DIFFERENTIAL OPERATORS AND THE DOUBLING ARCHIMEDEAN ZETA INTEGRALS

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ABSTRACT. In this survey, we explain Shimura's theory in [Shi90] on the differential operators and the Lie algebra action on the automorphic forms. We also explain how his theory is used in choosing the archimedean sections for constructing p-adic L-functions through the doubling method, and how the corresponding archimedean zeta integrals can be computed.

CONTENTS

1. THE MAASS–SHIMURA DIFFERENTIAL OPERATORS

We recall Shimura's theory on differential operators for symplectic and unitary groups. Following [Shi90], we recall the definition of the Maass–Shimura differential operators and the proof of their equivalence to the Lie algebra action. The Maass–Shimura differential operators are defined on the symmetric domain and can be interpreted as the Gauss–Manin connection, so they are very useful for arithmetic applications. On the other hand, the Lie algebra action is very convenient for applying representation theory. Therefore, the equivalence of the two is very useful in the study of algebraicity and p-adic properties of critical L-values.

Besides Shimura, with the motivation of studying critical L-values, the differential operators which increase the weights of automorphic forms have been studied in many works, *e.g.* [Har85, Har86, Böc85a, BD13, Ibu99, Eis12, Urb14, Liu19b, Ich15, EFMV18, AI17] from different point of views.

1.1. **Some notation and definitions.**

Let $J_{n,n} = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}$. We denote by G one of the following real Lie groups.

$$\begin{aligned}
\mathrm{Sp}(2n) &= \left\{ g \in \mathrm{GL}(2n, \mathbb{R}) \,:\, {}^{\mathrm{t}}g J_{n,n} g = J_{n,n} \right\}, \\
\mathrm{U}(J_{n,n}) &= \left\{ g \in \mathrm{GL}(2n, \mathbb{C}) \,:\, {}^{\mathrm{t}}\bar{g} J_{n,n} g = J_{n,n} \right\}, \\
\mathrm{U}(p,q) &= \left\{ g \in \mathrm{GL}(p+q, \mathbb{C}) \,:\, {}^{\mathrm{t}}\bar{g} \begin{pmatrix} \mathbf{1}_p & \\ & -\mathbf{1}_q \end{pmatrix} \right\}.
\end{aligned}$$

When $q=0$, we also write $\mathrm{U}(p,0)$ as $\mathrm{U}(p)$. Let

$$K_{\mathrm{Sp}(2n)} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,:\, ai+b \in \mathrm{U}(n) \right\},$$
$$K_{\mathrm{U}(J_{n,n})} = \left\{ \begin{pmatrix} \frac{a+b}{2} & \frac{a-b}{2i} \\ -\frac{a-b}{2i} & \frac{a+b}{2} \end{pmatrix} \,:\, a,b \in \mathrm{U}(n) \right\}$$
$$K_{\mathrm{U}(p,q)} = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \,:\, a \in \mathrm{U}(p),\, b \in \mathrm{U}(q) \right\}.$$

The K_G is a maximal compact subgroup of G. The symmetric domain of G is defined as

$$\mathcal{H}_{\mathrm{Sp}(2n)} = \left\{ z \in M_{n,n}(\mathbb{C}) \,:\, {}^{\mathrm{t}}z = z,\, i(\overline{z}-z) > 0 \right\},$$
$$\mathcal{H}_{\mathrm{U}(J_{n,n})} = \left\{ z \in M_{n,n}(\mathbb{C}) \,:\, i({}^{\mathrm{t}}\overline{z}-z) > 0 \right\},$$
$$\mathcal{H}_{\mathrm{U}(p,q)} = \left\{ z \in M_{p,q}(\mathbb{C}) \,:\, \mathbf{1}_q - {}^{\mathrm{t}}\overline{z}z > 0 \right\}.$$

The group G acts on $\mathcal{H}_G$ by

$$G \times \mathcal{H}_G \longrightarrow \mathcal{H}_G \tag{1.1.0}$$
$$\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \longmapsto g \cdot z = (az+b)(cz+d)^{-1},$$

and this action factors through K_G.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathcal{H}_G$, define the automorphy factors

$$\mu_G(g,z) = cz+d, \qquad \lambda_G(g,z) = \begin{cases} \overline{c}{}^{\mathrm{t}}z + \overline{d}, & G = \mathrm{Sp}(2n),\, \mathrm{U}(J_{n,n}), \\ \overline{a} + \overline{b}{}^{\mathrm{t}}z, & G = \mathrm{U}(p,q). \end{cases}, \tag{1.1.1}$$

and

$$\Lambda_G(g,z) = \begin{cases} \mu_G(g,z), & G = \mathrm{Sp}(2n), \\ \big(\lambda_G(g,z),\, \mu_G(g,z)\big), & G = \mathrm{U}(J_{n,n}),\, \mathrm{U}(p,q), \end{cases}. \tag{1.1.2}$$

For $z \in \mathcal{H}_G$, define

$$\Xi_G(z) = \begin{cases} i(\overline{z}-z), & G = \mathrm{Sp}(2n), \\ \big(i(\overline{z} - {}^{\mathrm{t}}z),\, i({}^{\mathrm{t}}\overline{z} - z)\big), & G = \mathrm{U}(J_{n,n}), \\ \big(\mathbf{1}_p - \overline{z}{}^{\mathrm{t}}z,\, \mathbf{1}_q - {}^{\mathrm{t}}\overline{z}z\big), & G = \mathrm{U}(p,q). \end{cases}$$

We view Λ_G (resp. Ξ_G) as a map from $G \times \mathcal{H}_G$ (resp. $\mathcal{H}_G$) to the group

$$R_G = \begin{cases} \mathrm{GL}(n,\mathbb{C}), & G = \mathrm{Sp}(2n), \\ \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}), & G = \mathrm{U}(J_{n,n}), \\ \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}), & G = \mathrm{U}(p,q), \end{cases} \tag{1.1.3}$$

For $g \in G$ and $z \in \mathcal{H}_G$, one can easily check that

$$\Xi_G(g\cdot z) = {}^{\mathrm{t}}\overline{\Lambda_G(g,z)}^{-1}\, \Xi_G(z)\, \Lambda_G(g,z)^{-1}. \tag{1.1.4}$$

We fix a base point

$$\mathbf{o}_G = \begin{cases} i \cdot \mathbf{1}_n, & G = \mathrm{Sp}(2n),\, \mathrm{U}(J_{n,n}), \\ \mathbf{0}_{p,q}, & G = \mathrm{U}(p,q), \end{cases}$$

in the symmetric domain $\mathcal{H}_G$. It is easy see that $\mathbf{o}_G$ is fixed by K_G for the action of G on $\mathcal{H}_G$ in (1.1.0), and the restriction of the map $\Lambda_G(\cdot\,,\mathbf{o}_G) : G \to \mathcal{H}_G$ to K_G is a group homomorphism.

Let $\big(\rho, L_\rho(\mathbb{C})\big)$ be a finite dimensional algebraic representation of R_G. We consider the following two spaces of smooth functions:

$$C^\infty_\rho(\mathcal{H}_G) = \{f : \mathcal{H}_G \to L_\rho(\mathbb{C}) \text{ smooth}\},$$
$$C^\infty_\rho(G) = \Big\{F : G \to L_\rho(\mathbb{C}) \text{ smooth} \,:\, F(gk) = \rho\big(\Lambda_G(k,\mathbf{o}_G)\big)^{-1} \cdot F(g) \text{ for all } g \in G,\, k \in K_G\Big\}.$$

If F is a totally real field and $\mathbf{G}$ is an algebraic group defined over $\mathcal{O}_F$ with $\mathbf{G}(\mathbb{R}) \cong G$, the invariant subspace of $\prod_{\substack{\text{archimedean}\\ \text{places of } F}} C^\infty_\rho(\mathcal{H}_G)$ (resp. $\prod_{\substack{\text{archimedean}\\ \text{places of } F}} C^\infty_\rho(G)$) under the action of (resp. the left translation by) a congruence subgroup of $\mathbf{G}(\mathcal{O}_F)$ can be viewed as automorphic forms on $\mathbf{G}$. The map

$$\begin{aligned} \mathscr{T}_{G,\rho} : C^\infty_\rho(\mathcal{H}_G) &\longrightarrow C^\infty_\rho(G) \\ f &\longmapsto \mathscr{T}_{G,\rho}(f)(g) = \rho\big(\Lambda_G(g,\mathbf{o}_G)\big)^{-1} f(g\cdot\mathbf{o}_G) \end{aligned} \tag{1.1.5}$$

is a bijection with its inverse given as

$$\mathscr{T}^{-1}_{G,\rho}(F)(z) = \rho\big(\Lambda_G(g_z,\mathbf{o}_G)\big) \cdot F(g_z),$$

where

$$g_z = \begin{cases} \begin{pmatrix} \left(\frac{z-{}^{\mathrm{t}}\overline{z}}{2i}\right)^{\frac12} & \frac{z+{}^{\mathrm{t}}\overline{z}}{2}\left(\frac{z-{}^{\mathrm{t}}\overline{z}}{2i}\right)^{-\frac12} \\ 0 & \left(\frac{z-{}^{\mathrm{t}}\overline{z}}{2i}\right)^{-\frac12} \end{pmatrix}, & G = \mathrm{Sp}(2n),\ \mathrm{U}(J_{n,n}), \\ \begin{pmatrix} (1-z{}^{\mathrm{t}}\overline{z})^{\frac12} & z(1-{}^{\mathrm{t}}\overline{z}z)^{-\frac12} \\ {}^{\mathrm{t}}\overline{z}(1-z{}^{\mathrm{t}}\overline{z})^{\frac12} & (1-{}^{\mathrm{t}}\overline{z}z)^{-\frac12} \end{pmatrix}, & G = \mathrm{U}(p,q). \end{cases} \tag{1.1.6}$$

Here when A is a positive Hermitian matrix, $A^{\frac12}$ denotes the unique positive Hermitian matrix whose square is A. It is easily seen that $g_z \cdot \mathbf{o}_G = z$.

