REMARKS ON STRICHARTZ ESTIMATES FOR SCHRÖDINGER EQUATIONS WITH SLOWLY DECAYING POSITIVE POTENTIALS

HARUYA MIZUTANI

Abstract. We discuss a recent progress [28] concerning Strichartz estimates for Schrödinger equations with real-valued slowly decaying positive potentials. Our admissible class of potentials particularly includes the positive Coulomb potential in three and higher space dimensions.

1. Introduction

This is a survey article based on a recent result [28] of the author concerning the (global-in-time) Strichartz estimates for the Schrödinger equation:

$$i\partial_t u(t,x) = Hu(t,x) + F(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n; \quad u|_{t=0} = u_0 \tag{1.1}$$

where $u_0 = u_0(x)$ and $F = F(t,x)$ are given data and

$$H = -\Delta + V(x)$$

is a Schrödinger operator with Δ being the Laplacian and $V(x)$ being a real-valued potential decaying at infinity. A typical example we have in mind is the repulsive Coulomb potential and its smooth approximation of the forms

$$V(x) = Z|x|^{-\mu} \quad or \quad Z\langle x\rangle^{-\mu}$$

where $Z > 0$, $\mu \in (0,2)$ and $\langle x\rangle = \sqrt{1+|x|^2}$.

The Strichartz estimate is a family of space-time inequalities for the solution $u = u(t,x)$ to (1.1) of the form

$$\|u\|_{L_t^{p_1}L_x^{q_1}} \lesssim \|u_0\|_{L_x^2} + \|F\|_{L_t^{p_2'}L_x^{q_2'}}, \tag{1.2}$$

where (p_j, q_j), $j = 1,2$, satisfy the following admissible condition:

$$p, q \ge 2, \quad 2/p = n(1/2 - 1/q), \quad (n,p,q) \neq (2,2,\infty). \tag{1.3}$$

Here $L_x^q = L^q(\mathbb{R}^n)$, $L_t^p = L^p(\mathbb{R})$, $L_t^pL_x^q = L^p(\mathbb{R}; L^q(\mathbb{R}^n))$ and $p' = p/(p-1)$ is the Hölder conjugate of p. The special case with $(p_1,q_1) = (p_2,q_2) = (2, \frac{2n}{n-2})$ in dimension $n \ge 3$ is called the (double) endpoint estimate. Note that the endpoint estimate, combined with the trivial one for $(p_1,q_1) = (p_2,q_2) = (\infty, 2)$ and the complex interpolation, implies (1.2) for all admissible pairs.

Before stating the main result, we first give a brief summary of the existing literature. The Strichartz estimate (1.2) for the free case $H = -\Delta$ has been established by [38, 14, 40, 23] and is known to be one of fundamental tools in the study of the nonlinear Schrödinger equation (NLS):

$$i\partial_t u + \Delta u = \lambda |u|^{p-1} u.$$

We refer to the textbook [39] for applications to NLS. Since then, the Strichartz estimate has been extensively studied by many authors and extended to various settings. In particular, if $n \geq 3$ and the real-valued potential V satisfies the very short-range condition

$$V \in L^{n/2}(\mathbb{R}^n)$$

and $H = -\Delta + V$ has neither zero eigenvalue nor zero resonance (see [27, Section 2] for the definition of zero resonance under the condition $V \in L^{n/2}$), then the continuous part $P_c(H)u$ of the solution u to (1.1) satisfies (1.2) for all admissible pairs ([34, 15, 2, 27]). Note that, in case of the point-wisely decaying potential

$$|V(x)| \lesssim \langle x \rangle^{-\mu},$$

the above very short-range condition corresponds to the condition $\mu > 2$. In contrast with the very short-range case, there is a counterexample if $\mu < 2$. Precisely, it wad proved in [16] that if $V \in C^3(\mathbb{R}^n \setminus \{0\}; \mathbb{R})$ satisfies

