A NOTE ON A QUESTION OF KOMJÁTH

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Abstract. We show that every graph of finite coloring number on a set of reals has an everywhere non-meager independent subset.

1. Introduction

Let $G = (V, E)$ be a graph. The **coloring number** of G is the least κ such that there is a well-ordering $\prec$ of V satisfying the following: For every $x \in V$, $|\{y \in V : y \prec x \text{ and } xEy\}| < \kappa$. The following fact was proved in [5].

Fact 1.1. *Let $G = (X, E)$ be a graph where $X \subseteq [0, 1]$. If every connected component of G is countable, then there exists an E-independent $Y \subseteq X$ such that $\mu^\star(Y) = \mu^\star(X)$.*

Here, $\mu^\star$ denotes Lebesgue outer measure on $\mathbb{R}$. A similar result holds in the case of category. P. Komjáth asked if Fact 1.1 could be generalized to graphs of countable coloring number.

Question 1.2. *Let (X, E) be a graph of countable coloring number where $X \subseteq [0, 1]$. Must there exist an E-independent $Y \subseteq X$ such that $\mu^\star(Y) = \mu^\star(X)$?*

Recall that for $Y \subseteq X \subseteq [0, 1]$, we say that Y is everywhere non-meager in X iff for every Borel $B \subseteq [0, 1]$, if $B \cap X$ is non-meager, then $B \cap Y$ is non-meager (equivalently, for every rational interval $J \subseteq [0, 1]$, if $J \cap X$ is non-meager, then $J \cap Y$ is non-meager). The category analogue of Question 1.2 would be the following.

Question 1.3. *Let (X, E) be a graph of countable coloring number where $X \subseteq [0, 1]$. Must there exist an E-independent $Y \subseteq X$ such that Y is everywhere non-meager in X?*

In this note, we show that the category version has a positive answer for graphs of **finite** coloring number. We also show that a counterexample to either one of Questions 1.2 and 1.3 produces $\aleph_1$-saturated ideals. For similar problems and results see [3, 4, 5]. For some background on generic ultrapowers, see [1].

2. Category

Given a graph $G = (V, E)$ and $X \subseteq V$, we denote the set of neighbors of X in G by $N_G(X) = \{v \in V : (\exists x \in X)(xEv)\}$.

Lemma 2.1. *Suppose $G = (X, E)$ is a graph where $X \subseteq [0, 1]$. Suppose for every $X' \subseteq X$ and for every rational interval J in which X' is non-meager, there exists a non-meager $Y \subseteq X' \cap J$ such that Y is E-independent and $X' \setminus N_G(Y)$ is everywhere non-meager in X'. Then there exists $Y \subseteq X$ such that Y is E-independent and everywhere non-meager in X.*

Proof. Let $\langle J_n : n < \omega\rangle$ list all rational intervals in which X is non-meager. Using the assumption, we can inductively construct $\langle Y_n : n < \omega\rangle$ such that for every $n < \omega$,

(1) $Y_n \subseteq X \cap J_n$,
(2) $\bigcup_{k\leq n} Y_k$ is E-independent,
(3) $J_n \cap Y_n$ is non-meager and
(4) $X \setminus N_G(\bigcup\{Y_k : k \leq n\})$ is everywhere non-meager in X.

At stage n, by applying the hypothesis to $X' = X \setminus N_G(\bigcup\{Y_k : k < n\})$, choose a non-meager $Y_n \subseteq X' \cap J_n$ such that Y_n is E-independent and $X' \setminus N_G(Y_n)$ is everywhere non-meager in X'. Having constructed $\langle Y_n : n < \omega\rangle$, put $Y = \bigcup_{n<\omega} Y_n$. It is clear that Y is an E-independent and everywhere non-meager in X. □

Fact 2.2 (Gitik-Shelah). *Suppose that the meager ideal $\mathcal{I}$ restricted to some non-meager set X is $\aleph_1$-saturated. Then forcing with $\mathcal{P}(X)/\mathcal{I}$ makes some non-meager subset of X meager. In particular, it cannot be isomorphic to Cohen forcing.*