1.2. **The Maass–Shimura differential operators and the action of the Lie algebra.** Denote by $\big(\tau_G, L_{\tau_G}(\mathbb{C})\big)$ the algebraic representation of R_G given as

$$\begin{aligned} &\mathrm{Sym}^2\,\mathrm{St}_{\mathrm{GL}(n)}(\mathbb{C}), && G = \mathrm{Sp}(2n), \\ &\mathrm{St}_{\mathrm{GL}(n)}(\mathbb{C}) \boxtimes \mathrm{St}_{\mathrm{GL}(n)}(\mathbb{C}), && G = \mathrm{U}(J_{n,n}) \\ &\mathrm{St}_{\mathrm{GL}(p)}(\mathbb{C}) \boxtimes \mathrm{St}_{\mathrm{GL}(q)}(\mathbb{C}), && G = \mathrm{U}(p,q), \end{aligned}$$

where for a positive integer m, the representation $\mathrm{St}_{\mathrm{GL}(m)}$ is the standard m-dimensional representation of $\mathrm{GL}(m)$ with basis $e_1,\dots,e_m$ and the action of $a \in \mathrm{GL}(m,\mathbb{C})$ given as

$$(a\cdot e_1,\dots,a\cdot e_m) = (e_1,\dots,e_m)\,a.$$

We fix the following basis for $L_{\tau_G}(\mathbb{C})$:

$$\begin{aligned} &\mathcal{E}_{jj},\ 1\le j\le n, \quad \mathcal{E}_{jk} = \mathcal{E}_{kj},\ 1\le j<k\le n && G = \mathrm{Sp}(2n), \\ &\mathcal{E}_{jk},\ 1\le j,k\le n, && G = \mathrm{U}(J_{n,n}), \\ &\mathcal{E}_{jk},\ 1\le j\le p,\ 1\le k\le q, && G = \mathrm{U}(p,q). \end{aligned} \tag{1.2.1}$$

with the action of R_G given by

$$\begin{aligned} \tau_G(a)\cdot\underline{\mathcal{E}} &= {}^{\mathrm{t}}a\,\underline{\mathcal{E}}\,a, & \underline{\mathcal{E}} &= (\mathcal{E}_{jk})_{1\le j,k\le n}\,, & G&=\mathrm{Sp}(2n),\\ \tau_G(a,b)\cdot\underline{\mathcal{E}} &= {}^{\mathrm{t}}a\,\underline{\mathcal{E}}\,b, & \underline{\mathcal{E}} &= (\mathcal{E}_{jk})_{1\le j,k\le n}\,, & G&=\mathrm{U}(J_{n,n}),\\ \tau_G(a,b)\cdot\underline{\mathcal{E}} &= {}^{\mathrm{t}}a\,\underline{\mathcal{E}}\,b, & \underline{\mathcal{E}} &= (\mathcal{E}_{jk})_{1\le j\le p,1\le k\le q}\,, & G&=\mathrm{U}(p,q).\end{aligned} \tag{1.2.2}$$

We consider certain weight-raising operators sending functions valued in ρ to functions valued in $\rho\otimes\tau_G$. For functions on $\mathcal{H}_G$, there is the Maass–Shimura differential operator

$$D_{G,\rho}: C^\infty_\rho(\mathcal{H}_G)\longrightarrow C^\infty_{\rho\otimes\tau_G}(\mathcal{H}_G)$$

defined as

$$(D_{G,\rho}f)(z)=\rho\big(\Xi_G(z)\big)^{-1}\sum_{j,k}\mathcal{E}_{jk}\frac{\partial}{\partial z_{jk}}\Big(\rho\big(\Xi_G(z)\big)f(z)\Big), \tag{1.2.3}$$

where the sum runs over the indices of our fixed basis in (1.2.1) of $L_{\tau_G}(\mathbb{C})$ (i.e. $1\le j\le k\le n$ for $G=\mathrm{Sp}(2n)$, $1\le j,k\le n$ for $G=\mathrm{U}(J_{n,n})$, $1\le j\le p$, $1\le k\le q$ for $G=\mathrm{U}(p,q)$).

For functions on G, the weight-raising operators come from the action of the Lie algebra of G. Denote by $\mathrm{Lie}\,G$ the Lie algebra of the real Lie group G. Given $w=w_1+iw_2\in\mathbb{C}^\times$, define

$$h_G(w)=\begin{cases}\begin{pmatrix} w_1\cdot\mathbf{1}_n & w_2\cdot\mathbf{1}_n\\ -w_2\cdot\mathbf{1}_n & w_2\cdot\mathbf{1}_n\end{pmatrix}, & G=\mathrm{Sp}(2n),\ \mathrm{U}(J_{n,n})\\ \begin{pmatrix} w\cdot\mathbf{1}_p & \\ & \overline{w}\cdot\mathbf{1}_q\end{pmatrix}, & G=\mathrm{U}(p,q).\end{cases}$$

The torus $\mathbb{C}^\times$ acts on G by

$$\begin{aligned}\mathbb{C}^\times\times G &\longmapsto G\\ (w,g)&\longmapsto w\cdot g=h_G(w)\,g\,h_G(w)^{-1},\end{aligned}$$

inducing an action of $\mathbb{C}^\times$ on $\mathrm{Lie}\,G$. Let $(\mathrm{Lie}\,G\otimes_{\mathbb{R}}\mathbb{C})^{a,b}$ be the subspace of $\mathrm{Lie}\,G$ on which $w\in\mathbb{C}^\times$ acts by the scalar $w^{-a}\overline{w}^{-b}$. Put

$$\mathfrak{k}_{G,\mathbb{C}}=(\mathrm{Lie}\,G\otimes_{\mathbb{R}}\mathbb{C})^{0,0},\qquad \mathfrak{p}_G^+=(\mathrm{Lie}\,G\otimes_{\mathbb{R}}\mathbb{C})^{-1,1},\qquad \mathfrak{p}_G^-=(\mathrm{Lie}\,G\otimes_{\mathbb{R}}\mathbb{C})^{1,-1}.$$

Then we have the decomposition

$$\mathrm{Lie}\,G\otimes_{\mathbb{R}}\mathbb{C}=\mathfrak{p}_G^+\oplus\mathfrak{k}_{G,\mathbb{C}}\oplus\mathfrak{p}_G^-. \tag{1.2.4}$$