$$V(x) = |x|^{-\mu} U(\theta), \quad \theta = x/|x|, \quad \mu \in [0, 2),$$

and U has a non-degenerate minimum point so that $\min_{S^{n-1}} U = 0$ then, for any admissible pair, (global-in-time) Strichartz estimates cannot hold in general. In the critical case $V(x) = O(\langle x \rangle^{-2})$, the Strichartz estimate is also known to hold under some repulsive conditions and smallness of the negative part V_- of the potential. A typical example satisfying such conditions is the inverse-square potential $V(x) = a|x|^{-2}$ with the subcritical constant $a > -\frac{(n-2)^2}{4}$ (see [8, 9, 3, 26] and references therein). It is worth noting that there is no existing positive result for the slowly decaying case

$$|V(x)| \sim \langle x \rangle^{-\mu}, \quad \mu \in (0, 2).$$

The Strichartz estimates have been also extensively established for more general operators than the Schrödinger operator with a scalar potential on the Euclidean space, e.g.,

- Schrödinger operator with short-range magnetic potentials ([13, 25]);
- Laplace-Beltrami operator on the asymptotically conic or hyperbolic manifolds under the nontrapping or moderate trapping conditions ([25, 19, 11, 4]);
- Schrödinger operator on a star graph or tree ([17, 1]);
- Fractional and higher-order Schrödinger operators or more general elliptic operators ([24, 18, 29, 30]).

2. Main result

Now we state the main results in the paper [28].

Theorem 2.1 (Smooth potential)**.** *Let $n \ge 2$ and $\mu \in (0,2)$. Suppose that V is a real-valued smooth function satisfying the following conditions:*

(H1) $|\partial_x^\alpha V(x)| \le C_\alpha(1+|x|)^{-\mu-|\alpha|}$ *on $\mathbb{R}^n$ for any multi index α;*
(H2) $V(x) \ge C_1(1+|x|)^{-\mu}$ *on $\mathbb{R}^n$ with some $C_1 > 0$;*
(H3) $-x\cdot\nabla V(x) \ge C_2(1+|x|)^{-\mu}$ *for $|x| \ge R_0$ with some $C_2, R_0 > 0$.*

Then the solution u to (1.1) *satisfies* (1.2) *for any admissible pairs.*

A typical example of V satisfying the above (H1)–(H3) is

$$V(x) = Z\langle x\rangle^{-\mu}, \quad Z > 0.$$

Theorem 2.2 (Singular potential)**.** *Let $n \ge 3$, $Z > 0$ and $\mu \in (0,2)$. Suppose that $V_S \in C^\infty(\mathbb{R}^n;\mathbb{R})$ satisfies*

$$|\partial_x^\alpha V_S(x)| \le C_\alpha(1+|x|)^{-1-\mu-|\alpha|}.$$

Let $V(x) = -\Delta + Z|x|^{-\mu} + \varepsilon V_S(x)$. Then there exists $\varepsilon_* = \varepsilon_*(Z,\mu,V_S) > 0$ such that for all $\varepsilon \in [0,\varepsilon_*)$, the solution u to (1.1) *satisfies* (1.2) *for any admissible pairs.*

Remark 2.3. We give some remarks on these theorems:

- In both cases of Theorems 2.1 and 2.2, H is purely absolutely continuous and $P_c(H) = \mathrm{Id}$.
- One can choose $\varepsilon_* = \min(Z/M_0,\ \mu Z/M_1)$ in Theorem 2.2, where $M_\ell = \||x|^\mu(x\cdot\nabla)^\ell V_S\|_{L^\infty}$.
- These theorems do not contradict with the counterexample due to [16] in the previous section. Indeed, both of conditions (H1)–(H3) and conditions in Theorem 2.2 do not intersect with one for the counterexample.
- The restriction $n \ge 3$ on the space dimension in Theorem 2.2 is due to the use of the following $L_t^2L_x^2$-estimate with a singular weight

$$\|\chi(x)|x|^{-\mu/2}e^{-it(-\Delta+\tilde V)}u_0\|_{L^2(\mathbb{R}^{1+n})} \le C\|u_0\|_{L^2(\mathbb{R}^n)}$$

for a smooth approximation $\tilde V$ of the potential V in Theorem 2.2, where $\tilde V$ satisfies (H1)–(H3) and $\chi \in C_0^\infty(\mathbb{R}^n)$. This estimate immediately follows from the endpoint Strichartz estimate in Theorem 2.1 and Hölder's inequality if $n \ge 3$ since $\chi(x)|x|^{-\mu/2} \in L^n$, but this is not the case if $n = 2$ since the endpoint Strichartz estimate cannot hold in two space dimensions. However, such a restriction seems to be not essential. it would be interesting to investigate whether Theorem 2.2 (with $p > 2$) also holds in two space dimensions or not. It is also an interesting question if Theorem 2.1 holds in the one space dimension in which case our proof does not work.

