Study on a Relaxation for Theorems of the Alternative for Sets

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Abstract. In the literature, characterization for set relations are of various application to optimality conditions of set optimization problems, variational principles for set-valued maps, theorems of the alternative, certain robustness of vector optimization problems, and so on. In this paper, the author presents properties of scalarization functions as dual expression of set relations. Comparing to existing results, one can confirm their uniqueness in their relaxed conditions using convex cone-compactness and closedness. Also, we show the results implies generalized Gordan's theorems of the alternative at the last part of the thesis.

1. Introduction

Set relation, that is one of main factors in set optimization and interestingly targeted by researchers for finding criteria of set-to-set comparisons, is defined to be a binary relation between sets. Nishnianidze ([22]) and Young ([28]) introduced multi-valued comparisons and Kuroiwa et.al. ([18]) proposed practical six types as set relations. Today, we have many kinds of relations for sets as set relations (e.g., [5]) based on the six types. However, set relations usually focus on infinitely many elements in given sets so that it is difficult to figure out which set is preferred to the other in a practical period of time. Here, we have a reason to consider set functions called scalarization functions.

In 1983, Tammer introduced this concept by a type of Minkowski functional with a given convex cone in [6] and properties of the functional have been researched in papers (e.g., [7, 8, 17]). The function gives nonlinear level sets that enable to separate not necessarily convex sets in a topological vector space. As a generalization of the functional, Hamel ([1]) and Kuwano et.al. ([19]) introduced set scalarization functions and their properties (e.g., convexity, continuity). Their results show the set scalarization functions play important roles as dual expressions for set relations in deciding which one is better than (preferred to) the other set in set optimization (e.g., [3, 4, 19, 20]) by nonlinearly separating the sets.

2020 *Mathematics Subject Classification.* 90C46, 90C29, 49J53.

Key words and phrases. Set optimization, set relation, nonlinear scalarization, theorem of the alternative.

In recent study, the scalarization functions are used in set-valued Gordan's type theorems of the alternative which is applicable to robustness of multi-valued optimization problems ([26]). Moreover, the calculation process of their values in a finite dimensional space by computer has been investigated ([29, 30]) if given sets and an ordering cone are convex polyhedra. We should referred to the existence of oriented distance functions that work similarly to the scalarization functions just as the other side of the coin in normed spaces (e.g., [2, 10, 11, 14–16]).

This thesis is to have a relaxation for criteria of theorems of the alternative given in [26] by using convex conical conditions. Firstly in Section 2, we recall basic notions of set relations and scalarization functions. As a main part of this , we propose relaxed theorems of the alternative for sets in Section 3. Section 4 presents some application of this thesis to Gordan's type theorems of the alternative for set-valued maps.

2. Preliminaries

Throughout this paper, we let X be a topological vector space, C a convex solid (i.e., $\mathrm{int}C \neq \emptyset$) cone in X. Also, $\leq_C$ is the pointwise ordering between two vectors in X ($x \leq_C y \;\; \fallingdotseq \;\; y - x \in C$ for $x, y \in X$) and $\preceq_C$ is a binary relation between two subsets of X.

Let us introduce convex conical properties as weak ones of compactness and closedness. S is C-closed if $S + C$ is closed. S is C-compact if any cover of S of the form of $\{U_\lambda + C \mid U_\lambda \text{ are open}\}$ admits a finite subcover. We easily confirm that C-compactness leads to C-closedness.

Next, we would like to show canonical six set relations.

Definition 2.1 (set relations ([18])). Let $S_1, S_2 \subset X$ be nonempty sets.

- $S_1 \preceq_C^{(1)} S_2 \overset{\text{def}}{\iff} S_1 \subset \bigcap_{s \in S_2}(s - C)$;
- $S_1 \preceq_C^{(2)} S_2 \overset{\text{def}}{\iff} S_1 \cap \bigcap_{s \in S_2}(s - C) \neq \emptyset$;
- $S_1 \preceq_C^{(3)} S_2 \overset{\text{def}}{\iff} S_2 \subset S_1 + C$;
- $S_1 \preceq_C^{(4)} S_2 \overset{\text{def}}{\iff} S_1 \cap \bigcap_{s \in S_1}(s + C) \neq \emptyset$;
- $S_1 \preceq_C^{(5)} S_2 \overset{\text{def}}{\iff} S_1 \subset S_2 - C$;
- $S_1 \preceq_C^{(6)} S_2 \overset{\text{def}}{\iff} S_2 \cap (S_1 + C) \neq \emptyset$.