One can easily check that $\mathfrak{k}_{G,\mathbb{C}}=(\mathrm{Lie}\,K_G)\otimes_{\mathbb{R}}\mathbb{C}$ with

$$\mathrm{Lie}\,K_G=\begin{cases}\left\{\begin{pmatrix} X & Y\\ -Y & X\end{pmatrix}: X,Y\in M_{n,n}(\mathbb{R}),\ {}^{\mathrm{t}}X=-X,\ {}^{\mathrm{t}}Y=Y\right\}, & G=\mathrm{Sp}(2n),\\ \left\{\begin{pmatrix} X & Y\\ -Y & X\end{pmatrix}: X,Y\in M_{n,n}(\mathbb{C}),\ {}^{\mathrm{t}}\overline{X}=-X,\ {}^{\mathrm{t}}\overline{Y}=Y\right\}, & G=\mathrm{U}(J_{n,n}),\\ \left\{\begin{pmatrix} X & \\ & Y\end{pmatrix}: X\in M_{p,p}(\mathbb{C}),\ Y\in M_{q,q,}(\mathbb{C})\ {}^{\mathrm{t}}\overline{X}=-X,\ {}^{\mathrm{t}}\overline{Y}=-Y\right\}, & G=\mathrm{U}(p,q),\end{cases}$$

and

$$\mathfrak{p}_G^+ = \begin{cases} \mathrm{Span}_{\mathbb{C}} \left\{ \begin{pmatrix} & X \\ X & \end{pmatrix} + \begin{pmatrix} -X & \\ & X \end{pmatrix} \otimes i : X \in M_{n,n}(\mathbb{R}),\ {}^{\mathrm{t}}X = X \right\}, & G = \mathrm{Sp}(2n), \\ \mathrm{Span}_{\mathbb{C}} \left\{ \begin{pmatrix} & X \\ X & \end{pmatrix} + \begin{pmatrix} -X & \\ & X \end{pmatrix} \otimes i : X \in M_{n,n}(\mathbb{C}),\ {}^{\mathrm{t}}\overline{X} = X \right\}, & G = \mathrm{U}(J_{n,n}), \\ \mathrm{Span}_{\mathbb{C}} \left\{ \begin{pmatrix} & X \\ {}^{\mathrm{t}}X & \end{pmatrix} + \begin{pmatrix} & -iX \\ i{}^{\mathrm{t}}X & \end{pmatrix} \otimes i : X \in M_{p,q}(\mathbb{R}) \right\}, & G = \mathrm{U}(p,q). \end{cases} \tag{1.2.5}$$

The conjugation action

$$h \cdot L \longmapsto hLh^{-1}, \qquad h \in K_G,\ L \in (\mathrm{Lie}\, G) \otimes_{\mathbb{R}} \mathbb{C}$$

fixes the decomposition in (1.2.4), and the conjugation action of K_G on $\mathfrak{p}_G^+$ is isomorphic to $\tau_G|_{K_G}$.

The action of G by the right translation on the space of smooth functions on G induces an action of $\mathrm{Lie}\, G$ on that space. For a smooth function on G on which K_G acts by ρ through the right translation, $\mathfrak{p}_G^+$ sends it to a smooth function on which K_G acts by ρ tensored with the conjugation action of K_G on $\mathfrak{p}_G^+$. Hence, like the effect of $D_{G,\rho}$ on smooth functions on $\mathcal{H}_G$, the Lie algebra $\mathfrak{p}_G^+$ raises the weight of smooth functions on G by τ_G.

Put

$$\mu_{G,\sigma}^+ = \begin{cases} \begin{pmatrix} & \sigma \\ \sigma & \end{pmatrix} + \begin{pmatrix} -\sigma & \\ & \sigma \end{pmatrix} \otimes i, & G = \mathrm{Sp}(2n),\ \sigma \in M_{n,n}(\mathbb{R}),\ {}^{\mathrm{t}}\sigma = \sigma, \\ \begin{pmatrix} -i\frac{\sigma - {}^{\mathrm{t}}\sigma}{2} & \frac{\sigma + {}^{\mathrm{t}}\sigma}{2} \\ \frac{\sigma + {}^{\mathrm{t}}\sigma}{2} & i\frac{\sigma - {}^{\mathrm{t}}\sigma}{2} \end{pmatrix} + \begin{pmatrix} -\frac{\sigma + {}^{\mathrm{t}}\sigma}{2} & -i\frac{\sigma - {}^{\mathrm{t}}\sigma}{2} \\ -i\frac{\sigma - {}^{\mathrm{t}}\sigma}{2} & \frac{\sigma + {}^{\mathrm{t}}\sigma}{2} \end{pmatrix} \otimes i, & G = \mathrm{U}(J_{n,n}),\ \sigma \in M_{n,n}(\mathbb{R}), \\ \begin{pmatrix} & \sigma \\ {}^{\mathrm{t}}\sigma & \end{pmatrix} + \begin{pmatrix} & -i\sigma \\ i{}^{\mathrm{t}}\sigma & \end{pmatrix} \otimes i, & G = \mathrm{U}(p,q),\ \sigma \in M_{p,q}(\mathbb{R}), \end{cases} \tag{1.2.6}$$

and

$$\mu_{G,jk}^+ = \begin{cases} \mu_{G,E_{jk}+E_{kj}}^+, & G = \mathrm{Sp}(2n),\ 1 \le j,k \le n, \\ \mu_{G,E_{jk}}^+, & G = \mathrm{U}(J_{n,n}),\ 1 \le j,k \le n, \\ \mu_{G,E_{jk}}^+, & G = \mathrm{U}(p,q),\ 1 \le j \le p, 1 \le k \le q, \end{cases} \tag{1.2.7}$$

where the notation E_{ml} denotes the matrix with 1 as the (m,l)-entry and 0 elsewhere of size $n \times n$ when $G = \mathrm{Sp}(2n)$, $\mathrm{U}(J_{n,n})$ and size $p \times q$ when $G = \mathrm{U}(p,q)$.

Theorem 1.2.1 ([Shi84][Proposition 7.3])**.** *With $\mathscr{T}_{G,\rho}$, $\mathscr{T}_{G,\rho\otimes\tau_G}$ defined as in (1.1.5), $D_{G,\rho}$ defined in (1.2.3), and $\mu_{G,jk}^+$ defined in (1.2.7), we have*

$$\mathscr{T}_{G,\rho\otimes\tau_G}(D_{G,\rho} f) = \frac{1}{2} \sum_{j,k} \mathcal{E}_{jk} \mu_{G,jk}^+ \cdot \mathscr{T}_{G,\rho}(f),$$

where the sum runs over the indices of our fixed basis $\mathcal{E}_{jk}$ in (1.2.1) of $L_{\tau_G}(\mathbb{C})$ (i.e. $1 \le j \le k \le n$ for $G = \mathrm{Sp}(2n)$, $1 \le j,k \le n$ for $G = \mathrm{U}(J_{n,n})$, $1 \le j \le p$, $1 \le k \le q$ for $G = \mathrm{U}(p,q)$).

Proof. Define

$$\mathscr{P}_G : C^\infty(\mathcal{H}_G) \longrightarrow C_\rho^\infty(G)$$
$$f \longmapsto \mathscr{P}_G(f)(g) = f(g \cdot \mathbf{o}_G).$$

It is easily seen that

$$\mathscr{T}_{G,\rho} = \rho\big(\Lambda_G(\cdot, \mathbf{o}_G)\big)^{-1} \cdot \mathscr{P}_G, \tag{1.2.8}$$

and the representation ρ does not appear explicitly in the defining formula for $\mathscr{P}_G$. (For this reason, omit the ρ from its subscript.)

The theorem follows from the following two lemmas whose proofs are straightforward and omitted here. Let $\boldsymbol{\mu}_G^+$ (resp. $\frac{\partial}{\partial z}$) denote the matrix of the same size $\underline{\mathcal{E}}$ with the (j,k)-entry equal to

$$\begin{aligned} &\frac{1}{2-\delta_{jk}}\mu^+_{G,jk} \quad (\text{resp. } \frac{\partial}{\partial z_{jk}}), && G = \mathrm{Sp}(2n), \\ &\mu^+_{G,jk} \quad (\text{resp. } \frac{\partial}{\partial z_{jk}}), && G = \mathrm{U}(J_{n,n}),\ \mathrm{U}(p,q). \end{aligned}$$

Lemma 1.2.2. $\underline{\mathcal{E}}\left(\boldsymbol{\mu}_G^+ \cdot \mathscr{P}_G(f)\right) = 2\left(\tau_G(\Lambda_G(\cdot,\mathbf{o}_G))^{-1}\cdot\underline{\mathcal{E}}\right)\mathscr{P}_G\left(\frac{\partial}{\partial z}f\right)$, *where both sides are viewed as matrices with entries being elements in* $C^\infty_{\rho\otimes\tau_G}(G)$.

Lemma 1.2.3. $\mu^+_{G,jk}\cdot\rho\left(\overline{\Lambda_G(g,\mathbf{o}_G)}\right) = 0$.