We conclude this section with a simple application of the above theorems to nonlinear scattering theory. Let $n \ge 3$ and V satisfy the assumption in Theorem 2.1 or 2.2. Consider the mass-critical NLS with the potential V:

$$(i\partial_t + \Delta - V)v = \sigma|v|^{4/n}v, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n; \quad v|_{t=0} = v_0. \tag{2.1}$$

Corollary 2.4. *Let $\sigma \in \mathbb{R}$. Then for any $v_0 \in L^2$ with sufficiently small L^2-norm $\|v_0\|_{L^2} \ll 1$, (2.1) admits a unique global mild solution*

$$v \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^{2+4/n}(\mathbb{R}^{1+n}).$$

Moreover, if $1 < \mu < 2$ then there exist unique $v_\pm \in L^2$ such that

$$\lim_{t\to\pm\infty} \|v(t) - e^{it\Delta}v_\pm\|_{L^2} = 0;$$

if $0 < \mu < 1$ then there exist unique $v_\pm \in L^2$ such that

$$\lim_{t\to\pm\infty} \|v(t) - e^{-iS(t,D)}v_\pm\|_{L^2} = 0,$$

where $S(t,D) = \mathcal{F}^{-1}S(t,\xi)\mathcal{F}$ is a Fourier multiplier by an approximate solution to the Hamilton-Jacobi equation

$$\partial_t S(t,\xi) = |\xi|^2 + V(\nabla_\xi S(t,\xi)).$$

The unique existence of the global solution v and the scattering of v to a linear solution $e^{-itH}\widetilde{v}_\pm$ can be proved by a standard method (see [39]) by means of Theorems 2.1 and 2.2. If $\mu > 1$ then V is of short-range type and the scattering of the linear solution $e^{-itH}\widetilde{v}_\pm$ to a free solution $e^{it\Delta}v_\pm$ is nothing but the asymptotic completeness of the wave operator

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{it\Delta}$$

which is well-known (see [33]). When $\mu \in (0,1)$ the above modified scattering result follows from the asymptotic completeness of the modified wave operator

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-iS(t,D)}$$

(see e.g. [12]).

3. Outline of the proof

In this section we give an outline of the proof of Theorem 2.1. We use a similar method based on the microlocal analysis as that in [4] where the Strichartz estimates on the long-range asymptotically conic manifold was studied. We may consider the homogeneous estimate only, namely we let $F \equiv 0$ and hence $u = e^{-itH}u_0$ for simplicity. Let $2^* = 2n/(n-2)$.

Step 1: Energy localization. Since $V \in L^1_{\mathrm{loc}}$ and $V \ge 0$, the kernel $e^{-tH}(x,y)$ of the semi-group e^{-tH} satisfies the Gaussian upper bound

$$0 \le e^{-tH}(x,y) \le e^{t\Delta}(x,y) \le (4\pi t)^{n/2}e^{-|x-y|^2/(4t)}, \quad t > 0 \tag{3.1}$$

(see [35]). Then an abstract theorem due to [10] implies that the following square function estimate holds:

$$\|u\|_{L_t^2 L_x^{2^*}} \lesssim \Big(\sum_{j\in\mathbb{Z}} \|f(2^{-j}H)u\|_{L_t^2 L_x^{2^*}}^2 \Big)^{1/2}$$

where $\{f(2^{-j}\lambda)\}_{j\in\mathbb{Z}}$ is a dyadic partition of unity on $(0,\infty)$, that is $f \in C_0^\infty(\mathbb{R})$, $0 \le f \le 1$, $\operatorname{supp} f \subset (1/2, 2)$ and

$$\sum_{j\in\mathbb{Z}} f(2^{-j}\lambda) = 1, \quad \lambda > 0.$$

Hence the proof is reduced to proving following energy localized estimate

$$\|\varphi(\lambda^{-2}H)u\|_{L_t^2 L_x^{2^*}} \lesssim \|u_0\|_{L^2}, \quad \lambda > 0, \tag{3.2}$$

where $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \Subset (0,\infty)$ and the implicit constant should be independent of λ. In what follows, we shall focus on the low energy case $\lambda \in (0,1]$ since the high energy case $\lambda \ge 1$ can be handled similarly.