Unless otherwise described, the relation $\preceq_C^{(i)}$ is denoted by "relation" (i) or just (i). Definition 2.1 can be expressed by the pointwise ordering that holds for any or some elements as follows.

Lemma 2.1. For $S_1, S_2 \subset X \setminus \{\emptyset\}$, it holds that

- $S_1 \preceq_C^{(1)} S_2 \iff \forall s_1 \in S_1,\ \forall s_2 \in S_2,\ s_1 \leq_C s_2$;
- $S_1 \preceq_C^{(2)} S_2 \iff \exists s_1 \in S_1,\ \forall s_2 \in S_2,\ s_1 \leq_C s_2$;

- $S_1 \preceq_C^{(3)} S_2 \Longleftrightarrow \forall s_2 \in S_2,\ \exists s_1 \in S_1,\ s_1 \leq_C s_2;$
- $S_1 \preceq_C^{(4)} S_2 \Longleftrightarrow \exists s_2 \in S_2,\ \forall s_1 \in S_1,\ s_1 \leq_C s_2;$
- $S_1 \preceq_C^{(5)} S_2 \Longleftrightarrow \forall s_1 \in S_1,\ \exists s_2 \in S_2,\ s_1 \leq_C s_2;$
- $S_1 \preceq_C^{(6)} S_2 \Longleftrightarrow \exists s_1 \in S_1,\ \exists s_2 \in S_2,\ s_1 \leq_C s_2.$

Note that relation (1) implies both (2) and (4) that show (3) and (5), respectively. Relation (6) is the weakest one which follows from any other one. Moreover, we remark that type (3) and (5) are preorders when $\mathbf{0} \in C$, that is, they are reflexive and transitive. These relations especially get wider attentions in papers as "set less order relations" (see [24] and the references cited therein) than the others. One should see [19] for detailed properties of them.

Definition 2.2 (scalarization functions ([26])). Let $S_1, S_2 \subset X$ be nonempty sets and $k \in \mathrm{int}C$. For $i = 1, \ldots, 6$, scalarization functions $Z_{C,k}^{(i)} : 2^X \to \mathbb{R} \cup \{\infty\}$ is defined by

$$Z_{C,k}^{(i)}(S_1, S_2) = \inf\{t \in \mathbb{R} \mid S_1 \preceq_C^{(i)} S_2 + tk\}.$$

One can easily check that $Z_{C,k}^{(i)}(S_1, S_2) \in \mathbb{R}$ under several conditions of S_1, S_2 (see [26] and the references cited therein). These functions are generalization of conical nonlinear functional in [6–8]. Also, if S_1, S_2, C are convex polyhedra, these functions are calculable by finitely many steps ([29, 30]).

Proposition 2.1 ([24]). Let $S_1, S_2 \in 2^X$. Then,

$$S_1 \preceq_{\mathrm{cl}C}^{(1)} S_2 \Longleftrightarrow Z_{C,k}^{(1)}(S_1, S_2) \leq 0.$$

Proposition 2.2 ([24]). Let $S_1, S_2 \in 2^X$. If S_1 is compact, then

$$S_1 \preceq_{\mathrm{cl}C}^{(2)} S_2 \Longleftrightarrow Z_{C,k}^{(2)}(S_1, S_2) \leq 0,$$

$$S_1 \preceq_{\mathrm{cl}C}^{(3)} S_2 \Longleftrightarrow Z_{C,k}^{(3)}(S_1, S_2) \leq 0.$$

Proposition 2.3 ([24]). Let $S_1, S_2 \in 2^X$. If S_2 is compact, then

$$S_1 \preceq_{\mathrm{cl}C}^{(4)} S_2 \Longleftrightarrow Z_{C,k}^{(4)}(S_1, S_2) \leq 0,$$

$$S_1 \preceq_{\mathrm{cl}C}^{(5)} S_2 \Longleftrightarrow Z_{C,k}^{(5)}(S_1, S_2) \leq 0.$$

Proposition 2.4 ([24]). Let $S_1, S_2 \in 2^X$. If S_2 and S_2 is compact, then

$$S_1 \preceq_{\mathrm{cl}C}^{(6)} S_2 \Longleftrightarrow Z_{C,k}^{(6)}(S_1, S_2) \leq 0.$$

Propositions 2.1–2.4 are called alternative theorems for sets in [26] and some examples of them and cases with a lack of compactness are depicted in [24]. The original forms are in [1, 23] (as far as the author knows) and [23] discovered that set-to-set comparisons can be done by calculating the value of scalarization functions: set-to-set comparisons are characterized by scalers, and thus the authors in [24, 26] use the term "characterization." Similar results are given in the form of oriented distance (e.g., see [2, 14–16]) that stands with a solid convex closed cone C in a normed space.

