crucial in the nonlinear ergodic theory for abstract semigroups (see also [11]). Let h be a continuous function of S into E such that the weak closure of $\{h(t) : t \in S\}$ is weakly compact. Then, for any $\mu \in C(S)^*$ there exists a unique element $h_\mu \in E$ such that

$$\langle h_\mu, x^* \rangle = (\mu)_t \langle h(t), x^* \rangle = \int \langle h(t), x^* \rangle \, d\mu(t)$$

for all $x^* \in E^*$. If μ is a mean on $C(S)$, then h_μ is contained in $\overline{\text{co}}\{h(t) : t \in S\}$ (for example, see [23, 24]). Sometimes, h_μ will be denoted by $\int h(t)d\mu(t)$.

Throughout this paper, S is a commutative semitopological semigroup with identity. Let C be a closed convex subset of a Banach space E. Then, a family $\mathscr{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(a) $T(s+t) = T(s)T(t)$ for all $s,t \in S$;

(b) $s \mapsto T(s)x$ is continuous;

(c) $\|T(s)x - T(s)y\| \leq \|x-y\|$ for all $x,y \in C$ and $s \in S$.

We denote by $F(\mathscr{S})$ the set of common fixed points of $T(t), t \in S$. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Assume that for each $x \in C$ and $x^* \in E^*$, the weak closure of $\{T(t)x : t \in S\}$ is weakly compact. Let μ be a mean on $C(S)$. Following [21], we also write $T_\mu x$ instead of $\int T(t)x\,d\mu(t)$ for $x \in C$. We remark that T_μ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathscr{S})$. If μ is a finite mean, i.e.,

$$\mu = \sum_{i=1}^{n} a_i \delta_{t_i} \; (t_i \in S, a_i \geq 0, \; \sum_{i=1}^{n} a_i = 1),$$

then

$$T_\mu x = \sum_{i=1}^{n} a_i T(t_i)x.$$

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x,y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x,y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \; : \; \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x,y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Let $S_E = \{x \in E : \|x\| = 1\}$ be unit sphere in a Banach space E. A Banach space E is said to be *locally uniformly rotund* if for each $x \in S_E$ and for each $\varepsilon \in (0,2]$,
there exists $\delta(x,\varepsilon) > 0$ such that
for each $y \in S_E$ with $\|x-y\| \geq \varepsilon$, we have

$$1 - \left\|\frac{x+y}{2}\right\| \geq \delta(x,\varepsilon)$$

For more details, see [18].

Let E be a Banach space, let C be a nonempty bounded closed and convex subset of E. Assume that C have nonempty interior, that is, $\mathrm{int}(C) \neq \emptyset$. We say that C is *locally uniformly rotund* if for each $x \in \partial C$ and for each $\varepsilon \in (0, d_x)$, where $d_x = \sup\{\|x-y\| : y \in C\}$, there exists $\delta(x,\varepsilon) > 0$ such that for each $y \in C$ with $\|x-y\| \geq \varepsilon$, we have

$$\mathrm{dist}\left(\frac{x+y}{2}, \partial C\right) := \inf\left\{\left\|\frac{x+y}{2} - x'\right\| : x' \in \partial C\right\} \geq \delta(x,\varepsilon).$$

Let C be a nonempty bounded closed and convex subset of a Banach space E. Assume that C have nonempty interior, that is, $\mathrm{int}(C) \neq \emptyset$. We say that C is uniformly convex if for each $\varepsilon \in (0, \mathrm{diam(C)})$, there exists $\eta_C(\varepsilon) > 0$ such that for each $x, y \in C$ with $\|x-y\| \geq \varepsilon$, we have

$$\mathrm{dist}\left(\frac{x+y}{2}, \partial C\right) := \inf\left\{\left\|\frac{x+y}{2} - x'\right\| : x' \in \partial C\right\} \geq \eta_C(\varepsilon).$$

Let C be a subset of a Banach space E and let T be mapping of C into E. The mapping T is said to be *demiclosed* if for any sequence $\{x_n\} \subset C$ the following implication hold:

$$\text{w-}\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \|Tx_n - y\| = 0$$

imply that

$$Tx = y$$

(see [8]).

Theorem 2.1 ([8])**.** *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E. Let T be nonexpansive mapping of C into itself and let I be the identity mapping. Then, $I-T$ is demiclosed at* 0*, that is,*

$$\text{w-}\lim_{n\to\infty} x_n = x \textit{ and } \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

imply that

$$Tx = x.$$

Throughout the rest of this paper, S is a commutative semitopological semigroup with identity. The following theorem has been essentially established in [9] (see also [5, 6, 24]).