Step 2: Reduction to a semiclassical problem. Next task is to approximate the energy localization $\varphi(\lambda^{-2}H)$ by a pseudodifferential operator (PDO). For each $\lambda > 0$, it can be shown by using Helffer–Sjöstrand formula [20]

$$\varphi(\lambda^{-2}H) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\varphi}}{\partial\bar{z}}(z)(\lambda^{-2}H - z)^{-1} dL(z)$$

(with $\tilde{\varphi}$ being an almost analytic extension of φ) and a microlocal parametrix construction of the resolvent $(\lambda^{-2}H - z)^{-1}$ that $\varphi(\lambda^{-2}H)$ is a PDO $\operatorname{Op}(a) = a(\lambda, x, D)$ with the principal symbol $a = \varphi(\lambda^{-2}|\xi|^2 + V(x))$. However, since $\|\partial_\xi a\|_{L^\infty_{t,x}} \sim \lambda^{-1}$, the operator norm $\|\operatorname{Op}(a)\|_{L_x^2 \to L_x^2}$ may blow up as $\lambda \searrow 0$ in general. Hence this rough observation cannot be used to show the above uniform estimate (3.2) in the low energy case $\lambda \in (0,1]$.

In order to overcome such a difficulty, we first decompose the energy localized $\varphi(\lambda^{-2}H)$ into two regions $\{\lambda|x| \le 1\}$ and $\{\lambda|x| \ge 1\}$:

$$\varphi(\lambda^{-2}H)u = \varphi(\lambda^{-2}H)\Big(\chi_{\{\lambda|x|\le 1\}} + \chi_{\{\lambda|x|\ge 1\}}\Big)u,$$

where χ_A is a smooth cut-off function supported near A and $\chi_A \equiv 1$ on A. In the compact region $\{\lambda|x| \le 1\}$, we use Bernstein's inequality

$$\|\varphi(\lambda^{-2}H)\|_{L_x^2 \to L_x^{2^*}} \lesssim \lambda$$

(which follows from (3.1) thanks to an abstract theorem by [10]) to deduce the desired Strichartz estimate for the first term $\varphi(\lambda^{-2}H)\chi_{\{\lambda|x|\le 1\}}u$ from the following weighted $L_t^2 L_x^2$-estimate

$$\|\langle x\rangle^{-1}u\|_{L_t^2 L_x^2} \lesssim \|u_0\|_{L^2},$$

where we have used the bound $\lambda \le \min(1, |x|^{-1})$. Then such a weighted $L_t^2 L_x^2$-estimate follows from the uniform resolvent estimate proved by [32] under the conditions (H1)–(H3) and Kato's smooth perturbation theory [22].

In the non-compact region $\{\lambda|x| \ge 1\}$, we approximate $\varphi(\lambda^{-2}H)\chi_{\{\lambda|x|\ge 1\}}$ by a suitable rescaled pseudodifferential operator

$$\mathcal{D}_\lambda \operatorname{Op}(a^\lambda)^* \mathcal{D}_\lambda^*$$

modulo an error term, where $\mathcal{D}_\lambda f(x) = \lambda^{n/2} f(\lambda x)$ is the usual dilation and the leading term of $a^\lambda(x,\xi)$ is given by

$$a_0^\lambda(x,\xi) = \varphi\left(|\xi|^2 + V^\lambda(x)\right)\chi_{\{|x|\ge 1\}}(x), \quad V^\lambda(x) = \lambda^{-2}V(\lambda^{-1}x)$$

and the error term can be handled by a similar argument as that for the compact part. Let us now introduce the semiclassical parameter h by