Proof. Fix $Z \subseteq X$ non-meager and a regular uncountable κ such that for every non-meager $Y \subseteq Z$, the additivity of $\mathcal{I} \restriction Y$ is κ. Put $\mathbb{P} = \mathcal{P}(Z)/\mathcal{I}$ and note that $\mathbb{P}$ is ccc. It suffices to show that $V^{\mathbb{P}} \models Z$ is meager. Let $Z = \bigcup\{Z_i : i < \kappa\}$ where Z_i's are meager and increasing with i. Let G be $\mathbb{P}$-generic over V and $j : V \to M \subseteq V[G]$, the generic ultrapower embedding with critical point κ. Since $V \models (\forall j < \kappa)(\bigcup\{Z_i : i < j\}$ is meager) and $j(\kappa) > \kappa$, $M \models \bigcup\{j(Z_i) : i < \kappa\}$ is meager. Since each $Z_i \subseteq j(Z_i)$, it follows that Z is meager in $V[G]$. □

Lemma 2.3. *For every $f : X \to Y$ where $X, Y \subseteq [0, 1]$ and X is non-meager, there exists $X' \subseteq X$ such that X' is non-meager and $Y \setminus f[X']$ is everywhere non-meager in Y.*

Proof. Use the fact that Y can be partitioned into two everywhere non-meager subsets - See Lusin [6] or use Fact 2.2. □

Lemma 2.4. *For every $f : X \to Y$, where $X, Y \subseteq [0, 1]$ and Y is non-meager, there exists $Y' \subseteq Y$ such that Y' is non-meager and $X \setminus f^{-1}[Y']$ is everywhere non-meager in X.*

Proof. Let $\mathcal{F}$ be a maximal family of pairwise disjoint meager subsets W of Y such that $X \setminus f^{-1}[W]$ is not everywhere non-meager in X. It is easy to see that $\mathcal{F}$ is countable and so $W_0 = \bigcup \mathcal{F}$ is meager. Let $Y_1 = Y \setminus W_0$. Note that for every meager $W \subseteq Y_1$, $X \setminus f^{-1}[W]$ is everywhere non-meager in X.

Towards a contradiction, assume that for every non-meager $Y' \subseteq Y_1$, $X \setminus f^{-1}[Y']$ is not everywhere non-meager in X. Then the meager ideal $\mathcal{I}$ restricted to Y_1 must be $\aleph_1$-saturated. It follows that $\mathbb{B} = \mathcal{P}(Y_1)/\mathcal{I}$ is a complete boolean algebra that satisfies ccc.

For each rational interval J in which X is non-meager, define $\mathbf{a}_J$ to be the infimum in $\mathbb{B}$ of all conditions $[Y']$ such that $Y' \subseteq Y_1$ and $(X \setminus f^{-1}[Y']) \cap J$ is meager. We claim that each $\mathbf{a}_J > 0_{\mathbb{B}}$. Suppose not. As $\mathbb{B}$ is ccc, there exists a countable family $\{Y'_n \subseteq Y_1 : n < \omega\}$ such that for each n, $(X \setminus f^{-1}[Y'_n]) \cap J$ is meager and $W = \bigcap\{Y'_n : n < \omega\}$ is meager. But now $X \setminus f^{-1}[W]$ is not everywhere non-meager in X while $W \subseteq Y_1$ is meager which is impossible. Therefore $\mathbf{a}_J > 0_{\mathbb{B}}$.

It follows that the family $\{\mathbf{a}_J : J$ rational interval and $X \cap J$ non-meager$\}$ is dense in $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$. So $\mathbb{B}$ must be isomorphic to Cohen forcing. But this contradicts Fact 2.2. □

Lemma 2.5. *Suppose* $Y, \langle (k_n, Y_n) : n < \omega \rangle$ *and* $\langle f_{n,k} : n < \omega, k < k_n \rangle$ *satisfy the following.*

(a) $1 \leq k_n < \omega$.
(b) $Y, Y_n \subseteq [0,1]$ *and* Y *is non-meager.*
(c) $f_{n,k} : Y_n \to Y$.