Applying Lemma 1.2.2 to the function $\rho(\Xi_G)f : z \mapsto \rho(\Xi_G(z))\, f(z)$, we get

$$\underline{\mathcal{E}}\left(\boldsymbol{\mu}_G^+\cdot\mathscr{P}_G\big(\rho(\Xi_G)f\big)\right) = 2\left(\tau_G\big(\Lambda_G(g_z,\mathbf{o}_G)\big)^{-1}\cdot\underline{\mathcal{E}}\right)\mathscr{P}_G\left(\frac{\partial}{\partial z}\rho(\Xi_G)f\right). \tag{1.2.9}$$

Plugging $z=\mathbf{o}_G$, $g=g_z$ (defined in (1.1.6)) into (1.1.4) and noting that $\Xi_G(\mathbf{o}_G)=1$, we get

$$\Xi_G(z) = {}^{\mathrm{t}}\overline{\Lambda_G(g_z,\mathbf{o}_G)}^{-1}\Lambda_G(g_z,\mathbf{o}_G)^{-1}.$$

Hence,

$$\begin{aligned} \mu^+_{G,jk}\cdot\mathscr{P}_G\big(\rho(\Xi_G)f\big) &= \mu^+_{G,jk}\cdot\left(\rho\big({}^{\mathrm{t}}\overline{\Lambda_G(\cdot,\mathbf{o}_G)}^{-1}\big)\rho\left(\Lambda_G(\cdot,\mathbf{o}_G)^{-1}\right)\cdot\mathscr{P}_G(f)\right) \\ &\overset{(1.2.8)}{=} \mu^+_{G,jk}\cdot\left(\rho\big({}^{\mathrm{t}}\overline{\Lambda_G(\cdot,\mathbf{o}_G)}^{-1}\big)\cdot\mathscr{T}_{G,\rho}(f)\right) \\ &\overset{(1.2.3)}{=} \rho\big({}^{\mathrm{t}}\overline{\Lambda_G(\cdot,\mathbf{o}_G)}^{-1}\big)\left(\mu^+_{G,jk}\cdot\mathscr{T}_{G,\rho}(f)\right), \end{aligned} \tag{1.2.10}$$

and

$$\mathscr{P}_G\left(\frac{\partial}{\partial z}\rho(\Xi_G)f\right) = \rho\big({}^{\mathrm{t}}\overline{\Lambda_G(\cdot,\mathbf{o}_G)}^{-1}\big)\rho\left(\Lambda_G(\cdot,\mathbf{o}_G)^{-1}\right)\cdot\mathscr{P}_G\left(\rho(\Xi_G)^{-1}\frac{\partial}{\partial z}\rho(\Xi_G)f\right). \tag{1.2.11}$$

Therefore,

$$\begin{aligned} \frac{1}{2}\underline{\mathcal{E}}\left(\boldsymbol{\mu}_G^+\cdot\mathscr{T}_{G,\rho}(f)\right) &\overset{(1.2.10)}{=} \underline{\mathcal{E}}\left(\rho\big({}^{\mathrm{t}}\overline{\Lambda_G(\cdot,\mathbf{o}_G)}\big)\cdot\boldsymbol{\mu}_G^+\cdot\mathscr{P}_G\big(\rho(\Xi_G)f\big)\right) \\ &\overset{(1.2.9)}{=} \left(\tau_G\big(\Lambda_G(g_z,\mathbf{o}_G)\big)^{-1}\cdot\underline{\mathcal{E}}\right)\rho\big({}^{\mathrm{t}}\overline{\Lambda_G(\cdot,\mathbf{o}_G)}\big)\cdot\mathscr{P}_G\left(\frac{\partial}{\partial z}\rho(\Xi_G)f\right) \\ &\overset{(1.2.11)}{=} \left(\tau_G\big(\Lambda_G(g_z,\mathbf{o}_G)\big)^{-1}\cdot\underline{\mathcal{E}}\right)\rho\left(\Lambda_G(\cdot,\mathbf{o}_G)^{-1}\right)\cdot\mathscr{P}_G\left(\rho(\Xi_G)^{-1}\frac{\partial}{\partial z}\rho(\Xi_G)f\right) \\ &= \rho\otimes\tau_G\big(\Lambda_G(g_z,\mathbf{o}_G)^{-1}\big)\cdot\left(\underline{\mathcal{E}}\,\mathscr{P}_G\left(\rho(\Xi_G)^{-1}\frac{\partial}{\partial z}\rho(\Xi_G)f\right)\right). \end{aligned}$$

Taking the traces of both sides proves the equality in the theorem. □

2. The doubling archimedean zeta integrals

2.1. **The doubling method.** The doubling method provides an integral representation for L-functions of automorphic representations of classical groups. It was discovered by Garrett [Gar84] and independently by Piatetski-Shapiro and Rallis [PSR87]. Garrett studied the pullback of the Siegel Eisenstein series on the Siegel upper half space of degree $n+m$ to the product of Siegel upper half spaces of degrees n and m with respect to the embedding $(z,w) \mapsto \begin{pmatrix} z & \\ & w \end{pmatrix}$, and discovered a formula for its projection into an irreducible automorphic representation of $\mathrm{Sp}(n,\mathbb{A}) \times \mathrm{Sp}(m,\mathbb{A})$. When $n=m$, the formula is the standard doubling method formula. When $n \neq m$, the formula gives an integral representation for Klingen Eisenstein series. The starting point of the work by Piatetski–Shapiro and Rallis is the Rallis inner product formula [Ral84], which shows that the Petersson inner product of a theta lifting is equal to a special value of the relevant global doubling zeta integral. In [PSR87], all classical groups are treated. They unfolded the global adelic integral, showed that it is equal to a product of local zeta integrals, and computed the local zeta integrals at unramified places.

Later, the work by Garrett was used to study the analytic properties of the standard L-functions for Siegel modular forms [Böc85b], the basis problem [B8̈3, Böc85a], the algebraicity of critical values of L-functions [Böc85a, Shi95, Shi97, Shi00], and the algebraicity of Klingen Eisenstein series (evaluated at certain half integers) for irreducible cuspidal automorphic representations whose archimedean components are holomorphic discrete series. The work by Piatetski–Shapiro and Rallis was used to define the local L-, γ- and ϵ-factors of representations of classical groups [PSR86, LR05, Yam14]. It is also a cornerstone of Rallis' program which aims to prove a local-global criterion for the nonvanishing of the global theta lifting with a prototype statement saying that the nonvanishing of a global theta lift is equivalent to the nonvanishing of the local theta lifts at all places plus the nonvanishing of a special value of the relevant L-function.

The doubling method has also found applications in Iwasawa theory and the theory of p-adic automorphic forms. It was used to construct the p-adic L-functions for symplectic and unitary groups [BS00, HLS06, Wan15, EW16, Liu16, EHLS20, LR20]. The doubling method for the groups $\mathrm{SL}(2)$, $\mathrm{U}(1,1)$, $\mathrm{U}(1)$ and $\mathrm{U}(2)$ constitutes a crucial ingredient in the proof of many cases of the Iwasawa–Greenberg main conjectures for elliptic curves [Urb06, SU14, Hsi14, Wan20]. It is also applied to Yoshida lifts on $\mathrm{GSp}(4)$ to study the Bloch–Kato conjecture for the critical values of the Rankin–Selberg L-function for a pair of modular forms of weights with equal parity [Jia10, BDSP12, AK13]

Let F be a number field and $E=F$ or E be a quadratic extension of F. Let $\mathcal{V}=(V,\langle,\rangle)$ be a finite dimensional vector space V over E with a non-degenerate sesqui-linear form $\langle,\rangle$ which is either symmetric or anti-symmetric if $E=F$ and is Hermitian if $E \neq F$. Let $G=\mathrm{Isom}(\mathcal{V})$ viewed as an algebraic group defined over F. Let $\mathcal{V}^{\square}=(V\oplus V, \langle,\rangle \oplus -\langle,\rangle)$ and $H=\mathrm{Isom}(\mathcal{V}^{\square})$ (sometimes called the doubled group). The group $G\times G$ are naturally embedded in H.

The space $V^{\diamond}=\{(v,v) : v\in V\}$ is a maximal totally isotropic subspace of $\mathcal{V}^{\square}$. Denote by Q the Siegel parabolic subgroup of H which stabilizes $\mathcal{V}^{\diamond}\subset\mathcal{V}^{\square}$. Let $I(s,\chi)$ be the (normalized) induction from $Q(\mathbb{A}_F)$ to $H(\mathbb{A}_F)$ of the character $\chi|\cdot|^s : E^{\times}\backslash\mathbb{A}_E^{\times}\to\mathbb{C}^{\times}$. The induced representation $I(s,\chi)$ is often called a degenerate principal series. For a given section $f(s,\chi)\in I(s,\chi)$, define the Siegel Eisenstein series as

$$E(h,f(s,\chi)) = \sum_{\gamma\in Q(F)\backslash H(F)} f(s,\chi)(\gamma h), \qquad h\in H(\mathbb{A}_F).$$

Suppose that π is an irreducible cuspidal automorphic representation of $G(\mathbb{A}_F)$ with unitary central character. Given $f(s,\chi)\in I(s,\chi)$ and $\varphi_1,\varphi_2\in\pi$, the global doubling zeta integral is defined as

$$Z\left(f(s,\chi),\varphi_1,\overline{\varphi_2}\right)=\int_{G(F)\times G(F)\backslash G(\mathbb{A}_F)\times G(\mathbb{A}_F)}\chi(\det g_2)^{-1}\varphi_1(g_1)\overline{\varphi_2(g_2)}E\big((g_1,g_2),f(s,\chi)\big)\,dg$$