3. Relaxed theorems of the alternative

The first implication for the values of the scalarization functions by the set relations is naturally give by [24] as follows.

Proposition 3.1 ([24])**.** Let $S_1, S_2 \in 2^X$ and $k \in \mathrm{int}C$. Then, it holds that

$$S_1 \preceq^{(i)}_{\mathrm{cl}C} S_2 \Longrightarrow Z^{(i)}_{C,k}(S_1, S_2) \leq 0, \quad \forall i = 1, \ldots, 6.$$

However, this thesis realizes the converse implication being true with relaxed conditions.

Proposition 3.2. Let $S_1, S_2 \in 2^X$ and $k \in \mathrm{int}C$. Then, it holds that

$$Z^{(1)}_{C,k}(S_1, S_2) \leq 0 \Longrightarrow S_1 \preceq^{(1)}_{\mathrm{cl}C} S_2.$$

The above proposition is true by Theorem 4.2 in [24].

Theorem 3.1. Let $S_1, S_2 \in 2^X$ and $k \in \mathrm{int}C$. if S_1 is C-compact, then

$$Z^{(2)}_{C,k}(S_1, S_2) \leq 0 \Longrightarrow S_1 \preceq^{(2)}_{\mathrm{cl}C} S_2.$$

Proof. We assume it holds that $S_1 \not\preceq^{(2)}_{\mathrm{cl}C} S_2$. Then, we have $S_1 \subset \left(\bigcap_{s\in S_2}(s - \mathrm{cl}C)\right)^c$. Since $\left(\bigcap_{s\in S_2}(s - \mathrm{cl}C)\right)^c$ is open, then for all $\tilde{s} \in S_1$, there exists $t_{\tilde{s}}$ such that $\tilde{s} - t_{\tilde{s}}k \in (\bigcap_{s\in S_2}(s - \mathrm{cl}C))^c$. Thus, $\left\{\left(\bigcap_{s\in S_2}(s - \mathrm{cl}C)\right)^c + t_{\tilde{s}}k\right\}_{\tilde{s}\in S_1} = \left\{\left(\bigcap_{s\in S_2}(s - \mathrm{cl}C)\right)^c + t_{\tilde{s}}k + C\right\}_{\tilde{s}\in S_1}$ is an open cover of S_1. Since S_1 is C-compact, there exists $x_1, \ldots, x_n \in S_1$ such that $S_1 \subset \bigcup_{i=1}^n \left(\left(\bigcap_{s\in S_2}(s - \mathrm{cl}C)\right)^c + t_{\tilde{s}}k\right)$. This implies that we have $Z^{(2)}_{C,k}(S_1, S_2) \geq \bar{t} > 0$ for $\bar{t} := \min\{t_{x_1}, \ldots, t_{x_n}\}$. □

Proposition 3.3. Let $S_1, S_2 \in 2^X$ and $k \in \mathrm{int}C$. If S_1 is C-closed, then

$$Z^{(3)}_{C,k}(S_1, S_2) \leq 0 \Longrightarrow S_1 \preceq^{(3)}_{\mathrm{cl}C} S_2.$$

Proof. We have $tk + S_2 \subset S_1 + \mathrm{cl}C$ for all $t > 0$. Thus, $S_2 \subset \bigcap_{x\in -k\mathbb{R}_+}(x + S_1 + \mathrm{cl}C) = \mathrm{cl}(S_1 + \mathrm{cl}C) = S_1 + \mathrm{cl}C$ since S_1 is C-closed and free-disposal with respect to C. □

Proposition 3.4. Let $S_1, S_2 \in 2^X$ and $k \in \text{int}C$. If S_2 is $(-C)$-closed, then

$$Z^{(5)}_{C,k}(S_1, S_2) \leq 0 \Longrightarrow S_1 \preceq^{(5)}_{\text{cl}C} S_2.$$

Proposition 3.4 directly follows from Proposition 3.3 by replacing S_1,S_2, and C with S_2,S_1, and $-C$, respectively.

Proposition 3.3, 3.4 has been also studied in different ways or forms (e.g., [7, 12]). The above proof is a relatively (and expectedly) more simple one.