Suppose further that either $\{k_n : n < \omega\}$ *is bounded in* ω *or the meager ideal restricted to* Y *is not* $\aleph_1$*-saturated. Then there exists* $Y' \subseteq Y$ *such that* Y' *is non-meager and for every* $n < \omega$, $Y_n \setminus \bigcup\{f_{n,k}^{-1}[Y'] : k < k_n\}$ *is everywhere non-meager in* Y_n.

Proof. Suppose no such Y' exists. We first show that the meager ideal restricted to Y must be $\aleph_1$-saturated. Suppose not and let $\{A_i : i < \omega_1\}$ be a family of pairwise disjoint non-meager subsets of Y. For each $i < \omega_1$, choose $n_i < \omega$ and a rational interval J_i such that $Y_{n_i} \cap J_i$ is non-meager and $(Y_{n_i} \setminus \bigcup\{f_{n_i,k}^{-1}[A_i] : k < k_{n_i}\}) \cap J_i$ is meager. By shrinking $\{A_i : i < \omega_1\}$, we can assume that $n_i = n_\star$ and $J_i = J_\star$ do not depend on i. Let W be a meager set such that for each $i < \omega$, $(Y_{n_\star} \setminus \bigcup\{f_{n_\star,k}^{-1}[A_i] : k < k_{n_\star}\}) \cap J$ is contained in W. Fix $x \in (Y_{n_\star} \setminus W) \cap J$. Then $x \in \bigcup\{f_{n_\star,k}^{-1}[A_i] : k < k_{n_\star}\}$ for every $i < \omega$. It follows that for some $i < j < \omega$ and $k < k_{n_\star}$, we have $x \in f_{n_\star,k}^{-1}[A_i] \cap f_{n_\star,k}^{-1}[A_j]$ which is impossible since $A_i \cap A_j = \emptyset$.

Using the hypothesis, we can fix $K < \omega$ such that each $k_n \leq K$. By replacing each Y_n with a similar copy of Y_n, we can assume that there is a disjoint family $\{J_n : n < \omega\}$ of intervals such that each $Y_n \subseteq J_n$. Put $X = \bigcup\{Y_n : n < \omega\}$ and define $F_k : X \to Y$ for $k < K$, such that for every $n < \omega$ and $k < k_n$, $F_k \restriction Y_n = f_{n,k}$. Applying Lemma 2.4 K-times, choose $Y' \subseteq Y$ such that Y' is non-meager and $X \setminus \bigcup\{F_k^{-1}[Y'] : k < K\}$ is everywhere non-meager in X. Since Y_n's are separated by pairwise disjoint intervals, it also follows that for every $n < \omega$, $Y_n \setminus \bigcup\{f_{n,k}^{-1}[Y'] : k < k_n\}$ is everywhere non-meager in Y_n. But this contradicts our assumption that no such Y' exists. □

Now suppose $X \subseteq [0,1]$ and $G = (X, E)$ has coloring number $\theta \leq \aleph_0$. Suppose either $\theta < \aleph_0$ or the meager ideal restricted to X is nowhere $\aleph_1$-saturated. We'll show that G satisfies the hypothesis of Lemma 2.1. Let J be a rational interval and $X' \subseteq X$ be such that $X' \cap J$ is non-meager. We need to find a non-meager $Y \subseteq X' \cap J$ such that Y is E-independent and $X' \setminus N_G(Y)$ is everywhere non-meager in X'. We can clearly assume that $X' = X$.

Fix a well-ordering $\prec$ of X such that for every $x \in X$, $|\{y \in X : y \prec x \wedge yEx\}| < \theta$. Let $F : X \to [X]^{<\theta}$ be defined by $F(x) = \{y \in X : y \prec x \wedge yEx\}$. Choose a partition $\{X_m : m < \theta\}$ of X such that each X_m is E-independent. Let $\{Y_n : n < \theta^2\}$ be a refinement of $\{X_m : m < \theta\}$ such that for each n, $|F(x)| = k_n$ does not depend on $x \in Y_n$. Choose $f_{n,k} : Y_n \to X$, for $n < \theta^2$ and $k < k_n$, such that for every $x \in Y_n$, $F(x) = \{f_{n,k}(x) : k < k_n\}$.