Unfolding the Eisenstein series as in [PSR87], one gets

$$Z\left(f(s,\chi),\varphi_1,\overline{\varphi_2}\right)=\int_{G(\mathbb{A}_F)}f(s,\chi)\big((g,1)\big)\left\langle\pi(g)\varphi_1,\overline{\varphi_2}\right\rangle dg, \tag{2.1.1}$$

where $\langle\,,\rangle:\pi\times\overline{\pi}\to\mathbb{C}$ is the Petersson inner product between π and $\overline{\pi}=\{\overline{\xi}:\xi\in\pi\}$ defined as $\left\langle\xi_1,\overline{\xi_2}\right\rangle=\int_{G(F)\backslash G(\mathbb{A}_F)}\xi_1(g)\overline{\xi_2(g)}\,dg$. One can fix isomorphisms $\pi\cong\otimes_v\pi_v$ and $\overline{\pi}\cong\otimes_v\widetilde{\pi}_v$, where π_v is an irreducible admissible representation of $G(F_v)$ and $\widetilde{\pi}_v$ is its admissible dual, such that if $\xi_1\in\pi$ and $\overline{\xi_2}\in\overline{\pi}$ are identified with $\otimes_v\xi_{1,v}$ and $\otimes_v\overline{\xi}_{2,v}$ under the fixed isomorphism and $\langle\,,\rangle_v$ denotes the tautological pairing between π_v and $\widetilde{\pi}_v$, then $\left\langle\xi_1,\overline{\xi_2}\right\rangle=\prod_v\left\langle\xi_{1,v},\overline{\xi}_{2,v}\right\rangle_v$. With the fixed isomorphisms, we can write the right hand side of (2.1.1) as a product of local integrals. For $\xi_v\in\pi_v$, $\widetilde{\xi}_v\in\widetilde{\pi}_v$ and $f_v(s,\chi_v)\in I_v(s,\chi_v)$, define the local doubling zeta integral as

$$Z_v\left(f_v(s,\chi),\xi_v,\widetilde{\xi}_v\right)=\int_{G(F_v)}f_v(s,\chi)\big((g,1)\big)\left\langle\pi_v(g)\xi_v,\widetilde{\xi}_v\right\rangle dg.$$

If $f(s,\chi)=\otimes_vf_v(s,\chi)$ and φ_1 (resp. $\overline{\varphi_v}$) is identified with $\otimes_v\varphi_{1,v}$ (resp. $\overline{\varphi}_{2,v}$ under the fixed isomorphism $\pi\cong\otimes\pi_v$ (resp. $\overline{\pi}\cong\otimes_v\widetilde{\pi}_v$), then we have

$$Z\left(f(s,\chi),\varphi_1,\overline{\varphi_2}\right)=\prod_v Z_v\left(f_v(s,\chi),\varphi_{1,v},\overline{\varphi}_{2,v}\right).$$

Let v be a finite place of F where G, π and χ are all unramified. Denote by $f_v^{\mathrm{ur}}(s,\chi_v)$, ξ_v^{ur} and $\widetilde{\xi}_v^{\mathrm{ur}}$ the spherical the spherical vectors in $I_v(s,\chi_v)$, π_v and $\widetilde{\pi}_v$ with $f_v^{\mathrm{ur}}(s,\chi_v)(1)=1$ and $\left\langle\xi_v^{\mathrm{ur}},\widetilde{\xi}_v^{\mathrm{ur}}\right\rangle_v=1$. It is proved in [PSR87, LR05, Li92] that

$$Z_v\left(f_v^{\mathrm{ur}}(s,\chi),\xi_v^{\mathrm{ur}},\widetilde{\xi}_v^{\mathrm{ur}}\right)=d_{H,v}(s,\chi_v)^{-1}\cdot L_v\left(s+\frac{1}{2},\pi_v\times\chi_v\right),$$

where $d_{H,v}(s,\chi_v)$ is a product of L-factors of characters of $F_v^\times$. (See Remark 3 in [LR05] for its precise definition.)

For many applications, it is necessary to study the doubling local zeta integrals at the archimedean places. In [KR90], in order to locate possible poles of the L-functions, the authors proved that for all $s_0\in\mathbb{C}$, one can always choose sections at the archimedean place such that the local doubling zeta integral does not have a pole or zero at $s=s_0$. In [Böc85a, Shi95, Shi97, Shi00, Har08] where the doubling method is applied to study the algebraicity of critical L-values, specific choices of section inside the degenerate principal series at the archimedean places are made based on different theories of differential operators, and the archimedean doubling zeta integrals are computed for special cases.

In order to obtain complete interpolation formulas for the p-adic L-functions constructed in [Liu16, EHLS20] and verify the conjecture of Coates and Perrin–Riou in [CPR89, Coa91], one needs to calculate the doubling archimedean zeta integrals for holomorphic discrete series of symplectic and unitary groups and the specific sections in the degenerate principal series chosen in [Liu16, EHLS20]. In the following, we explain the choice of the sections and how the archimedean zeta integrals are computed.

2.2. **The choice of archimedean sections for p-adic L-functions.** We focus on the archimedean place and use the notation in §1. Assume that $p+q=n$. We consider the following two cases:

(Sp) $G=\mathrm{Sp}(2n)$, $H=\mathrm{Sp}(4n)$,

(U) $G=\mathrm{U}(p,q)$, $H=\mathrm{U}(J_{n,n})$.

Denote by χ the character of $\mathbb{R}^\times$ in case (Sp) (resp. $\mathbb{C}^\times$ in case (U)) defined as $\chi(z)\mapsto z/|z|$. Let r be an integer We use $I(s,\chi^r)$ to denote the degenerate principal series of the Lie group H inducing the character χ^r from its Siegel parabolic $Q=\left\{\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}\in H\right\}$. ($\chi^r$ is viewed as a character of Q by composing it with $\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}\mapsto \det A$.) Let

$$\mathcal{D}_{\underline{t}}=\begin{cases}\text{holomorphic discrete series of } \mathrm{Sp}(2n) \text{ of weight } \underline{t}=(t_1,\dots,t_n), & \text{(Sp)}\\ \text{holomorphic discrete series of } \mathrm{U}(p,q) \text{ weight } \underline{t}=(\tau_1,\dots,\tau_p;\nu_1,\dots,\nu_q), & \text{(U)}\end{cases}\tag{2.2.1}$$

and $v_{\underline{t}}$ be the highest weight vector inside the lowest K_G-type of $\mathcal{D}_{\underline{t}}$. Denote by $v_{\underline{t}}^*$ the dual vector of $v_{\underline{t}}$ in the dual representation of $\mathcal{D}_{\underline{t}}$. The integral we need to consider is

$$Z\left(f(s,\chi^r),v_{\underline{t}}^*,v_{\underline{t}}\right)=\int_{G(\mathbb{R})} f(s,\chi^r)\big(\imath(g,1)\big)\langle g\cdot v_{\underline{t}}^*,v_{\underline{t}}\rangle\, dg\tag{2.2.2}$$

where $f(s,\chi^r)$ is a section inside $I(s,\chi^r)$ described in the following, and $\imath$ is given by

$$\imath: G\times G\longrightarrow H$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\times\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\longmapsto\begin{cases}\begin{pmatrix}\mathbf{1}_n & & & \\ & \mathbf{1}_n & & \\ & \mathbf{1}_n & \mathbf{1}_n & \\ \mathbf{1}_n & & & \mathbf{1}_n\end{pmatrix}^{-1}\begin{pmatrix}a & & b & \\ & d' & & c'\\ c & & d & \\ & b' & & a'\end{pmatrix}, & \text{(Sp)}\\ \frac{1}{2}\begin{pmatrix}\mathbf{1}_n & -i\cdot\mathbf{1}_n\\ \mathbf{1}_n & i\cdot\mathbf{1}_n\end{pmatrix}^{-1}\begin{pmatrix}a & & & b\\ & d' & c' & \\ & b' & a' & \\ c & & & d\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & -i\cdot\mathbf{1}_n\\ \mathbf{1}_n & i\cdot\mathbf{1}_n\end{pmatrix}. & \text{(U)}\end{cases}$$

Now we describe the type of sections $f(s,\chi^r)$ used in [Liu16, EHLS20]. For each integer k congruent to r modulo 2, there is a classical section $f_k(s,\chi^r)\in I(s,\chi^r)$ defined as

$$f_k(s,\chi^r)\left(h=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\right)=\begin{cases}\det(Ci+D)^{-k}|\det(Ci+D)|^{-s+k-\frac{2n+1}{2}}, & \text{(Sp)}\\ (\det h)^{\frac{r+k}{2}}\det(Ci+D)^{-k}|\det(Ci+D)|_{\mathbb{C}}^{-s+\frac{k}{2}-\frac{n}{2}}. & \text{(U)}\end{cases}$$

Let $\mathfrak{p}_H^+$ be the sub-Lie algebra of $(\mathrm{Lie}\, H)\otimes_{\mathbb{R}}\mathbb{C}$ defined in the same way as in (1.2.5). We consider $f(s,\chi^r)$ obtained by applying the the Lie algebra operators in $\mathfrak{p}_H^+$ to $f_k(s,\chi^r)\in I(s,\chi^r)$. The reason that one considers this special type of sections for arithmetic applications is that we know a lot about the Siegel Eisenstein series $E\big(\cdot,f_k(s,\chi^r)\big)$ thanks to the calculation in [Shi82], and about the arithmetic properties of the action of $\mathfrak{p}_H^+$ thanks to Theorem 1.2.1 and the moduli interpretation of the Maass–Shimura differential operators. We follow the ideas in [Har86] to choose the operators in $\mathbb{C}[\mathfrak{p}_H^+]$ to apply to $f_k(s,\chi^r)$.

