Non-self adjoint Hamiltonian and its applications

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Abstract

Theory of non-self adjoint operators and these applications are interested in various fields of mathematics and physics. These are many research results related to *pseudo-Hermitian operators.* In this filed, generalized Riesz systems can be used to construct some physical operators. From this fact, it seems to be important to consider under what conditions biorthogonal sequences are generalized Riesz systems. In this paper, we shall focus introduce the research I have done in the last few years.

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1 Introduction

A sequence $\{\varphi_n\}$ in a Hilbert space $\mathcal{H}$ is called a generalized Riesz system if there exist an orthonormal basis (from now on, ONB) $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$ and a densely defined closed operator T in $\mathcal{H}$ with densely defined inverse such that $\mathcal{F}_e \subset D(T) \cap D((T^{-1})^*)$ and $Te_n = \varphi_n$, $n = 0, 1, \cdots$. In this case $(\mathcal{F}_e, T)$ is called a constructing pair for $\{\varphi_n\}$, [3, 7, 10]. Then, if we put $\psi_n := (T^{-1})^* e_n$, $n = 0, 1, \cdots$, $\mathcal{F}_\varphi := \{\varphi_n\}$ and $\mathcal{F}_\psi := \{\psi_n\}$ are biorthogonal sequences in $\mathcal{H}$, that is, $\langle \varphi_n, \psi_m \rangle = \delta_{nm}$, $n, m = 0, 1, \cdots$.

The notion of generalized Riesz system is useful to investigate non-self-adjoint Hamiltonians constructed from $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$. More precisely, let $\mathcal{F}_\varphi$ be a generalized Riesz system with a constructing pair $(\mathcal{F}_e, T)$ and define ψ_n as above. Then we consider the operators

$$H_\varphi^{\boldsymbol{\alpha}} := T H_{\boldsymbol{e}}^{\boldsymbol{\alpha}} T^{-1}, \quad A_\varphi^{\boldsymbol{\alpha}} := T A_{\boldsymbol{e}}^{\boldsymbol{\alpha}} T^{-1} \text{ and } B_\varphi^{\boldsymbol{\alpha}} := T B_{\boldsymbol{e}}^{\boldsymbol{\alpha}} T^{-1},$$

together with

$$H_\psi^{\boldsymbol{\alpha}} := (T^*)^{-1} H_{\boldsymbol{e}}^{\boldsymbol{\alpha}} T^*, \quad A_\psi^{\boldsymbol{\alpha}} := (T^*)^{-1} A_{\boldsymbol{e}}^{\boldsymbol{\alpha}} T^* \text{ and } B_\psi^{\boldsymbol{\alpha}} := (T^{-1})^* B_{\boldsymbol{e}}^{\boldsymbol{\alpha}} T^*,$$

where $\boldsymbol{\alpha} = \{\alpha_n\} \subset \mathbb{C}$. Here

$$H_{\boldsymbol{e}}^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_n e_n \otimes \bar{e}_n, \quad A_{\boldsymbol{e}}^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_{n+1} e_n \otimes \bar{e}_{n+1}, \quad B_{\boldsymbol{e}}^{\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \alpha_{n+1} e_{n+1} \otimes \bar{e}_n$$

are the self-adjoint Hamiltonian, the lowering operator and the raising operator for $\{e_n\}$, respectively (if, $x, y, z \in \mathcal{H}$, $(y \otimes \bar{z})x := \langle x, z \rangle y$).

Since $H_\varphi^{\boldsymbol{\alpha}} \varphi_n = \alpha_n \varphi_n$, $A_\varphi^{\boldsymbol{\alpha}} \varphi_n = \alpha_n \varphi_{n-1}$ (0 if $n = 0$) and $B_\varphi^{\boldsymbol{\alpha}} \varphi_n = \alpha_{n+1} \varphi_{n+1}$, $n = 0, 1, \cdots$, it seems natural to call the operators $H_\varphi^{\boldsymbol{\alpha}}$, $A_\varphi^{\boldsymbol{\alpha}}$ and $B_\varphi^{\boldsymbol{\alpha}}$ the non-self adjoint Hamiltonian, and the generalized lowering and raising operators for $\{\varphi_n\}$, respectively. Similarly, since $H_\psi^{\boldsymbol{\alpha}} \psi_n = \alpha_n \psi_n$, $A_\psi^{\boldsymbol{\alpha}} \psi_n = \alpha_n \psi_{n-1}$ (0 if $n = 0$) and $B_\psi^{\boldsymbol{\alpha}} \psi_n = \alpha_{n+1} \psi_{n+1}$, the operators $H_\psi^{\boldsymbol{\alpha}}$, $A_\psi^{\boldsymbol{\alpha}}$, $B_\psi^{\boldsymbol{\alpha}}$ are called the non-self adjoint Hamiltonian, generalized lowering operator and raising operator for $\{\psi_n\}$ respectively.
Then, it is interesting to understand under what conditions biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz system, which is what we will discuss in this paper.

Studies on this subject have been undertaken in [7, 8, 9, 10]. Here we want to explore this question in a more general framework.

