# Voros coefficients and the topological recursion for the hypergeometric differential equation of type $(2,3)$


By

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## § 1. Introduction

The exact WKB analysis is a method of analyzing differential equations with a small parameter $\hbar$ (or a large parameter $\eta = \hbar^{-1}$). It was first developed for second order ordinary differential equations and then has been extended to higher order ordinary differential equations. In this paper we discuss the hypergeometric differential equation of type $(2,3)$

$$\left\{4\hbar^3\frac{d^3}{dx^3} - 2x\hbar^2\frac{d^2}{dx^2} + 2(\lambda_\infty - (\nu_\infty + 1)\hbar)\hbar\frac{d}{dx} - t\right\}\psi = 0. \tag{1.1}$$

This equation is obtained from the class of hypergeometric systems of two variables studied in [15] by fixing the second variable.

A Voros coefficient is defined as a contour integral of the logarithmic derivative of WKB solutions. The explicit form of Voros coefficients enables us to describe parametric Stokes phenomena. The explicit forms of Voros coefficients are first computed for the Weber equation [16, 17], and now known for many equations such as the Whittaker equation [13], the Kummer equation [2], the Gauss hypergeometric equation [1], the hypergeometric equation of type $(1,4)$ [10], and so on.

On the other hand, the topological recursion introduced by Eynard and Orantin [7] is a generalization of the loop equations that the correlation functions of the matrix model satisfy. For a Riemann surface $\Sigma$ and meromorphic functions $x$ and $y$ on $\Sigma$, it produces an infinite tower of multidifferential $W_{g,n}(z_1,\ldots,z_n)$ on $\Sigma$ and free energies $F_g$. A triplet $(\Sigma, x, y)$ is called a spectral curve and $W_{g,n}(z_1,\ldots,z_n)$ is called a correlation function.

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The quantization scheme for the spectral curve connects WKB solutions with the topological recursion ([8, 5, 4] etc.). More precisely, it is found that WKB solutions can be constructed by correlation functions for the spectral curve when the spectral curve satisfies the "admissibility condition" in the sense of [4, Definition 2.7]. WKB solutions can be constructed by correlation functions also for spectral curves not satisfying the admissibility condition, in particular, for elliptic curves ([9]) and for any degree 2 spectral curve ([6, 14]). It means that the recursive relations satisfied by the coefficients of WKB solutions have some relationship with the topological recursion. Furthermore, in the case of the confluent family of the Gauss hypergeometric differential equations, we found that the Voros coefficients are expressed as the difference of the free energy of the spectral curve obtained as the classical limit of the equations [11, 12].

Taking these situations into account, we consider a similar problem for the hypergeometric differential equations of type $(1,4)$ and $(2,3)$ in [18]. The main results are the following:

1. We show that the Voros coefficients are expressed as a difference of free energies.
2. We also get the explicit forms of the free energy and Voros coefficients of such equations. Note that the Voros coefficients for the hypergeometric differential equation of type $(1,4)$ were computed by the paper [10].

In what follows we explain the precise claim of these results in the case of the hypergeometric differential equation of type $(2,3)$. For their proofs and the precise claims in the case of the hypergeometric differential equation of type $(2,3)$, we refer the reader to [18].

The paper is organized as follows: In section 2 we introduce WKB solutions and Voros coefficients. In section 3 we introduce the topological recursion. In section 4 we state the quantization for the hypergeometric differential equation of type $(2,3)$. In section 5 we state our main results for the hypergeometric differential equation of type $(2,3)$.

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## § 2. WKB solutions and Voros coefficients

We consider the following third order ordinary differential equation with a small parameter $\hbar \neq 0$

$$P\left(x, \hbar\frac{d}{dx}\right)\psi = \left\{\hbar^3\frac{d^3}{dx^3} + p(x)\hbar^2\frac{d^2}{dx^2} + q(x,\hbar)\hbar\frac{d}{dx} + r(x)\right\}\psi = 0. \tag{2.1}$$

Here

$$p(x) = -\frac{x}{2}, \quad q(x,\hbar) = q_0(x) + q_1(x)\hbar = \frac{\lambda_\infty}{2} - \frac{\nu_\infty + 1}{2}\hbar, \quad r(x) = -\frac{t}{4}, \tag{2.2}$$

and $t, \lambda_\infty \neq 0$, $\nu_\infty$ are parameters. We call this equation the hypergeometric differential equation of type $(2,3)$. We consider (2.1) as a differential equation on the Riemann sphere $\mathbb{P}^1$ with regular or irregular singular points.