In the case needed for the p-adic L-functions in [Liu16, EHLS20], the $k,r,\underline{t}$ satisfy:

$$\begin{aligned}&t_1\geq t_2\geq\cdots\geq t_n\geq k\geq n+1, && k\equiv r \mod 2, \quad \text{(Sp)}\\ &\tau_1\geq\cdots\geq\tau_q\geq\frac{k+r}{2}\geq\frac{r-k}{2}\geq\nu_1\geq\cdots\geq\nu_q,\quad k\geq n, && k\equiv r \mod 2,\quad \text{(U)}\end{aligned}\tag{2.2.3}$$

and we will assume this condition on $k, r, \underline{t}$ from now on. Since $v_{\underline{t}}$ in (2.2.2) is picked from the lowest K_G-type of $\mathcal{D}_{\underline{t}}$, in order for (2.2.2) not to vanish trivially, it is natural to require that the action of $K_G \times K_G$ on $f(s, \chi^r)$ is isomorphic to $\underline{t} \boxtimes \underline{t}^\vee$. We know that the action $K_G \times K_G$ on the classical section $f_k(s, \chi^r)$ is isomorphic to the representation of scalar weight

(2.2.4)

$$(\underbrace{k, \ldots, k}_{n}) \boxtimes (\underbrace{-k, \ldots, -k}_{n}), \qquad \text{(Sp)}$$

$$\Big(\underbrace{\frac{k+r}{2}, \ldots, \frac{k+r}{2}}_{p}\Big) \boxtimes \Big(\underbrace{\frac{r-k}{2}, \ldots, \frac{r-k}{2}}_{q}\Big) \boxtimes \Big(\underbrace{\frac{k-r}{2}, \ldots, \frac{k-r}{2}}_{p}\Big) \boxtimes \Big(\underbrace{-\frac{k+r}{2}, \ldots, -\frac{k+r}{2}}_{q}\Big). \qquad \text{(U)}$$

By applying operators in $\mathfrak{p}_H^+$ to $f_k(s, \chi^r)$, one can increase the $K_G \times K_G$-type by the $K_G \times K_G$-representations appearing in the conjugation action of $K_G \times K_G$ (as a subgroup of K_H) on $\mathbb{C}[\mathfrak{p}_H^+]$. The action of K_H on $\mathbb{C}[\mathfrak{p}_H^+]$ extends to an action of its complexification R_H (defined as in (1.1.3)) and is isomorphic to

$$\mathbb{C}[\mathrm{Sym}_{2n}] \circlearrowleft \mathrm{GL}(2n,\mathbb{C}) \qquad A \cdot P(X) = P({}^{\mathrm{t}}AXA), \qquad \text{(Sp)}$$

$$\mathbb{C}[M_{n,n}] \circlearrowleft \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) \qquad (A, B) \cdot P(X) = P(B^{-1}XA), \qquad \text{(U)}$$

where by

(2.2.5)
$$X_1 + iX_2 \longmapsto \mu^+_{\mathrm{Sp}(4n),X_1} + (1 \otimes i)\mu^+_{\mathrm{Sp}(4n),X_2}, \quad X_1, X_2 \in \mathrm{Sym}_{2n}(\mathbb{R}), \quad \text{(Sp)}$$
$$X_1 + iX_2 \longmapsto \mu^+_{\mathrm{U}(J_{n,n}),X_1} + (1 \otimes i)\mu^+_{\mathrm{U}(J_{n,n}),X_2}, \quad X_1, X_2 \in M_{n,n}(\mathbb{R}), \quad \text{(U)}$$

we identify $\mathfrak{p}_H^+$ with $\mathrm{Sym}_{2n}(\mathbb{C})$, the symmetric $2n \times 2n$ matrices, in case (Sp), and with $M_{n,n}(\mathbb{C})$ in case (U). See (1.2.6) for the definition the elements in $\mathfrak{p}_H^+$ appearing on the right hand side of (2.2.5). The subgroup $R_G \times R_G$ of R_H is given as

$$\left\{ \begin{pmatrix} a & \\ & a' \end{pmatrix} : a, a' \in \mathrm{GL}(n, \mathbb{C}) \right\}, \qquad \text{(Sp)}$$

$$\left\{ \left(\begin{pmatrix} a & \\ & b \end{pmatrix}, \begin{pmatrix} a' & \\ & b' \end{pmatrix} \right) : a, a' \in \mathrm{GL}(p, \mathbb{C}),\, b, b' \in \mathrm{GL}(q, \mathbb{C}) \right\}, \qquad \text{(U)}$$

and it is easy to see that $\mathbb{C}[\mathfrak{p}_H^+]|_{R_G \times R_G}$, and therefore $\mathbb{C}[\mathfrak{p}_H^+]|_{K_G \times K_G}$ is not multiplicity free. Hence one would not be able to pick a canonical operator by only considering the K_G-types. Instead, we need to consider the decomposition of $I(s, \chi^r)|_{G \times G}$.

Let

(2.2.6)
$$s_0 = \begin{cases} k - \frac{2n+1}{2}, & \text{(Sp)} \\ \frac{k-n}{2}. & \text{(U)} \end{cases}$$

It is the evaluation at $s = s_0$ of (2.2.2) that is needed for studying the critical L-values. Denote by $U(\mathrm{Lie}\, H)$ the universal enveloping algebra of $\mathrm{Lie}\, H$, and by $U(\mathrm{Lie}\, H) \cdot f_k(s_0, \chi^r)$ the $(\mathrm{Lie}\, H, K_H)$-submodule inside $I(s, \chi^r)$ generated by $f_k(s_0, \chi^r)$. We consider the decomposition of

(2.2.7)
$$U(\mathrm{Lie}\, H) \cdot f_k(s_0, \chi^r)|_{\mathrm{Lie}\, G \times \mathrm{Lie}\, G}.$$

The idea in [Har86] is to use the results in [JV79, Proposition 2.2, Corollary 2.3], which reduces the decomposition of (2.2.7) to the decomposition of

(2.2.8)
$$\mathbb{C}[\mathfrak{p}_H^+/\mathfrak{p}_G^+ \times \mathfrak{p}_G^+]|_{R_G \times R_G}.$$

Under our identification (2.2.5), $\mathfrak{p}_G^+ \times \mathfrak{p}_G^+$ is identified with

$$\left\{\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} : X_1, X_2 \in \mathrm{Sym}_n(\mathbb{C})\right\}, \qquad \text{(Sp)}$$
$$\left\{\begin{pmatrix} 0 & X_1 \\ {}^{\mathrm{t}}X_2 & 0 \end{pmatrix} : X_1, X_2 \in M_{p,q}(\mathbb{C})\right\}, \qquad \text{(U)}$$

so we can identify the quotient $\mathfrak{p}_H^+/\mathfrak{p}_G^+ \times \mathfrak{p}_G^+$ with

$$M_{n,n}(\mathbb{C}) \cong \left\{\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} : X \in M_{n,n}(\mathbb{C})\right\}, \qquad \text{(Sp)}$$
$$M_{p,q}(\mathbb{C}) \times M_{p,q}(\mathbb{C}) \cong \left\{\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} : X \in M_{p,p}(\mathbb{C}),\, Y \in M_{q,q}(\mathbb{C})\right\}), \qquad \text{(U)}$$

with the action of $R_G \times R_G$ isomorphic to
(2.2.9)
$$\mathbb{C}[M_{n,n}] \circlearrowleft \mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C}) \qquad (a,a')\cdot P(X) = P(a'^{-1}Xa), \qquad \text{(Sp)}$$
$$\mathbb{C}[M_{p,q}\times M_{p,q}] \circlearrowleft \mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})\times\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C}) \qquad (a,d,a',d')\cdot P(X,Y) = (a'^{-1}Xa, d'^{-1}Yd'). \qquad \text{(U)}$$

By the basic theory of algebraic representations of general linear groups, we know that (2.2.9), hence (2.2.8), has a multiplicity free decomposition. Denote by Δ_j (resp. Δ'_j) the determinant of the upper left (resp. lower right) $j \times j$ block of a matrix. Define the polynomial