Theorem 3.2. Let $S_1, S_2 \in 2^X$ and $k \in \text{int}C$. If S_2 is $(-C)$-compact, then

$$Z^{(4)}_{C,k}(S_1, S_2) \leq 0 \Longrightarrow S_1 \preceq^{(4)}_{\text{cl}C} S_2.$$

Proof. We assume it holds that $S_1 \not\preceq^{(4)}_{\text{cl}C} S_2$. Then, we have $S_2 \subset \left(\bigcap_{s\in S_1}(s + \text{cl}C)\right)^c$. Since $\left(\bigcap_{s\in S_1}(s + \text{cl}C)\right)^c$ is open, then for all $\tilde{s} \in S_2$, there exists $t_{\tilde{s}}$ such that $\tilde{s}+t_{\tilde{s}}k \in (\bigcap_{s\in S_1}(s+\text{cl}C))^c$. Thus, $\left\{\left(\bigcap_{s\in S_1}(s + \text{cl}C)\right)^c - t_{\tilde{s}}k\right\}_{\tilde{s}\in S_2} = \left\{\left(\bigcap_{s\in S_1}(s + \text{cl}C)\right)^c - t_{\tilde{s}}k - C\right\}_{\tilde{s}\in S_1}$ is an open cover of S_2. Since S_2 is $(-C)$-compact, there exists $x_1, \ldots, x_n \in S_2$ such that $S_2 \subset \bigcup_{i=1}^n \left(\left(\bigcap_{s\in S_1}(s + \text{cl}C)\right)^c - t_{\tilde{s}}k\right)$. This implies that we have $Z^{(4)}_{C,k}(S_1, S_2) \geq \bar{t} > 0$ for $\bar{t} := \max\{t_{x_1}, \ldots, t_{x_n}\}$. □

Theorem 3.3. Let $S_1, S_2 \in 2^X$ and $k \in \text{int}C$. If S_1 is C-closed and S_2 is $(-C)$-compact, or S_1 is C-compact and S_2 is $(-C)$-closed, then

$$Z^{(6)}_{C,k}(S_1, S_2) \leq 0 \Longrightarrow S_1 \preceq^{(6)}_{\text{cl}C} S_2.$$

Proof. When $i = 6$, we assume $S_2 \subset (S_1 + \text{cl}C)^c$. We need to note that $(S_1 + \text{cl}C)^c$ is an open set since S_1 is C-closed. For all $s \in S_2$, there exists $t_s > 0$ such that $s - t_s k \in (S_1 + \text{cl}K)^c$. Thus, $S_2 \subset \bigcup_{s\in S_2}((S_1 + \text{cl}C)^c + t_s k)$. Since $(S_1 + \text{cl}C)^c + t_s k$ is free-disposal with respect to $-C$ for all $s \in S_2$, it holds that $(S_1+\text{cl}C)^c+t_s k = (S_1+\text{cl}C)^c+t_s k-C$. □

4. Application

In the previous section, the scalarization functions perform as characterization of the set relations. Ogata et.al. ([25, 26]) suggested this kind of characterization can be applied to have generalized theorems of the alternative. To begin with, let us see Gordan's theorem.

Proposition 4.1 (Gordan's theorem of the alternative ([9])**.** Let A be an $m \times n$ matrix. Then exactly one of the followings is consistent:

(i) There exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} < \boldsymbol{0}$;
(ii) There exists $\boldsymbol{y} \in \mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}$ such that $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

The above proposition implies a duality for negativity of linear functions and easily gives the duality of linear programming problems. From this point of view, one might see some generalized forms of Proposition 4.1 in [13, 21, 23, 27] by replacing a linear function with vector-valued or set-valued maps. These results use linear separation to identify the existence of negative values of a given function so that convexity (or convexlikeness) is required in some parts. However, [1, 23] study an interesting idea by using the scalarization functions. In the paper, the authors succeeded to remove any conditions related to convexity. The following generalized Gordan's theorem is a relaxed version of one given in [24].

Theorem 4.1. Let S be a nonempty set, $F : S \to 2^X \backslash \{\mathbf{0}\}$, $V \in 2^X \backslash \{\mathbf{0}\}$. If

- F is C-compact-valued in case of $i = 2$;
- F is C-closed-valued in case of $i = 3$;
- V is $(-C)$-compact-valued in case of $i = 4$;
- V is $(-C)$-closed in case of $i = 5$;
- F is C-closed-valued and V is $(-C)$-compact, or F is C-compact-valued and V is $(-C)$-closed in case of $i = 6$,

then exactly one of the followings is consistent for $i = 1, \ldots, 6$:

(i) there exists $s \in S$ such that $F(s) \preceq^{(i)}_{\mathrm{cl}C} V$;

(ii) there exists $k \in \mathrm{int}C$ such that $Z^{(i)}_{C,k}(F(s), V) > 0$ for all $s \in S$.

Acknowledgment

This work was supported by Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Number JP21K13842 (Grant-in-Aid for Early-Career Scientists).

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