## § 2.1. WKB solutions

For (2.1) we construct a formal solution, called a WKB solution, of the form

$$\psi(x,\hbar) = \exp\left[\int^x S(x,\hbar)dx\right]. \tag{2.3}$$

We substitute this into (2.1) and have the following equation

$$\begin{aligned} &\hbar^3\left(\frac{d^2}{dx^2}S(x,\hbar) + 3S(x,\hbar)\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^3\right) \\ &\quad + p(x)\hbar^2\left(\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^2\right) + \hbar q(x,\hbar)S(x,\hbar) + r(x) = 0. \end{aligned} \tag{2.4}$$

This is a counterpart of the Riccati equation in the second-order case. We seek for a formal series solution of the form

$$S(x,\hbar) = \hbar^{-1}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \cdots = \sum_{m=-1}^{\infty} \hbar^m S_m(x). \tag{2.5}$$

By substituting (2.5) into (2.4), and equating like powers of both sides with respect to $\hbar$, we obtain

$$S_{-1}^3 + p(x)S_{-1}^2 + q_0(x)S_{-1} + r(x) = 0, \tag{2.6}$$

$$\left(3{S_{-1}}^2 + 2p(x)S_{-1} + q_0(x)\right)S_0 + 3S_{-1}\frac{dS_{-1}}{dx} + p(x)\frac{dS_{-1}}{dx} + q_1(x)S_{-1} = 0, \tag{2.7}$$

and

$$\begin{aligned} &\left(3{S_{-1}}^2 + 2p(x)S_{-1} + q_0(x)\right)S_{m+1} + \sum_{\substack{i+j+k=m-1 \\ i,j,k\geq 0}} S_iS_jS_k \\ &\quad + 3\sum_{j=0}^{m-1} S_{m-j-1}S_j + 3S_m\frac{dS_{-1}}{dx} + 3S_{-1}\frac{dS_m}{dx} + \frac{d^2S_{m-1}}{dx^2} \\ &\quad + p(x)\sum_{j=0}^{m} S_{m-j}S_j + p(x)\frac{dS_m}{dx} + q_1(x)S_m = 0 \quad (m\geq 0). \end{aligned} \tag{2.8}$$

Eq. (2.6) has three solutions, and once we fix one of them, we can determine $S_m$ for $m \geq 0$ uniquely and recursively by (2.7) and (2.8). $S_m(x)$ for $m \geq 0$ are functions on the Riemann surface defined by $S_{-1}(x)$.

### § 2.2. Voros coefficients

In this subsection we define Voros coefficients of the hypergeometric differential equation of type $(2,3)$.

By straightforward computations, we obtain asymptotic behaviors of characteristic roots of the quantum $(2,3)$ curve; these are solutions of $P(x,\xi)=0$,

(2.9)
$$\begin{cases} \xi^{(0)}(x,\lambda_\infty) = \dfrac{i\sqrt{t}}{\sqrt{2}} x^{-1/2} + \dfrac{\lambda_\infty}{2} x^{-1} + O(x^{-3/2}), \\ \xi^{(1)}(x,\lambda_\infty) = -\dfrac{i\sqrt{t}}{\sqrt{2}} x^{-1/2} + \dfrac{\lambda_\infty}{2} x^{-1} + O(x^{-3/2}), \\ \xi^{(2)}(x,\lambda_\infty) = \dfrac{1}{2} x - \lambda_\infty x^{-1} + O(x^{-2}). \end{cases}$$

Let $\gamma_{\infty,j}$ ($j \in \{0,1\}$) be a path on the Riemann surface of $S_{-1}(x)$ which starts from $x=\infty$ on the third sheet (i.e., $\xi(x,\lambda_\infty) = \xi^{(2)}(x,\lambda_\infty)$ on the sheet), turn around a turning point, and returns to $x=\infty$ on the $(j+1)$-th sheet (i.e., $\xi(x,\lambda_\infty) = \xi^{(j)}(x,\lambda_\infty)$ on the sheet). A turning point is defined by a point at which two characteristic roots coincide.