(2.2.10)
$$\mathfrak{Q}_{k,\underline{t}} = \left(\prod_{j=1}^{n-1} \Delta_j^{t_j - t_{j+1}}\right) \Delta_n^{t_n - k}, \qquad \text{(Sp)}$$
$$\mathfrak{Q}_{k,r,\underline{t}} = \left(\prod_{j=1}^{p-1} \Delta^{\tau_j - \tau_{j+1}}\right) \Delta_p^{\tau_p - \frac{k+r}{2}} \cdot \left(\prod_{j=1}^{q-1} \Delta^{\nu_j^* - \nu_{j+1}^*}\right) \Delta_q^{\nu_q^* - \frac{k-r}{2}}, \qquad \text{(U)}$$

where

(2.2.11)
$$(\nu_1^*, \nu_2^*, \dots, \nu_q^*) = (-\nu_q, -\nu_{q-1}, \dots, -\nu_1).$$

This polynomial is a highest (resp. lowest) weight vector for the first (resp. second) copy of R_G and it generates the irreducible representation of $R_G \times R_G$ whose tensor product with the representation in (2.2.4) is isomorphic to the product of the lowest K_G-type in $\mathcal{D}_{\underline{t}}$ with its dual. Therefore, a natural choice of a section in $I(s, \chi^r)$ for given $k, r, \underline{t}$ as in (2.2.3) is

(2.2.12)
$$f_{k,\underline{t}}(s,\chi^r) = \begin{cases} \mathfrak{Q}_{k,\underline{t}}\left(\dfrac{\boldsymbol{\mu}_H^{+,\text{ up-right}}}{2\pi i}\right) \cdot f_k(s,\chi^r), & \text{(Sp)} \\ \mathfrak{Q}_{k,r,\underline{t}}\left(\dfrac{\boldsymbol{\mu}_H^{+,\text{ up-left}}}{2\pi i}, \dfrac{\boldsymbol{\mu}_H^{+,\text{ low-right}}}{2\pi i}\right) \cdot f_k(s,\chi^r), & \text{(U)} \end{cases}$$

where $\boldsymbol{\mu}_H^{+,\text{ up-right}}$ (resp. $\boldsymbol{\mu}_H^{+,\text{ up-left}}$, $\boldsymbol{\mu}_H^{+,\text{ low-right}}$) is the upper right $n \times n$ (resp. upper left $p \times p$, lower right $q \times q$) block of $\boldsymbol{\mu}_H^+$, and $\boldsymbol{\mu}_H^+$ is the matrix whose (j,k) entry is the element in $\mathfrak{p}_H^+$ defined as in (1.2.7) for $H = \mathrm{Sp}(4n)$ (resp. $H = \mathrm{U}(J_{n,n})$). The $2\pi i$ on the denominator is to make the corresponding Maass–Shimura differential operator algebraic with respect to the algebraic structure of the relevant Shimura variety.

2.3. **The zeta integral for the chosen sections.**

Theorem 2.3.1. *[Liu19a, EL] Suppose that $k, r, \underline{t}$ satisfy the condition (2.2.3), $f_{k,\underline{t}}(s,\chi^r)$ is the section in $I(s,\chi^r)$ defined as in (2.2.12), $v_{\underline{t}}$ is the highest vector inside the lowest K_G-type of the holomorphic discrete series in (2.2.1), $v^*_{\underline{t}}$ is the dual vector of $v_{\underline{t}}$, and s_0 is given as in (2.2.6). Then*

$$Z\left(f_{k,\underline{t}}\left(s,\chi^r\right), v^*_{\underline{t}}, v_{\underline{t}}\right)\big|_{s=s_0}$$
$$=\begin{cases} \dfrac{2^{-3nk+2n^2+3n}\pi^{\frac{n^2+2n}{2}}(2\pi i)^{-\sum t_j+nk}}{\dim\left(\mathrm{GL}(n),\underline{t}\right)}\dfrac{\prod_{j=1}^{n}\Gamma(t_j-j+k-n)}{\prod_{j=1}^{2n}\Gamma\left(k-\frac{j-1}{2}\right)}, & \text{(Sp)}\\ \dfrac{2^{pq-\frac{n}{2}}\pi^{pq}(2\pi i)^{-\sum\tau_j-\sum\nu^*_j+\frac{p(k+r)}{2}+\frac{q(k-r)}{2}}}{\dim\left(\mathrm{GL}(p)\times\mathrm{GL}(q),\underline{t}\right)}\dfrac{\prod_{j=1}^{p}\Gamma\left(\tau_j-j+\frac{k-r}{2}-q+1\right)\prod_{j=1}^{q}\Gamma\left(\nu^*_j-j+\frac{k+r}{2}-p+1\right)}{\prod_{j=1}^{n}\Gamma(k-j+1)}, & \text{(U)}\end{cases}$$

with ν^*_j as in (2.2.11), $\dim\left(\mathrm{GL}(n),\underline{t}\right)$ the dimension of the $\mathrm{GL}(n)$-representation of highest weight $\underline{t}$, and similarly $\dim\left(\mathrm{GL}(p)\times\mathrm{GL}(q),\underline{t}\right)$ the product of the dimensions of the $\mathrm{GL}(p)$-representation of highest weight $(\tau_1,\dots,\tau_p)$ and the $\mathrm{GL}(q)$-representation of highest weight $(\nu_1,\dots,\nu_q)$.

Note that the section $f_{k,\underline{t}}$ in [Liu19a] is set to be $\mathfrak{Q}_{k,r,\underline{t}}\left(\frac{\boldsymbol{\mu}_H^{+,\,\text{up-right}}}{4\pi i}\right)\cdot f_k(s_0,\chi^r)$, so the formula above for the case (Sp) differs from the formula in *loc. cit.* by $2^{\sum t_j-nk}$.

Sketch of the proof. There are two steps in the proof. First, by viewing $f_{k,\underline{t}}(s_0,\chi^r)$ as a Siegel–Weil section for the Weil representation of $\mathrm{Sp}(4n)\times\mathrm{O}(2k)$ in case (Sp) and $\mathrm{U}(J_{n,n})\times\mathrm{U}(k)$ in case (U), we see that $f_{k,\underline{t}}(s_0,\chi^r)\big(\imath(g,1)\big)$ equals a matrix coefficient of the Weil representation, so the zeta integral is the integral of a matrix coefficient of the Weil representation against a matrix coefficient of the holomorphic discrete series $\mathcal{D}_{\underline{t}}$. By the results on the decomposition of the Weil representation [KV78], Harish-Chandra's formula on formal degrees of discrete series [HII08], and our specific choice of $f_{k,\underline{t}}(s_0,\chi^r)$, in case (Sp), we reduce the computation of $Z\left(f_{k,\underline{t}}\left(s,\chi^r\right), v^*_{\underline{t}}, v_{\underline{t}}\right)\big|_{s=s_0}$ to the computation of the integral

$$\int_{M_{n,2k}(\mathbb{R})} P^{\mathrm{hol,inv}}_{k,\underline{t}}\begin{pmatrix}x\\x\end{pmatrix}e^{-2\pi\mathrm{Tr}x{}^{\mathrm{t}}x}\,dx, \tag{2.3.1}$$

where $P^{\mathrm{hol,inv}}_{k,\underline{t}}$ is the unique polynomial on $M_{2n,2k}=M_{n,2k}\times M_{n,2k}$ satisfying:

(1) $\sum_{j=1}^{2k}\frac{\partial^2}{\partial x_{aj}\partial x_{bj}}P^{\mathrm{hol,inv}}_{k,\underline{t}}\begin{pmatrix}x\\y\end{pmatrix}=\sum_{j=1}^{2k}\frac{\partial^2}{\partial y_{aj}\partial y_{bj}}P^{\mathrm{hol,inv}}_{k,\underline{t}}\begin{pmatrix}x\\y\end{pmatrix}=0$ for all $1\le a,b\le n$,

(2) $P^{\mathrm{hol,inv}}_{k,\underline{t}}\begin{pmatrix}x\\y\end{pmatrix}=P^{\mathrm{hol,inv}}_{k,\underline{t}}\left(\begin{pmatrix}x\\y\end{pmatrix}h\right)$ for all $h\in\mathrm{O}(2k)$,

(3) $P^{\mathrm{hol,inv}}_{k,\underline{t}}$ is a highest weight vector of weight $(t_1-k,t_2-k,\dots,t_n-k),(t_1-k,t_2-k,\dots,t_n-k)$ for the action of $\mathrm{GL}(n)\times\mathrm{GL}(n)$ on $\mathbb{C}[M_{2n,2k}]$ by $(a_1,a_2)\cdot P\begin{pmatrix}x\\y\end{pmatrix}=P\begin{pmatrix}{}^{\mathrm{t}}a_1x\\{}^{\mathrm{t}}a_2y\end{pmatrix}$, and its evaluation at

$$\begin{array}{c c} & \begin{array}{cccc} n & k-n & n & k-n\end{array}\\ \begin{array}{c}n\\n\end{array} & \begin{pmatrix}\mathbf{1}_n & 0 & 0 & 0\\ 0 & 0 & \mathbf{1}_n & 0\end{pmatrix}\end{array}\begin{pmatrix}\mathbf{1}_k & \mathbf{1}_k\\ i\mathbf{1}_k & -i\mathbf{1}_k\end{pmatrix}^{-1} \tag{2.3.2}$$

equals 1.