$$h = \lambda^{2/\mu - 1} \in (0,1],$$

where recall that we have assumed $\mu \in (0,2)$. By rescaling, the main term $\mathcal{D}_\lambda \operatorname{Op}(a^\lambda)^* \mathcal{D}_\lambda^* e^{-itH}$ then can be written in the form

$$\mathcal{D}_\lambda \operatorname{Op}(a^\lambda)^* \mathcal{D}_\lambda^* e^{-itH} = \mathcal{D}_{\lambda^{2/\mu}} \operatorname{Op}_h(a_h)^* e^{-it\lambda^2 H_h} \mathcal{D}_{\lambda^{2/\mu}}^* \tag{3.3}$$

where H_h is a semiclassical Schrödinger operator given by

$$H_h = -h^2\Delta + V_h(x), \quad V_h(x) = \lambda^{-2}V(\lambda^{-2/\mu}x)$$

and $\operatorname{Op}_h(a_h) = a_h(x, hD)$ is a semiclassical PDO with the symbol a_h supported in

$$\{(x,\xi) \in \mathbb{R}^{2n} \mid |x| \ge 1,\ 1/4 < |\xi|^2 + V_h(x) < 4\}.$$

The main advantage to introduce the parameter h is that, since V_h obeys

$$|V_h(x)| \lesssim (\lambda^{2/\mu} + |x|)^{-\mu} \lesssim \langle x \rangle^{-\mu}$$

for $|x| \ge 1$ uniformly in λ (and hence in h), a_h belongs to the symbol class $S(\langle x\rangle^{-1}\langle \xi\rangle^{-\infty}, \langle x\rangle^{-2}dx^2 + \langle \xi\rangle^{-2}d\xi^2)$ uniformly in $h \in (0,1]$, namely they satisfy, for any $N \ge 0, \alpha, \beta$,

$$|\partial_x^\alpha \partial_\xi^\beta a_h(x,\xi)| \le C_{\alpha\beta N}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-N-|\beta|}$$

uniformly in $h \in (0,1]$. Therefore, we can use the semiclassical analysis to handle the operator (3.3). Then, by virtue of a scaling argument, the problem is reduced to showing the following semiclassical Strichartz estimate

$$\| \operatorname{Op}_h(a_h)^* e^{-itH_h/h}\psi_0\|_{L_t^2 L_x^{2^*}} \lesssim h^{-1/2}\|\psi_0\|_{L_x^2} \tag{3.4}$$

with the implicit constant being independent of $h \in (0,1]$.

Step 3: Semiclassical Strichartz estimate. In order to obtain (3.4), we further decompose the support of the symbol a_h into the compact part $\{1 \le |x| \le 2R\}$ and non-compact part $\{|x| \ge R\}$ for sufficiently large R (which is independent of h): $a_h = a_h^{\mathrm{com}} + a_h^\infty$, where

$$\operatorname{supp} a_h^{\mathrm{com}} \subset \operatorname{supp} a_h \cap \{1 \le |x| \le 2R\},$$
$$\operatorname{supp} a_h^{\infty} \subset \operatorname{supp} a_h \cap \{|x| \ge R\}.$$

For the compact part, we employ a similar idea as that in [37] which yields that the desired Strichartz estimate can be deduced from the semiclassical dispersive estimate

$$\| \operatorname{Op}_h(a_h^{\mathrm{com}})^* e^{-itH_h/h} \operatorname{Op}_h(a_h^{\mathrm{com}})\|_{L^1_x \to L^\infty_x} \lesssim |th|^{-n/2}, \quad |t| \le t_0, \tag{3.5}$$

for sufficiently small $t_0 \ll 1$ (independent of h), and the local smoothing estimate of the form

$$\| \operatorname{Op}_h(a_h^{\mathrm{com}})^* e^{-itH_h/h} \psi_0\|_{L^2_t L^2_x} \lesssim \|\psi_0\|_{L^2}. \tag{3.6}$$

(3.5) can be proved by using the semiclassical WKB parametrix construction of $e^{-itH_h/h} \operatorname{Op}_h(a_h^{\mathrm{com}})$ and the stationary phase method (see e.g. [41]), while one can prove (3.6) by using a semiclassical version of Mourre's theory ([31]).