*Remark* 1. We will consider $x$ as a branched covering map $x(z)$ from $\mathbb{P}^1$ to itself below. We can verify that $\gamma_{\infty,0}$ is defined as the image by $x$ of a path from $\infty$ to $0$ on $z$-plane and $\gamma_{\infty,1}$ is defined as the image by $x$ of a path from $\infty$ to $\infty$ on $z$-plane.

Then, we define Voros coefficients of the hypergeometric differential equation of type $(2,3)$

(2.10)
$$V^{(\infty,j)}(\lambda_\infty, t, \nu_{\infty,0}, \nu_{\infty,1}; \hbar) = \int_{\gamma_{\infty,j}} \Big( S(x,\hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \Big) dx \quad (j \in \{0,1\}).$$

## § 3. Topological recursion

### § 3.1. Topological recursion

In this section we briefly explain the global topological recursion. It was shown in [3] that it is indeed equivalent to the usual local formulation of the topological recursion introduced by Eynard and Orantin [7] when the ramification points are all simple.

Let us start from the definition of genus 0 spectral curves. (See [7] for general definition of spectral curves.)

**Define 3.1.** A spectral curve of genus 0 is a pair $(x(z), y(z))$ of non-constant rational functions on $\mathbb{P}^1$, such that their exterior differentials $dx$ and $dy$ never vanish simultaneously.

Let $R$ be the set of ramification points of $x(z)$, i.e., $R$ consists of zeros of $dx(z)$ of any order and poles of $x(z)$ whose orders are greater than or equal to two (here we consider $x$ as a branched covering map from $\mathbb{P}^1$ to itself). We need to introduce some notation to define the topological recursion.

**Define 3.2.** Let $A \subseteq_k B$ if $A \subseteq B$ and $|A| = k$.

**Define 3.3.** Let $\mathcal{S}(\boldsymbol{t})$ be the set of set partitions of an ensemble $\boldsymbol{t}$.

Then, we define the recursive structure:

**Define 3.4** ([4, Definition 3.4])**.** Let $\{W_{g,n}\}$ be an arbitrary collection of symmetric multidifferential on $(\mathbb{P}^1)^n$ with $g \geq 0$ and $n \geq 1$. Let $k \geq 1$, $\boldsymbol{t} = \{t_1, \ldots, t_k\}$ and $\boldsymbol{z} = \{z_1, \ldots, z_n\}$. Then, we define

$$\begin{aligned} &\mathcal{R}^{(k)} W_{g,n+1}(\boldsymbol{t}; \boldsymbol{z}) \\ &:= \sum_{\mu \in \mathcal{S}(\boldsymbol{t})} \sum_{\sqcup_{i=1}^{l(\mu)} J_i = \boldsymbol{z}} \sideset{}{'}\sum_{\sum_{i=1}^{l(\mu)} g_i = g + l(\mu) - k} \left\{ \prod_{i=1}^{l(\mu)} W_{g_i, |\mu_i| + |J_i|}(\mu_i, J_i) \right\}. \end{aligned} \tag{3.1}$$

The first summation in (3.1) is over set partitions of $\boldsymbol{t}$, $l(\mu)$ is the number of subsets in the set partition $\mu$. The third summation in (3.1) is over all $l(\mu)$-tuple of non-negative integers $(g_1, \ldots, g_{l(\mu)})$ such that $\sum_{i=1}^{l(\mu)} g_i = g + l(\mu) - k$. $\sqcup$ denotes the disjoint union, and the prime $'$ on the summation symbol in (3.1) means that we exclude terms with $(g_i, |\mu_i| + |J_i|) = (0, 1)$ $(i = 1, \ldots, l(\mu))$ (so that $W_{0,1}$ does not appear) from the sum. We also define

$$\mathcal{R}^{(0)} W_{g,n+1}(\boldsymbol{z}) := \delta_{g,0} \delta_{n,0}, \tag{3.2}$$

where $\delta_{i,j}$ is the Kronecker delta symbol.

We now define the topological recursion.

