# SOLUTIONS FOR A FRACTIONAL-ORDER DIFFERENTIAL EQUATION WITH BOUNDARY CONDITIONS

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## 1. Introduction

In [5], the authors considered the fractional-order boundary value problem, we consider existence and uniqueness of solutions of the fractional-order boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{1.1}$$

Inspiring the result of Ma and Silva [4], we consider the problem (1.2). The problem (1.1) is the case that $g \equiv 0$ of the problem (1.2). Due to the restriction of $g$, results in [5] cannot deal with the problem (1.3). where $3 < \alpha \leq 4$, $f$ is a continuous function of $[0, 1] \times \mathbb{R}$ into $\mathbb{R}$, $g$ is a function of $\mathbb{R}$ into itself and $D^{\alpha}_{0+}$ denotes the Riemann-Liouville fractional derivative of order $\alpha$ which is defined in Section 2. A function $u \in C[0, 1]$, where $C[0, 1]$ is the set of all continuous functions of $[0, 1]$ into $\mathbb{R}$, is called a solution of the problem (1.1) if $D^{\alpha}_{0+}u \in C[0, 1]$ and $u$ satisfies (1.1).

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)), \end{cases} \tag{1.2}$$

A function $u \in C[0, 1]$, where $C[0, 1]$ is the set of all continuous functions of $[0, 1]$ into $\mathbb{R}$, is called a solution of the problem (1.2) if $D^{\alpha}_{0+}u \in C[0, 1]$ and $u$ satisfies (1.2).

When $\alpha = 4$, the problem (1.1) is the boundary value problems for cantilever beam equations and the problem (1.2) is the following fourth-order boundary value problem

$$\begin{cases} u''''(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)). \end{cases} \tag{1.3}$$

In the problem (1.3), since $u''$ represents the shear force at $t = 1$, the sonditions $u''(1) = 0, u'''(1) = g(u(1))$ means that the vertical force is equal to , which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u$. Furthermore, since $u''(1) = 0$ indicates that there is no bending moment at $t = 1$, the beam is resting on the bearing $g$. Existence and iterative schemes to solve the problem (1.3) were studied by Ma and da Silva [4]. The purpose of this article is to follow the results of the fractional-order problems (1.1) and (1.2).

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## 2. Preliminaries

In this section, we introduce preliminary facts. Especially, we construct the Green function $G(s,t)$ for the boundary value problems (1.1) and (1.2), and we discuss some properties of the function.

We start with the definition of the Riemann-Liouville fractional integral and fractional derivative. Let $\alpha > 0$ and $u$ be a continuous function of $[0,1]$ into $\mathbb{R}$. The Riemann-Liouville fractional integral of order $\alpha$ of $u$, denoted $I^{\alpha}_{0+}u$, is defined by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$$

for $0 \le t \le 1$. The Riemann-Liouville fractional derivative of order $\alpha$ of $u$, denoted $D^{\alpha}_{0+}u$, is defined by

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds$$

for $0 \le t \le 1$, where $n$ denotes a positive integer such that $n-1 < \alpha \le n$. For $\alpha \ge 0$ and $\beta > -1$, we have

$$D^{\alpha}_{0+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha};$$

see [3]. For the case of (1.2), let $h$ be a continuous mapping of $[0,1]$ into $\mathbb{R}$. Let $3 < \alpha \le 4$. Then the unique solution of the boundary value problem is

$$u(t) = \int_0^1 G(t,s)h(s)ds,$$

where

(2.1)

$$G(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\big((t-s)^{\alpha-1} + t^{\alpha-1}(1-s)^{\alpha-4}(4s-\alpha s-1) + (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s\big) & (0 \le s \le t < 1), \\ \dfrac{1}{\Gamma(\alpha)}\big(t^{\alpha-1}(1-s)^{\alpha-4}(4s-\alpha s-1) + (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-4}s\big) & (0 \le t \le s < 1); \end{cases}$$

see [5]. Also for the case of (1.2)

$$\begin{cases} D^{\alpha}_{0+}u(t) = h(t), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = \gamma \end{cases}$$

if and only if $u$ is a solution of the integral equation

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\gamma t^{\alpha-1}}{(\alpha-1)(\alpha-2)} - \frac{\gamma t^{\alpha-2}}{(\alpha-2)(\alpha-3)} \tag{2.2}$$

for $0 \le t \le 1$, where $G(t,s)$ is defined by (2.1); see [6].

## 3. Main results

In this section, we consider the boundary value problems (1.1) and (1.2). By the Banach fixed point theorem, we obtain a sufficient condition for uniqueness and existence of solutions of the problems.

**Theorem 1.** *Let $3 < \alpha \le 4$. Let $f$ be a continuous function of $[0,1] \times \mathbb{R}$ into $\mathbb{R}$. Let $g$ be a Lipschitz continuous function of $\mathbb{R}$ into itself with a nonnegative constant $L$. Assume that there exists a nonnegative constant $\lambda$ with*

$$\lambda\Lambda + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)} < 1$$

*such that for any $0 \le t \le 1$ and $u_1, u_2 \in \mathbb{R}$,*

$$|f(t,u_1) - f(t,u_2)| \le \lambda |u_1 - u_2|,$$

*where $\Lambda$ is the constant*

$$\Lambda = \sup_{0 \le t \le 1} \int_0^1 |G(t,s)| ds.$$

*Then the boundary value problem (1.2) has a unique solution.*

*Proof.* See [6, Theorem 1]. □

For the case that $g = 0$ in Theorem 1, we have the following; see [5, Theorem 3.1].