To deal with the non-compact part a_h^∞, we decompose a_h^∞ into the outgoing part a_h^+ and incoming part a_h^-. Thanks to the abstract TT^*-argument by [23], it suffices to show the following long-time dispersive estimate

$$\| \operatorname{Op}_h(a_h^\pm)^* e^{-itH_h/h} \operatorname{Op}_h(a_h^\pm)\|_{L^1_x \to L^\infty_x} \lesssim |th|^{-n/2}, \quad t \neq 0.$$

To this end, we essentially follow the idea of [6]. The main ingredient in the proof is the construction of the semiclassical Isozaki-Kitada (IK) parametrix of $e^{-itH_h/h} \operatorname{Op}_h(a_h^+)$ whose main term is given by

$$J_h^+(c^+) e^{ith\Delta} J_h^+(d^+)^*,$$

where $J_h^+(w)$, which is called the IK modifier, is a semiclassical Fourier integral operator with a time-independent phase function

$$S^+(x,\xi) = x\cdot\xi + O(\langle x\rangle^{1-\mu})$$

and the amplitudes c^+ and d^+ are supported in some outgoing regions. The dispersive estimate for the main part of the IK parametrix of the form

$$\| \operatorname{Op}_h(a_h^+)^* J_h^+(c^+) e^{ith\Delta} J_h^+(d^+)^*\|_{L^1_x \to L^\infty_x} \lesssim |th|^{-n/2}, \quad t \neq 0,$$

then can be proved by using the standard stationary phase method, while several microlocal propagation estimates will be used in order to deal with the error term. Since the calculation is rather involved, we omit the details (see [28, Subsection 3.1]). To prove such propagation estimates, we employ the local decay estimate for the propagator $e^{-itH_h/h} \operatorname{Op}_h(a_h^\pm)$ which can be obtained by means of the semiclassical version studied again by [31] of the multiple commutator method of [21].

Acknowledgments

The author is partially supported by JSPS KAKENHI Grant-in-Aid for Young Scientists (B) #JP17K14218 and Grant-in-Aid for Scientific Research (B) #JP17H02854. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

[1] V. Banica, L. I. Ignat, *Dispersion for the Schrödinger equation on networks*, J. Math. Phys. **52** (2011), 083703, 14 pp.

[2] M. Beceanu, *New estimates for a time-dependent Schrödinger equation*, Duke Math. J. **159** (2011), 351–559.

[3] J. -M. Bouclet, H. Mizutani, *Uniform resolvent and Strichartz estimates for Schrödinger equations with critical singularities*, Trans. Amer. Math. Soc. **370** (2018), 7293–7333.

[4] J. -M. Bouclet, H. Mizutani, *Global in time Strichartz inequalities on asymptotically flat manifolds with temperate trapping*, preprint, arXiv:1602.06287.

[5] J. -M. Bouclet, N. Tzvetkov, *Strichartz estimates for long range perturbations*, Amer. J. Math. **129** (2007) 1565–1609.

[6] J. -M. Bouclet, N. Tzvetkov, *On global Strichartz estimates for non trapping metrics*, J. Funct. Analysis **254** (2008) 1661–1682.

[7] N. Burq, P. Gérard, N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004) 569–605.

[8] N. Burq, F. Planchon, J. G. Stalker, A.S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203** (2003), 519–549.

[9] N. Burq, F. Planchon, J. G. Stalker, A.S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay*, Indiana Univ. Math. J. **53** (2004), 1665–1680.

[10] F. Cacciafesta, P. D'Ancona, *Weighted L^p estimates for powers of selfadjoint operators*, Adv. Math. **229** (2012) 501–530

[11] X. Chen, *Resolvent and spectral measure on non-trapping asymptotically hyperbolic manifolds III: Global-in-time Strichartz estimates without loss*, Ann. Inst. H. Poincaré Anal. Non Linéaire **35** (2018), 803–829.