One of the Y_n's is non-meager in J, say Y_0. Since either $\mathsf{sup}\{k_n : n < \omega\} < \theta < \aleph_0$ or the meager ideal restricted to X is nowhere $\aleph_1$-saturated, we can use Lemmas 2.3 and 2.5 to obtain $Y' \subseteq Y_0 \cap J$ such that Y' is non-meager and for every $1 \leq n < \theta^2$, $Y_n \setminus \bigcup(\{f_{n,k}^{-1}[Y'] : k < k_n\} \cup \{f_{0,k}[Y'] : k < k_0\})$ is everywhere non-meager in Y_n. Since Y_0 is E-independent, $X \setminus N_G(Y')$ is also everywhere non-meager in X. Applying Lemma 2.1, it follows that X has an everywhere non-meager E-independent subset. □

Since the rational distance graph $E = \{\{x, y\} : x, y \in \mathbb{R}^2, ||x - y|| \in \mathbb{Q}\}$ in plane has countable coloring number (see Theorem 2.3 in [3]), we get the following: Suppose X is a subset of plane such that the meager ideal restricted to X is nowhere $\aleph_1$-saturated. Then X has an everywhere non-meager subset that avoids rational distances.

3. Measure

By modifying the arguments of the previous section, we can show that unless the null ideal restricted to some non-null subset of X is $\aleph_1$-saturated, Question 1.2 has a positive answer. Let us indicate the main steps.

Lemma 3.1. *Suppose $X \subseteq [0, 1]$ and $G = (X, E)$ is a graph. Suppose for every $X' \subseteq X$, there exists $Y \subseteq X'$ such that Y is E-independent, $\mu^\star(Y) \geq 0.5\mu^\star(X')$ and $\mu^\star(X' \setminus N_G(Y)) = \mu^\star(X')$. Then there exists $Y \subseteq X$ such that Y is E-independent and $\mu^\star(Y) = \mu^\star(X)$.*

Proof. For $A \subseteq [0, 1]$, let $\mathsf{env}(A)$ be a G_δ set containing A such that $\mu^\star(A) = \mu(\mathsf{env}(A))$. Construct $\langle Y_n : n < \omega\rangle$ as follows.

(1) $Y_0 \subseteq X$, $\mu^\star(Y_0) \geq 0.5\mu^\star(X)$, Y_0 is E-independent and $\mu^\star(X \setminus N_G(Y_0)\}) = \mu^\star(X)$.
(2) Suppose $\langle Y_k : k \leq n\rangle$ has been constructed. Let $Z_n = \bigcup_{k\leq n} Y_k$. Define $W_n = X \setminus (N_G(Z_n) \cup \bigcup_{k\leq n} \mathsf{env}(Y_k))$. Choose $Y_{n+1} \subseteq W_n$ such that
   (a) $\mu^\star(Y_{n+1}) \geq 0.5\mu^\star(W_n)$,
   (b) Y_{n+1} is E-independent and
   (c) $\mu^\star(W_n \setminus N_G(Y_{n+1})) = \mu^\star(W_n)$.

Now it is easy to check that $Y = \bigcup\{Y_n : n < \omega\}$ is E-independent and $\mu^\star(Y) = \mu^\star(X)$. □

The following can be proved exactly like Fact 2.2.

Fact 3.2 (Gitik-Shelah)**.** *Suppose that the null ideal $\mathcal{I}$ restricted to some non-null set X is $\aleph_1$-saturated. Then forcing with $\mathcal{P}(X)/\mathcal{I}$ makes some non-null subset of X null. In particular, it cannot be isomorphic to random forcing.*

Lemma 3.3. *For every $f : X \to Y$, where $X, Y \subseteq [0, 1]$ and $\mu^\star(X) > 0$, for every $\varepsilon > 0$, there exists $Z \subseteq X$ such that $\mu^\star(Z) \geq (1-\varepsilon)\mu^\star(X)$ and $\mu^\star(Y \setminus f[Z]) = \mu^\star(Y)$.*