**Corollary 2.** *Let $3 < \alpha \le 4$. Let $f$ be a continuous mapping of $[0,1] \times \mathbb{R}$ into $\mathbb{R}$. Assume that there exists $\lambda \in \left[0, \frac{1}{\Lambda}\right)$ such that for any $u, v \in [0,\infty)$ and $t \in [0,1]$,*

$$|f(t,u) - f(t,v)| \le \lambda |u - v|,$$

*where*

$$\Lambda = \sup_{0 \le t \le 1} \int_0^1 |G(t,s)| ds$$

*and $G$ is the function given by (2.1). Then the boundary value problem represented by (1.1) has a unique solution.*

For the case that $\alpha = 4$ in Theorem 1, we have the following; see Theorem 1 in [4].

**Corollary 3.** *Let $f$ be a continuous function of $[0,1] \times \mathbb{R}$ into $\mathbb{R}$ with bounded partial derivative with respect to the second variable. Let $g$ be a Lipschitz continuous function of $\mathbb{R}$ into itself with a nonnegative constant $L$. Let*

$$\lambda = \max_{(t,u) \in [0,1] \times \mathbb{R}} \left| \frac{\partial f}{\partial u}(t,u) \right|.$$

*If*

$$\frac{\lambda}{8} + \frac{L}{3} < 1,$$

*then the boundary value problem (1.3) has a unique solution.*

*Proof.* By the mean value theorem, we have for any $0 \le t \le 1$ and $u_1, u_2 \in \mathbb{R}$,

$$|f(t,u_1) - f(t,u_2)| \le \lambda |u_1 - u_2|.$$

In the case that $\alpha = 4$, the function $G(t,s)$ reduces to

$$G(t,s) = \begin{cases} \frac{1}{6}s^2(3t-s) & (s < t), \\ \frac{1}{6}t^2(3s-t) & (t \le s). \end{cases}$$

Since

$$\Lambda = \sup_{0 \le t \le 1} \int_0^1 |G(t,s)| ds = \frac{1}{8},$$

we obtain the conclusion by Theorem 1. □

**Remark 4.** Let $3 < \alpha \le 4$. Since the function $G(t,s)$ satisfies

$$\int_0^1 |G(t,s)| ds \le \frac{1}{\Gamma(\alpha)} \left( \frac{1}{\alpha} + \frac{\alpha}{\alpha - 3} \right) \tag{3.1}$$

for all $0 \le t \le 1$, $\sup_{0 \le t \le 1} \int_0^1 |G(t,s)| ds$ is finite. The function $G(t,s)$ satisfies

$$l(t,s) \le G(t,s) \le m(t,s)$$

for $0 \le t \le 1$ and $0 \le s < 1$. where

$$l(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4}(2s + st - t) & (s < t), \\ \dfrac{\alpha-2}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4} s & (t \le s) \end{cases}$$

and

$$m(t,s) = \begin{cases} \dfrac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4} s & (s < t), \\ \dfrac{3}{\Gamma(\alpha)} t^{\alpha-2}(1-s)^{\alpha-4} s & (t \le s); \end{cases}$$

see [6].

To conclude the paper, we present an example demonstrating an application of Theorem 1.

**Example 5.** Let us consider the boundary value problem

$$\begin{cases} D_{0+}^{3.1} u(t) = \frac{3}{(54e^t + 1)(1 + |u(t)|)}, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \; u'''(1) = g(u(1)), \end{cases} \tag{3.2}$$

where

$$g(t) = \frac{1}{100} \sin t.$$

By (3.1), the constant $\Lambda$ in Theorem 1 satisfies

$$\Lambda \le \frac{1}{\Gamma(\alpha)} \left( \frac{1}{\alpha} + \frac{\alpha}{\alpha - 3} \right) = 14.2530 \cdots < 15.$$

Moreover we have, for any $0 \le t \le 1$ and $u_1, u_2 \in \mathbb{R}$,

$$|f(t,u_1) - f(t,u_2)| \le \frac{3}{55} |u_1 - u_2|,$$

5 where

$$f(t,u) = \frac{3}{(54e^t + 1)(1 + |u|)}$$

for $0 \leq t \leq 1$ and $u \in \mathbb{R}$; see Section 4 in [2]. Since the constants $\lambda = \frac{3}{55}$ and $L = \frac{1}{100}$ in Theorem 1, we have

$$\lambda\Lambda + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)} \leq \frac{3}{55} \times 15 + \frac{2 \times \frac{1}{100}}{2.1 \times 1.1 \times 0.1} = 0.\dot{9}0476\dot{1} < 1.$$

It follows from Theorem 1 that the problem (3.2) has a unique solution.

**Example 6.** We also consider the following.

$$\begin{cases} D_{0+}^{3.1}u(t) = \frac{1}{50e^t(1+u^2)}, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)), \end{cases} \tag{3.3}$$

where

$$g(t) = \frac{1}{100}\sin t.$$

For any $0 \leq t \leq 1$ and $u_1, u_2 \in \mathbb{R}$,

$$|f(t,u_1) - f(t,u_2)| \leq \frac{3}{50}|u_1 - u_2|,$$

where

$$f(t,u) = \frac{3}{50e^t(1+u^2)}$$

for $0 \leq t \leq 1$ and $u \in \mathbb{R}$. In this case we also have

$$\lambda\Lambda + \frac{2L}{(\alpha-1)(\alpha-2)(\alpha-3)} \leq \frac{3}{50} \times 15 + \frac{2 \times \frac{1}{100}}{2.1 \times 1.1 \times 0.1} \approx 0.9866 < 1.$$

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