The polynomials satisfying the first two conditions are called the pluri-harmonic polynomial and have been employed for studying the holomorphic differential operators in [Ibu99]. In case (U), we use a similar strategy but the Schrödinger model is inconvenient if $p\neq q$. We need to use the

Fock model and Bargmann transform. Then the computation of $Z\left(f_{k,\underline{t}}\left(s,\chi^r\right), v^*_{\underline{t}}, v_{\underline{t}}\right)\big|_{s=s_0}$ can be reduced to the computation of the integral

$$\int_{M_{n,k}(\mathbb{C})} P^{\mathrm{hol,inv}}_{k,r,\underline{t}}\begin{pmatrix} z \\ \overline{z}\end{pmatrix} e^{-\pi \operatorname{Tr}({}^{\mathrm{t}}\overline{z}z)} |dz d\overline{z}|, \tag{2.3.3}$$

where $P^{\mathrm{hol,inv}}_{k,r,\underline{t}}$ is the unique polynomial on $M_{2n,k} = M_{n,k}\times M_{N,k}$ satisfying:

(1) $\sum_{j=1}^{k} \frac{\partial^2}{\partial z_{1,aj}\partial z_{2,bj}} P^{\mathrm{hol,inv}}_{k,r,\underline{t}}\begin{pmatrix} z \\ w\end{pmatrix} = \sum_{j=1}^{k} \frac{\partial^2}{\partial w_{1,aj}\partial w_{2,bj}} P^{\mathrm{hol,inv}}_{k,r,\underline{t}}\begin{pmatrix} z \\ w\end{pmatrix}$ for all $1\leq a\leq p$, $1\leq b\leq q$, where write $z = \overset{k}{\begin{pmatrix} z_1 \\ z_2\end{pmatrix}}\begin{matrix} p \\ q\end{matrix}$, $w = \overset{k}{\begin{pmatrix} w_1 \\ w_2\end{pmatrix}}\begin{matrix} p \\ q\end{matrix}$,

(2) $P^{\mathrm{hol,inv}}_{k,r,\underline{t}}\begin{pmatrix} z \\ w\end{pmatrix} = P^{\mathrm{hol,inv}}_{k,r,\underline{t}}\left(\begin{pmatrix} z \\ w\end{pmatrix} h\right)$ for all $h\in \mathrm{U}(k)$,

(3) $P^{\mathrm{hol,inv}}_{k,r,\underline{t}}$ is a highest vector of weight $(\tau_1 - \frac{k+r}{2},\dots,\tau_p-\frac{k+r}{2})$, $(\nu_1+\frac{k-r}{2},\dots,\nu_q+\frac{k-r}{2})$, $(\tau_1 - \frac{k+r}{2},\dots,\tau_p-\frac{k+r}{2})$, $(\nu_1+\frac{k-r}{2},\dots,\nu_q+\frac{k-r}{2})$ for the action of $\mathrm{U}(p)\times U(q)\times \mathrm{U}(p)\times\mathrm{U}(q)$ on $\mathbb{C}[M_{2n,k}] = \mathbb{C}[M_{p,k}\times M_{q,k}\times M_{p,k}\times M_{q,k}]$ by $(a_1,b_1,a_2,b_2)\cdot P\begin{pmatrix} z_1 \\ z_2 \\ w_1 \\ w_2\end{pmatrix} = P\begin{pmatrix} {}^{\mathrm{t}}a_1 z_1 \\ b_1^{-1}z_2 \\ {}^{\mathrm{t}}a_2 w_1 \\ b_2^{-1} w_2\end{pmatrix}$, and its evaluation at

$$\begin{matrix} & p & k-n & q \\ p & & & \\ q & & & \\ p & & & \\ q & & & \end{matrix}\!\!\!\!\begin{pmatrix} \mathbf{1}_p & 0 & 0 \\ 0 & 0 & \mathbf{1}_q \\ \mathbf{1}_p & 0 & 0 \\ 0 & 0 & \mathbf{1}_q \end{pmatrix}$$

equals 1.

The second step is to compute the integrals in (2.3.1) and (2.3.3). However, the polynomials $P^{\mathrm{hol,inv}}_{k,\underline{t}}$ and $P^{\mathrm{hol,inv}}_{k,r,\underline{t}}$ are very difficult to write down explicitly. Let $\mathfrak{H}_{G,k}(\underline{t})\otimes\mathfrak{H}_{G,k}(\underline{t})$ denote the space of polynomials satisfying the conditions (1) and (3). The idea is to pick some other polynomial in $\mathfrak{H}_{G,k}(\underline{t})\otimes\mathfrak{H}_{G,k}(\underline{t})$ such that replacing $P^{\mathrm{hol,inv}}_{k,\underline{t}}$ and $P^{\mathrm{hol,inv}}_{k,r,\underline{t}}$ in (2.3.1) and (2.3.3) by the picked polynomial makes the integrals much easier to compute and one can relate the easy integrals to the original integrals. Our choice of such a polynomial is

$$\mathcal{I}_{k,\underline{t}}\left(\begin{pmatrix} x \\ y\end{pmatrix} = \overset{\begin{matrix} k & k\end{matrix}}{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2\end{pmatrix}}\begin{matrix} n \\ n\end{matrix}\right) = \mathfrak{Q}_{k,\underline{t}}\left({}^{\mathrm{t}}(x_1+ix_2)(y_1-iy_2)\right), \tag{Sp}$$

$$\mathcal{I}_{k,r,\underline{t}}\begin{pmatrix} z \\ w\end{pmatrix} = \mathfrak{Q}_{k,r,\underline{t}}\left({}^{\mathrm{t}}zw\right), \tag{U}$$

The integral for $\mathcal{I}_{k,\underline{t}}$ (resp. $\mathcal{I}_{k,r,\underline{t}}$) is easy to compute because of its invariance under the left translation by $\left\{\begin{pmatrix} u & \\ & \overline{u}\end{pmatrix} : u\in\mathrm{U}(n)\right\}$ (resp. $\left\{\begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \overline{u_1} & \\ & & & \overline{u_2}\end{pmatrix} : u_1\in\mathrm{U}(p),\ u_2\in\mathrm{U}(q)\right\}$), and differs from the integral for $P_{k,\underline{t}}$ (resp. $P_{k,r,\underline{t}}$) by

$$\frac{\dim \lambda_{k,\underline{t}}}{\dim\left(\mathrm{GL}(n), \underline{t}\right)} \qquad\qquad (\text{resp. } \frac{\dim \lambda_{k,r,\underline{t}}}{\dim\left(\mathrm{GL}(p)\times \mathrm{GL}(q), \underline{t}\right)}),$$

where $\lambda_{k,\underline{t}}$ (resp. $\lambda_{k,r,\underline{t}}$) is the theta lift of $\mathcal{D}_{\underline{t}}$ to $\mathrm{O}(2k)$ (resp. $\mathrm{U}(k)$). ☐

With the formulas above, one can verify that the interpolation results of the p-adic L-functions constructed in [Liu16, EHLS20] satisfy the conjecture of Coates and Perrin-Riou. By using the formulas above and the functional equations of the doubling local zeta integrals, one can also deduce the formulas for $Z\left(f_{k,\underline{t}}\left(s,\chi_{\mathrm{ac}}^r\right), v_{\underline{t}}^*, v_{\underline{t}}\right)\big|_{s=-s_0}$ [LR20, EL].

When the holomorphic discrete series is of scalar weight, a choice of archimedean sections is made in [Shi00] and in [BS00] (different from each other) and the corresponding zeta integrals are computed. In the unitary case, when either $(\tau_1,\ldots,\tau_p)$ or $(\nu_1,\ldots,\nu_q)$ is scalar, the zeta integral is computed in [Gar08] for a special choice of archimedean sections (different from ours here). When $q=1$ and the section is taken to be a matrix coefficient of a non-holomorphic discrete series of another unitary group of the same size, the zeta integral is computed in [Liu15, LL16]. In the symplectic case when the t_j's are of the same parity, the Lie algebra operators which lower the K_G-types are employed to choose the archimedean sections and the corresponding zeta integrals are computed in [PSS18].

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