Theorem 2.2 ([9])**.** *Let C be a nonempty bounded closed and convex subset of a uniformly convex Banach space E. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Then, $F(\mathscr{S})$ is nonempty.*

The following theorem has been essentially established in [5] (see also [6, 9, 24]).

Theorem 2.3 ([5])**.** *Let C be a closed and convex subset of a strictly convex Banach space E. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathscr{S}) \neq \emptyset$. Then, the set $F(\mathscr{S})$ is closed and convex.*

3 Convergence theorems for nonexpansive semigroups

In this section, we give the convergence theorems for nonexpansive semigroups with no common fixed points in the interior of their domains. Throughout the rest of this paper, S is a commutative semitopological semigroup with identity.

For $h \in (0,\infty)$, we denote by C_h the set $C \cap \{x \in E : \|x - z_0\| \geq h\}$. Let $z_0 \in C$ and let $x^* \in E^*$ with $\|x^*\| = 1$. We denote by V_{k,z_0} the hyperplane

$$\{x \in E : x^*(x) = k\}$$

which supports C at the point z_0, where $k \in (0,\infty)$, $x^*(z_0) = k$.

The following was proved in [10].

Lemma 3.1 ([10]). *Let E be a Banach space and let C be a bounded, closed and convex subset of E. Assume that* $\operatorname{int}(C)$ *is nonempty,* $0 \in \operatorname{int}(C)$ *and that C is locally uniformly rotund. Let $z_0 \in \partial C$, let $x^* \in E^*$ with $\|x^*\| = 1$ and let the hyperplane*

$$V_{k,z_0} = \{x \in E : x^*(x) = k\}$$

which supports C at the point z_0 be given, where $k \in (0,\infty)$. If $r \in (0,\infty)$ and the set

$$C_r = C \cap \{x \in E : \|x - z_0\| \geq r\}$$

is nonempty, then there exists $k_1 \in \mathbb{R}$ such that $0 < k_1 < k$ and

$$C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

The following lemma plays an important role in our main results (see [4, 11, 22]).

Lemma 3.2 ([4]). *Let C be a nonempty bounded, closed convex subset of a uniformly convex Banach space E. Let S be a commutative semitopological semigroup with identity. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that* $\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$ *for each $s \in S$. Then, for each $t \in S$,*

$$\lim_{n\to\infty} \sup_{y \in C} \left\| T_{\mu_n} y - T(t) T_{\mu_n} y \right\| = 0.$$

A sequence $\{x_n\}$ in C is said to be an *approximating sequence* of a nonexpansive mapping $T : C \to C$ if

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

(for example, see [10]). A sequence $\{x_n\}$ in C is said to be an *approximating sequence* of a nonexpansive semigroup $\mathscr{S} = \{T(t) : t \in S\}$ if

$$\lim_{n\to\infty} \|x_n - T(t) x_n\| = 0$$

for each $t \in S$ (for example, see [10]). We study the behavior of approximating sequences of nonexpansive semigroups (see [2]).

Theorem 3.3 ([2]). *Let E be a reflexive Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let S be a commutative semitopological semigroup with identity. Let $\mathscr{S} = \{T(t):t \in S\}$ be a nonexpansive semigroup on C. Assume that $I - T(t)$ is demiclosed at 0 for each $t \in S$. If $\mathscr{S} = \{T(t):t \in S\}$ has a unique common fixed point z_0 and z_0 lies on the boundary ∂C of C, then every approximating sequence $\{x_n\}$ of $\mathscr{S}$ converges strongly to z_0.*

We get convergence of orbits of nonexpansive semigroups with no common fixed points in the interior of their domains (see [2]).

Theorem 3.4 ([2]). *Let E be a uniformly convex Banach space and let C be a bounded closed and convex subset of E. Assume that C has nonempty interior and that it is locally uniformly rotund. Let S be a commutative semitopological semigroup with identity. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. If $\mathscr{S} = \{T(t) : t \in S\}$ has no common fixed point in the interior of C, then there exists a unique point z_0 on the boundary ∂C of C such that each orbit $\{T(t)x : t \in S\}$ converges strongly to z_0.*

Using theorems 3.3 and 3.4, we get some convergence theorems (see [10]).

Let C be a closed convex subset of a Banach space E. Then, a family $\mathscr{S} = \{T(s) : s \in \mathbb{R}^+\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(a) $T(s+t) = T(s)T(t)$ for all $s,t \in \mathbb{R}^+$;

(b) $T(0)x = x$ for each $x \in C$;

(c) $s \mapsto T(s)x$ is continuous;

(d) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in \mathbb{R}^+$

Using Theorem 3.3 and Lemma 3.2, we obtain the following convergence theorem (see also [2, 7]).