**Define 3.5** ([4, Definition 3.6])**.** Eynard-Orantin's correlation function $W_{g,n}(z_1, \cdots, z_n)$ for $g \geq 0$ and $n \geq 1$ is defined as a multidifferential on $(\mathbb{P}^1)^n$ using the recurrence relation

$$\begin{aligned} W_{g,n+1}(z_0, z_1, \cdots, z_n) := \sum_{r \in R} \operatorname*{Res}_{z=r} \Bigg\{ &\sum_{k=1}^{r-1} \sum_{\beta(z) \subseteq_k \tau'(z)} (-1)^{k+1} \frac{w^{z-\alpha}(z_0)}{E^{(k)}(z; \beta(z))} \\ &\times \mathcal{R}^{(k+1)} W_{g,n+1}(z, \beta(z); z_1, \cdots, z_n) \Bigg\} \end{aligned} \tag{3.3}$$

for $2g+n \geq 2$ with initial conditions

$$W_{0,1}(z_0) := y(z_0)dx(z_0), \quad W_{0,2}(z_0, z_1) = B(z_0, z_1) := \frac{dz_0 dz_1}{(z_0 - z_1)^2}. \tag{3.4}$$

Here we set $W_{g,n} \equiv 0$ for a negative $g$ and

$$E^{(k)}(z; t_1, \ldots, t_k) := \prod_{i=1}^{k} (W_{0,1}(z) - W_{0,1}(t_i)), \tag{3.5}$$

$$w^{z-\alpha}(z_0) := \left( \frac{1}{z_0 - z} - \frac{1}{z_0 - \alpha} \right) dz_0. \tag{3.6}$$

The second and third summations in (3.3) together mean that we are summing over all subsets of $\tau'(z)$, where $\tau'(z)$ is a set of preimages of $x(z)$ except for the original point $z$. $\alpha$ is an arbitrary base point on $\mathbb{P}^1$, but it can be checked (see [3]) that the definition is actually independent of the choice of base point $\alpha$.

See [7] for basic properties of $W_{g,n}$.

### § 3.2. Definition of free energies

The free energy $F_g$ ($g \geq 0$) is defined for the spectral curve, and one of the most important objects in Eynard-Orantin's theory.

**Define 3.6** ([7, Definition 4.3])**.** For $g \geq 2$, the $g$-th free energy $F_g$ is defined by

$$F_g := \frac{1}{2-2g} \sum_{r \in R} \operatorname*{Res}_{z=r} \left[ \Phi(z) W_{g,1}(z) \right] \quad (g \geq 2), \tag{3.7}$$

where $\Phi(z)$ is a primitive of $y(z)dx(z)$. The free energies $F_0$ and $F_1$ are also defined, but in a different manner (see [7, §4.2.2 and §4.2.3] for the definition).

## § 4. Quantum $(2,3)$ curve

We treat the quantization introduced by Bouchard and Eynard [4].

In what follows we restrict ourselves to the case of the $(2,3)$ curve

$$P(x,y) = 4y^3 - 2xy^2 + 2\lambda_\infty y - t = 0, \tag{4.1}$$

where $\lambda_\infty \neq 0$ and $t$ are parameters. We parametrize (4.1) as

$$\begin{cases} x = x(z) = \dfrac{4z^3 + 2\lambda_\infty z - t}{2z^2} = 2z + \dfrac{\lambda_\infty}{z} - \dfrac{t}{2z^2}, \\ y = y(z) = z \end{cases} \tag{4.2}$$

and regard it as a spectral curve in the sense of Definition 3.1.

We choose

$$
\begin{aligned}
D(z;\nu_\infty) &= [z] - (1-\nu_\infty)[0] - \nu_\infty[\infty] \\
&= (1-\nu_\infty)([z]-[0]) + \nu_\infty([z]-[\infty])
\end{aligned} \tag{4.3}
$$

as the divisor for the quantization, where $\nu_\infty$ is a parameter. Then, the quantum curve of the $(2,3)$ curve (quantum $(2,3)$ curve) is given by

$$
\left\{ 4\hbar^3 \frac{d^3}{dx^3} - 2x\hbar^2 \frac{d^2}{dx^2} + 2(\hat{\lambda}_\infty - \hbar)\hbar \frac{d}{dx} - t \right\} \psi = 0 \tag{4.4}
$$

according to [4, Lemma 5.14]. Here we used the notation

$$
\hat{\lambda}_\infty = \lambda_\infty - \nu_\infty \hbar. \tag{4.5}
$$

# § 5. Main results for the hypergeometric differential equation of type $(2,3)$

## § 5.1. Relations between the Voros coefficient and the free energy

In this subsection we descrive the Voros coefficient of the quantum $(2,3)$ curve in terms of the free energy of the $(2,3)$ curve with a parameter shift. Let

$$
F(\lambda_\infty, t;\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\lambda_\infty, t) \tag{5.1}
$$

be the generating function of the free energies of the $(2,3)$ curve. Then, the following relations hold:

**Theorem 5.1.**

(5.2)
$$
V^{(\infty,0)}(\lambda_\infty, t, \nu_\infty;\hbar) = F(\hat{\lambda}_\infty + \hbar, t;\hbar) - F(\hat{\lambda}_\infty, t;\hbar) - \frac{\partial F_0}{\partial \lambda_\infty}\hbar^{-1} + \frac{2\nu_\infty - 1}{2}\frac{\partial^2 F_0}{\partial {\lambda_\infty}^2},
$$

(5.3)
$$
V^{(\infty,1)}(\lambda_\infty, t, \nu_\infty;\hbar) = 0.
$$

*Here* $\hat{\lambda}_\infty = \lambda_\infty - \nu_\infty\hbar$ *as we have introduced in* (4.5).

We can prove Theorem 5.1 similarly to the case of the Gauss equation ([12, Theorem 3.1]).

Then, we get the three-term difference equation that the generating function of the free energies satisfies.

**Theorem 5.2.** *The free energy* (5.1) *satisfies the following difference equation.*

$$F(\lambda_\infty + \hbar, t; \hbar) - 2F(\lambda_\infty, t; \hbar) + F(\lambda_\infty - \hbar, t; \hbar) = -\frac{1}{2} \log(-2t) \tag{5.4}$$

To prove Theorem 5.2, we need the following identity.

*Lemma* 5.3.

$$V^{(\infty,0)}(\lambda_\infty, t, \nu_\infty + 1; \hbar) = V^{(\infty,0)}(\lambda_\infty, t, \nu_\infty; \hbar). \tag{5.5}$$

Using the following Lemma, we can prove Lemma 5.3.

*Lemma* 5.4. The WKB solution of the quantum (2, 3) curve satisfies

$$\frac{d}{dx}\psi(x; \nu_{\infty,0}, \nu_{\infty,1}, \hbar) = \psi(x; \nu_{\infty,0} + 1, \nu_{\infty,1}, \hbar). \tag{5.6}$$

*Proof of Theorem 5.2.* Substituting $\nu_\infty = 0$ into (5.5), we have

$$V^{(\infty,0)}(\lambda_\infty, t, 1; \hbar) = V^{(\infty,0)}(\lambda_\infty, t, 0; \hbar). \tag{5.7}$$

On the other hand, (5.2) with $\nu_\infty = 1$ and $\nu_\infty = 0$ gives

$$V^{(\infty,0)}(\lambda_\infty, t, 1; \hbar) = F(\lambda_\infty, t; \hbar) - F(\lambda_\infty - \hbar, t; \hbar) - \frac{1}{\hbar}\frac{\partial F_0}{\partial \lambda_\infty} + \frac{1}{2}\frac{\partial^2 F_0}{\partial {\lambda_\infty}^2}, \tag{5.8}$$

$$V^{(\infty,0)}(\lambda_\infty, t, 0; \hbar) = F(\lambda_\infty + \hbar, t; \hbar) - F(\lambda_\infty, t; \hbar) - \frac{1}{\hbar}\frac{\partial F_0}{\partial \lambda_\infty} - \frac{1}{2}\frac{\partial^2 F_0}{\partial {\lambda_\infty}^2}. \tag{5.9}$$

Then, the desired equality (5.4) follows immediately from (5.7), (5.8), (5.9) and

$$F_0 = -\frac{{\lambda_\infty}^2}{4} \log(-2t).$$

□

## § 5.2. The explicit forms of the free energy and Voros coefficient

As a corollary, we obtain explicit formulas for the coefficients of the free energy and Voros coefficient. In this subsection we finally provide the explicit forms for the free energy and Voros coefficients.

**Theorem 5.5.** (i) *For $g \geq 2$ the $g$-th free energy of the* (2, 3) *curve has the following expression:*

$$F_g(\lambda_\infty, t) = 0 \quad (g \geq 2). \tag{5.10}$$

(ii) *The Voros coefficients for the quantum* (2, 3) *curve are explicitly given by*

$$V^{(\infty,j)}(\lambda_\infty, t, \nu_\infty; \hbar) = 0 \quad (j \in \{0, 1\}). \tag{5.11}$$

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