Proof. By a result of Lusin [6] or by using Fact 3.2, we can decompose $Y = Y' \sqcup Y''$ such that $\mu^\star(Y') = \mu^\star(Y'') = \mu^\star(Y)$. Note that one of the sets $f^{-1}[Y']$, $f^{-1}[Y'']$ has outer measure $\geq 0.5\mu^\star(X)$. It follows that there exists $Z_0 \subseteq X$ such that $\mu^\star(Z_0) \geq 0.5\mu^\star(X)$ and $\mu^\star(Y \setminus f[Z_0]) = \mu^\star(Y)$. Put $X' = X \setminus \mathsf{env}(Z_0)$, $Y' = Y \setminus f[Z_0]$ and repeat this argument for $f \restriction X' : X' \to Y'$ to obtain $Z_0 \subseteq Z_1 \subseteq X$ such that $\mu^\star(Z_1) \geq 0.75\mu^\star(X)$ and $\mu^\star(Y \setminus f[Z_1]) = \mu^\star(Y)$. Repeating this n-times where $\varepsilon 2^n > 1$, we get the required $Z \subseteq X$. □

Lemma 3.4. *Suppose $Y, \langle (k_n, Y_n) : n < \omega\rangle$, $\langle f_{n,k} : n < \omega, k < k_n\rangle$ satisfy the following.*

(a) $1 \leq k_n < \omega$.
(b) $Y, Y_n \subseteq [0, 1]$.
(c) *Y is non-null.*
(d) $f_{n,k} : Y_n \to Y$.

Suppose further that for every Borel set B satisfying $\mu^\star(B \cap Y) > 0$, the null ideal restricted to $Y \cap B$ is not $\aleph_1$-saturated. Then there exists $Y' \subseteq Y$ such that $\mu^\star(Y') = \mu^\star(Y)$ and for every $n < \omega$, $\mu^\star(Y_n \setminus \bigcup\{f_{n,k}^{-1}[Y'] : k < k_n\}) = \mu^\star(Y_n)$.

Proof. First observe that there is an uncountable partition $\{A_i : i < \omega_1\}$ of Y such that for each $i < \omega_1$, $\mu^\star(A_i) = \mu^\star(Y)$. This follows from the following.

Claim 3.5. *For every Borel $B \subseteq \mathsf{env}(Y)$ of positive measure, there exist a Borel $B' \subseteq B$ of positive measure and a partition $\{Z_\alpha : \alpha < \omega_1\}$ of $Y \cap B'$ such that for every $\alpha < \omega$, $\mathsf{env}(Z_\alpha) = B'$.*

Proof. Since $\mu^\star(Y \cap B) = \mu(B) > 0$, we can fix a partition $\{W_i : i < \omega_1\}$ of $Y \cap B$ into sets of positive outer measure. Since random forcing is ccc, we can choose $\langle i_\alpha : \alpha < \omega_1\rangle$ such that i_α's are strictly increasing and cofinal in ω_1 and $\mathsf{env}\left(\bigcup\{W_j : i_\alpha < j < i_{\alpha+1}\}\right) = B'$ does not depend on $\alpha < \omega_1$. For each $\alpha < \omega_1$, define $Z_\alpha = B' \cap \bigcup\{W_j : i_\alpha < j < i_{\alpha+1}\}$. Then B', $\{Z_\alpha : \alpha < \omega_1\}$ are as required. □

Towards a contradiction, assume that for every $i < \omega_1$, $Y' = A_i$ does not satisfy the conclusion of the lemma. By passing to an uncountable subfamily of $\{A_i : i < \omega_1\}$, we can assume that there are $n_\star < \omega$ and a rational interval $J_\star$ such that for every $i < \omega_1$, $\mu^\star(Y_{n_\star} \cap J_\star) > (1 - 2^{-4k_{n_\star}})\mu(J_\star)$ and $\mu^\star\left((Y_{n_\star} \cap J_\star) \setminus \bigcup\{f_{n_\star,k}^{-1}[A_i] : k < k_{n_\star}\}\right) < 2^{-4k_{n_\star}}\mu(J_\star)$. It follows that

$$\bigcap_{i \le k_{n_\star}} \bigcup_{k < k_{n_\star}} f_{n_\star,k}^{-1}[A_i]$$

is nonempty and we get a contradiction as before. □

Now suppose $X \subseteq [0,1]$, $G = (X, E)$ has countable coloring number and the null ideal restricted to X is nowhere $\aleph_1$-saturated. We'll show that G satisfies the hypothesis of Lemma 3.1. Let $X' \subseteq X$. We need to find $Y \subseteq X'$ such that Y' is E-independent and $\mu^\star(Y) \ge \mu^\star(X')$. As before, we can assume that $X' = X$.