Theorem 3.5. *Let E be a uniformly convex Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let $\mathscr{S} = \{T(t):t \in S\}$ be a nonexpansive semigroup on C. Assume that $\mathscr{S} = \{T(t):t \in S\}$ has a unique common fixed point z_0 and that z_0 lies on the boundary ∂C of C. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$$

for each $s \in S$. Let $x \in C$ and let $\{z_n\}$ be the sequence defined by

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) T_{\mu_n} z_n \quad \text{for each} \quad n \in \mathbb{N}.$$

Then, $\{z_n\}$ converges strongly to z_0.

Using Theorem 3.3 and Lemma 3.2, we also obtain the following convergence theorem (see also [2, 28]).

Theorem 3.6. *Let E be a uniformly convex Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let $\mathscr{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Assume that $\mathscr{S} = \{T(t) : t \in S\}$ has a unique common fixed point z_0 and that z_0 lies on the boundary ∂C of C. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$$

for each $s \in S$. Let $u_0 = x \in C$ and let $\{u_n\}$ be the sequence defined by

$$u_n = \frac{1}{n} u_{n-1} + \left(1 - \frac{1}{n}\right) T_{\mu_n} u_n \quad \text{for each} \quad n \in \mathbb{N}.$$

Then, $\{u_n\}$ converges strongly to z_0.

Using theorem 3.4, we get the following theorems (see [2, 10]).

Theorem 3.7. *Let E be a uniformly convex Banach space and let C be a bounded closed and convex subset of E. Assume that C has nonempty interior and that it is locally uniformly rotund. Let T be a nonexpansive mapping of C into itself. If T has no fixed point in the interior of C, then there exists a unique point z_0 on the boundary ∂C of C such that each sequence $\{T^n x : n = 1, 2, 3, \dots\}$ converges strongly to z_0.*

Theorem 3.8. *Let E be a uniformly convex Banach space and let C be a bounded closed and convex subset of E. Assume that C has nonempty interior and that it is locally uniformly rotund. Let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C. If $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ has no common fixed point in the interior of C, then there exists a unique point z_0 on the boundary ∂C of C such that each orbit $\{T(t)x : t \in \mathbb{R}^+\}$ converges strongly to z_0.*

4 Weak convergence theorems

In this section, we establish a weak convergence theorem for two normally 2-generalized hybrid mappings. We also give a convergence theorem for a generic 2-generalized hybrid mapping.

Kohsaka and Takahashi [14], and Takahashi [26] introduced the following nonlinear mappings.

A mapping $T : C \to H$ is said to be nonspreading [14] if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is said to be hybrid [26] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings (see also [15, 27]). In general, nonspreading and hybrid mappings are not continuous mappings. Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced the class of λ-hybrid mappings in a Hilbert space. This class contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Kocourek, Takahashi and Yao [13] introduced a broader class of nonlinear mappings than the class of λ-hybrid mappings in a Hilbert space. A mapping $T : C \to E$ is said to be generalized hybrid [13] if there are real numbers α, β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Maruyama, Takahashi and Yao [19] introduced a broad class of nonlinear mappings called 2-generalized hybrid which contains generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H. A mapping $T : C \to C$ is said to be 2-generalized hybrid [19] if there exist real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. Kondo and Takahashi [16] introduced the following class of nonlinear mappings which covers 2-generalized hybrid mappings in a Hilbert space. A mapping $T : C \to C$ is said to be normally 2-generalized hybrid [16] if there exist real numbers $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0,$$

$$\alpha_2 + \alpha_1 + \alpha_0 > 0$$

and

$$\begin{aligned}\alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0\end{aligned}$$

for all $x, y \in C$. We call such a mapping T an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-normally 2-generalized hybrid mapping. We know that the class of $(1 - \alpha, -(1 - \beta), \alpha, -\beta, 0, 0)$-normally 2-generalized hybrid mappings is the class of generalized hybrid mappings. Now, we get the following theorems (see [3]).

Theorem 4.1. *Let C be a nonempty and convex subset of a Hilbert space H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for $a, b \in (0, 1)$ with $a \leq b$ and let $\{\beta_n\}$ be a sequence of real numbers such that $0 < c \leq \beta_n \leq d < 1$ for $c, d \in (0, 1)$ with $c \leq d$. Suppose*

$x_1 = x \in C$ and $\{x_n\}$ is given by

$$\begin{cases} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)\dfrac{1}{n}\sum_{k=0}^{n-1} S^k y_n, \\ y_n &= \beta_n x_n + (1-\beta_n)\dfrac{1}{n}\sum_{l=0}^{n-1} T^l x_n, \end{cases}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a common attractive point $z \in A(S) \cap A(T)$, where $z = \lim_{n\to\infty} P_A x_n$. Additionally, if C is closed, then x_n converges weakly to $\bar{z} = \lim_{n\to\infty} P_F x_n$, where P_F is the metric projection from H onto $F(S) \cap F(T)$.