[12] J. Dereziński, C. Gérard, *Scattering theory of classical and quantum N-particle systems*, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997

[13] M. B. Erdoğan, M. Goldberg, W. Schlag, *Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in $\mathbb{R}^3$*, J. Eur. Math. Soc. (JEMS) **10** (2008), 507–531.

[14] J. Ginibre, G. Velo, *The global Cauchy problem for the non linear Schrödinger equation*, Ann. lHP-Analyse non linéaire. **2** (1985), 309–327.

[15] M. Goldberg, *Strichartz estimates for the Schrödinger equation with time-periodic $L^{n/2}$ potentials*, J. Funct. Anal. **256** (2009), 718–746

[16] M. Goldberg, L. Vega, N. Visciglia, *Counterexamples of Strichartz inequalities for Schrödinger equations with repulsive potentials*, Int. Math. Res. Not. (2006), Art. ID 13927

[17] A. Grecu, L. I. Ignat, *The Schrödinger equation on a star-shaped graph under general coupling conditions*, J. Phys. A **52** (2019), 035202, 26 pp.

[18] Z. Guo, J. Li, K. Nakanishi, L. Yan, *On the boundary Strichartz estimates for wave and Schrödinger equations*, J. Differential Equations **265** (2018), 5656–5675.

[19] A. Hassell, J. Zhang, *Global-in-time Strichartz estimates on non-trapping asymptotically conic manifolds*, Anal. PDE **9** (2016) 151–192

[20] B. Helffer and J. Sjöstrand, *Equation de Schrödinger avec champ magnétique et équation de Harper*, In Schrödinger Operators, H. Holden and A. Jensen, eds., pp. 118-197: Lecture Notes in Physics 345, Springer-Verlag, 1989

[21] A. Jensen, É. Mourre, P. Perry *Multiple commutator estimates and resolvent smoothness in quantum scattering theory*, Ann. Inst. H. Poincaré Phys. Théor. **41** (1984) 207–225.

[22] T. Kato, *Wave operators and similarity for some non-self-adjoint operators*, Math. Ann. **162** (1965/1966), 258–279.
[23] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955-980.
[24] C. E. Kenig, G. Ponce, L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33–69.
[25] J. Marzuola, J. Metcalfe, and D. Tataru, Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations, J. Funct. Anal. **255** (2008), 1497–1553.
[26] H. Mizutani, *Remarks on endpoint Strichartz estimates for Schrödinger equations with the critical inverse-square potential*, J. Differential Equations **263** (2017) 3832–3853.
[27] H. Mizutani, *Uniform Sobolev estimates for Schrödinger operators with scaling-critical potentials and applications*, Anal. PDE **13** (2020) 1333–1369.
[28] H. Mizutani, *Strichartz estimates for Schrödinger equations with slowly decaying potentials*, J. Funct. Anal. **279** (2020) 108789.
[29] H. Mizutani, X. Yao, *Kato smoothing and Strichartz estimates for higher-order and fractional operators with sharp Hardy potentials*, preprint available at arXiv:2002.02163
[30] H. Mizutani, X. Yao, *Global Kato smoothing and Strichartz estimates for higher-order Schrödinger operators with rough decay potentials*, preprint. arXiv:2004.10115
[31] S. Nakamura, *Resolvent estimates and time-decay in the semiclassical limit*, Astérisque **210** (1992) 247–262.
[32] S. Nakamura, *Low energy asymptotics for Schrödinger operators with slowly decreasing potentials*, Comm. Math. Phys. **161** (1994) 63–76.
[33] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, IV, Academic Press, New York-London, 1978.
[34] I. Rodnianski, W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004), 451–513.
[35] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982) 447–526.
[36] H. F. Smith, C. D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations **25** (2000), 2171–2183.
[37] G. Staffilani, D. Tataru, *Strichartz estimates for a Schrödinger operator with non-smooth coefficients*, Comm. Partial Differential Equations **27** (2002) 1337–1372
[38] R. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–714
[39] T. Tao, *Nonlinear dispersive equations. Local and global analysis*, CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006.
[40] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm Math. Phys. **110** (1987), 415–426.
[41] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, vol. 138, AMS (2012)

(H. Mizutani) Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan.
Email address: haruya@math.sci.osaka-u.ac.jp