Fix a well-ordering $\prec$ of X such that for every $x \in X$, $\{y \in X : y \prec x \wedge yEx\}$ is finite. Let $F : X \to [X]^{<\aleph_0}$ be defined by $F(x) = \{y \in X : y \prec x \wedge yEx\}$. Choose a partition $\{X_m : m < \omega\}$ of X such that each X_m is E-independent. Let $\{Y_n : n < \omega\}$ be a refinement of $\{X_m : m < \omega\}$ such that for each n, $|F(x)| = k_n$ does not depend on $x \in Y_n$. Choose $f_{n,k} : Y_n \to X$, for $n < \omega$ and $k < k_n$, such that for every $x \in Y_n$, $F(x) = \{f_{n,k}(x) : k < k_n\}$.

Fix $n_\star$ such that $\mu^\star(\bigcup\{Y_n : n < n_\star\}) \ge 0.6\mu^\star(X)$. By applying Lemma 3.4 $n_\star$-times, we can find $\langle Y'_n : n < n_\star\rangle$ such that the following hold.

(1) For every $n < n_\star$, $\mu^\star(Y'_n) = \mu^\star(Y_n)$.
(2) Whenever $m, n < n_\star$ and $k < k_m$, we have $Y'_m \cap f_{m,k}^{-1}[Y'_n] = \emptyset$.
(3) For every $n_\star \le m < \omega$, $\mu^\star(Y_m) = \mu^\star(Y_m \setminus \bigcup\{f_{m,k}^{-1}[Y'_n] : n < n_\star \wedge k < k_m\})$.

Put $W = \bigcup\{Y'_n : n < n_\star\}$ and observe that W is E-independent and $\mu^\star(W) \ge 0.6\mu^\star(X)$. Put $X'' = \bigcup_{m \ge n_\star} Y_m \setminus \bigcup\{f_{m,k}^{-1}[Y'_n] : n < n_\star \wedge k < k_m\}$. Let $\mathcal{F}$ be a finite family of functions from W to X such that for every $n < n_\star$ and $k < k_n$, there exists $f \in \mathcal{F}$ such that $f \restriction Y'_n = f_{n,k} \restriction Y'_n$. Applying Lemma 3.3 $|\mathcal{F}|$-times, we can obtain $Y \subseteq W$ such that $\mu^\star(Y) \ge 0.9\mu^\star(W)$ and $\mu^\star(X'') = \mu^\star(X'' \setminus \bigcup\{f[Y] : f \in \mathcal{F}\})$. It follows now that $\mu^\star(X \setminus N_G(Y)) = \mu^\star(X)$ and hence Y is as required. □

We do not know if the measure analogue of Lemma 2.4 holds.

Question 3.6. *Suppose $f : X \to Y$ where $X, Y \subseteq [0,1]$ and Y is non-null. Must there exist $Y' \subseteq Y$ non-null such that $\mu^\star(X \setminus f^{-1}[Y']) = \mu^\star(X)$?*

A special case of this (Fact 1.1) was shown in [5] under the additional assumption that f is countable-to-one.

References

[1] M. Foreman, Ideals and generic elementary embeddings, Handbook of set theory Vol. 2 (M. Foreman, A. Kanamori, ed.), Springer 2010, 885-1147

[2] M. Gitik and S. Shelah, Forcing with ideals and simple forcing notions, Israel J. Math 68 1989, 129-160

[3] P. Komjáth, Set theoretic constructions in Euclidean spaces, New Trends in Discrete and Computational Geometry (J. Pach, ed.), Springer, 1993, 303-325

[4] A. Kumar, Avoiding rational distances, Real Analysis Exchange, Vol. 38(2) 2012/13, 493-498

[5] A. Kumar, S. Shelah, A transversal of full outer measure, Advances in Mathematics, Vol. 321 (2017), 475-485

[6] N. Lusin, Sur la décomposition des ensembles, C. R. Acad. Sci. Paris 198 1934, 1671-1674

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