Recently, Kondo and Takahashi [17] introduced a broad class of mappings. A mapping $T : C \to C$ is called generic 2-generalized hybrid if there exist if $\alpha_{i,j}, \beta_i, \gamma_i \in \mathbb{R}(i, j = 0, 1, 2)$ such that

$$\begin{aligned} &\alpha_{00}\|x-y\|^2 + \alpha_{01}\|x-Ty\|^2 + \alpha_{02}\|x-T^2y\|^2 \\ &+\alpha_{10}\|Tx-y\|^2 + \alpha_{11}\|Tx-Ty\|^2 + \alpha_{12}\|Tx-T^2y\|^2 \\ &+\alpha_{20}\|T^2x-y\|^2 + \alpha_{21}\|T^2x-Ty\|^2 + \alpha_{22}\|T^2x-T^2y\|^2 \\ &+\beta_0\|x-Tx\|^2 + \beta_1\|Tx-T^2x\|^2 + \beta_2\|T^2x-x\|^2 \\ &+\gamma_0\|y-Ty\|^2 + \gamma_1\|Ty-T^2y\|^2 + \gamma_2\|T^2y-y\|^2 \le 0 \end{aligned}$$

for all $x, y \in C$. For Theorem 4.3, we will assume that T satisfies (4.1a) or (4.1b);

$$\begin{aligned} &\alpha_{0k} + \alpha_{1k} \ge 0, \alpha_{20}, \alpha_{21}, \alpha_{22} \ge 0, \alpha_{1k} > 0, \\ &\qquad \beta_0, \beta_1, \beta_2 \ge 0, \gamma_0 + \gamma_1 \ge 0, \gamma_2 \ge 0; \end{aligned} \tag{4.1a}$$

$$\begin{aligned} &\alpha_{k0} + \alpha_{k1} \ge 0, \alpha_{02}, \alpha_{12}, \alpha_{22} \ge 0, \alpha_{k1} > 0, \\ &\qquad \beta_0 + \beta_1 \ge 0, \beta_2 \ge 0, \gamma_0, \gamma_1, \gamma_2 \ge 0, \end{aligned} \tag{4.1b}$$

where

$$\alpha_{ik} = \alpha_{i0} + \alpha_{i1} + \alpha_{i2} \text{ and } \alpha_{ki} = \alpha_{0i} + \alpha_{1i} + \alpha_{2i} \tag{4.1c}$$

for $i = 0, 1, 2$. The class of generic 2-generalized hybrid mappings that satisfies (4.1a) or (4.1b) contains nonexpansive mappings, generalized hybrid mappings and normally 2-generalized hybrid mappings as special cases.

Theorem 4.2 ([17]). *Let C be a nonempty subset of a Hilbert space H. Let T be a generic 2-generalized hybrid mappings such that $F(T) \neq \emptyset$. Suppose that T satisfies one of the following conditions :*

$$\alpha_{0k} + \alpha_{1k} \ge 0, \alpha_{2k} \ge 0, \alpha_{1k} > 0, \beta_0, \beta_1, \beta_2 \ge 0; \tag{4.2a}$$

$$\alpha_{k0} + \alpha_{k1} \ge 0, \alpha_{k2} \ge 0, \alpha_{k1} > 0, \gamma_0, \gamma_1, \gamma_2 \ge 0, \tag{4.2b}$$

where α_{ki} and $\alpha_{ik}(i = 0, 1, 2)$ are defined in (4.1c) . Then, T is quasi-nonexpansive.

We get the following theorem (see [3]).

Theorem 4.3. *Let C be a nonempty closed and convex subset of a Hilbert space H. Let S and T be commutative generic 2-generalized hybrid mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Suppose that S and T satisfy (4.1a) or (4.1b). Let P be the metric projection from H onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for $a, b \in (0,1)$ with $a \leq b$ and let $\{\beta_n\}$ be a sequence of real numbers such that $0 < c \leq \beta_n \leq d < 1$ for $c, d \in (0,1)$ with $c \leq d$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} x_{n+1} & = & \alpha_n x_n + (1-\alpha_n)\dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} S^k y_n, \\ y_n & = & \beta_n x_n + (1-\beta_n)\dfrac{1}{n}\displaystyle\sum_{l=0}^{n-1} T^l x_n, \end{cases}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a common fixed point $z \in F(S) \cap F(T)$, where $z = \lim_{n\to\infty} Px_n$.

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