Let D_φ and D_ψ be the linear spans of the biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$, respectively, and define the subspaces $D(\varphi)$ and $D(\psi)$ in $\mathcal{H}$ by

$$D(\varphi) = \{x \in \mathcal{H}; \sum_{n=0}^{\infty} |\langle x, \varphi_n \rangle|^2 < \infty\},$$
$$D(\psi) = \{x \in \mathcal{H}; \sum_{n=0}^{\infty} |\langle x, \psi_n \rangle|^2 < \infty\}.$$

Clearly, $D_\psi \subset D(\varphi)$ and $D_\varphi \subset D(\psi)$. In [9], we have shown that if both D_φ and D_ψ are dense in $\mathcal{H}$ (this case is called *regular*), then $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems. After that, in [10], it was proved that, if either D_φ and $D(\varphi)$, or D_ψ and $D(\psi)$, are dense in $\mathcal{H}$ (the case is called *semiregular*), again $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems. Hence we will consider under what conditions $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems when none of the above conditions is satisfied. In [3], we have proved that this holds under the assumptions that $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are biorthogonal and, at once, $\mathcal{D}$-quasi bases, in the sense that

$$\sum_{n=0}^{\infty} \langle x, \varphi_n \rangle \langle \psi_n, y \rangle = \langle x, y \rangle, \quad \forall x, y \in \mathcal{D},$$

where $\mathcal{D}$ is a dense subspace in $\mathcal{H}$ such that $\mathcal{F}_\varphi \cup \mathcal{F}_\psi \subset \mathcal{D} \subset D(\varphi) \cap D(\psi)$, with some additional assumptions. In this paper we shall show that this result holds in a more general case. In Section 3 we define the notion of $(\mathcal{D}, \mathcal{E})$-quasi bases which is a generalization of $\mathcal{D}$-quasi bases as follows:

$$\sum_{n=0}^{\infty} \langle x, \varphi_n \rangle \langle \psi_n, y \rangle = \langle x, y \rangle, \quad \forall x \in \mathcal{D},\ y \in \mathcal{E}$$

where $\mathcal{D}$ and $\mathcal{E}$ are dense subspaces in $\mathcal{H}$ such that $D_\psi \subset \mathcal{D} \subset D(\varphi)$ and $D_\varphi \subset \mathcal{E} \subset D(\psi)$, and we show in Theorem 3.2 that, under this condition, $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems.

In Section 4, we shall introduce some physical examples and relationships between the examples and generalized Riesz systems.

2 Preliminaries

In this section we review some results on generalized Riesz systems needed in the rest of the paper. By Lemma 3.2, [10], we have the following

Lemma 2.1. *Let $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\mathcal{F}_e, T)$. Then, we have the following statements.*

(1) T^* has a densely defined inverse and $(T^*)^{-1} = (T^{-1})^*$.

(2) Let $\psi_n := (T^{-1})^* e_n$, $n = 0, 1, \cdots$. Then, $\{\varphi_n\}$ and $\{\psi_n\}$ are biorthogonal and $(T^{-1})^*$ is a densely defined closed operator in $\mathcal{H}$ with densely defined inverse T^*. Hence $\{\psi_n\}$ is a generalized Riesz system with a constructing pair $(\mathcal{F}_e, (T^{-1})^*)$.

(3) $D(\varphi) \cap D(\psi)$ is dense in $\mathcal{H}$.

Next, for any ONB $\{e_n\}$ in $\mathcal{H}$ and a sequence $\{\varphi_n\}$ in $\mathcal{H}$, we introduce the operators $T^0_{\varphi,\boldsymbol{e}}$, $T_{\varphi,\boldsymbol{e}}$ and $T_{\boldsymbol{e},\varphi}$ as follows:

$$\begin{aligned} T^0_{\varphi,\boldsymbol{e}} &:= \text{the linear operator defined by } \; T^0_{\varphi,\boldsymbol{e}} e_n = \varphi_n, \quad n = 0, 1, \cdots, \\ T_{\varphi,\boldsymbol{e}} &:= \sum_{n=0}^{\infty} \varphi_n \otimes \bar{e}_n, \\ T_{\boldsymbol{e},\varphi} &:= \sum_{n=0}^{\infty} e_n \otimes \bar{\varphi}_n. \end{aligned}$$

Similarly we can introduce, for the set $\{\psi_n\}$ in Lemma 2.1, the operators $T^0_{\psi,\boldsymbol{e}}$, $T_{\psi,\boldsymbol{e}}$ and $T_{\boldsymbol{e},\psi}$. These operators had a role in [10] and will also be relevant here. By Lemmas 2.1, 2.2 in [10] we get the following

Lemma 2.2. *(1) $T_{\varphi,\boldsymbol{e}}$ is a densely defined linear operator in $\mathcal{H}$ such that*

$$T_{\varphi,\boldsymbol{e}} \supseteq T^0_{\varphi,\boldsymbol{e}} \quad \text{and} \quad T^0_{\varphi,\boldsymbol{e}} e_n = T_{\varphi,\boldsymbol{e}} e_n = \varphi_n, \quad n = 0, 1, \cdots .$$

(2) $D(T_{\boldsymbol{e},\varphi}) = D(\varphi)$ and $(T^0_{\varphi,\boldsymbol{e}})^* = T^*_{\varphi,\boldsymbol{e}} = T_{\boldsymbol{e},\varphi}$.

(3) $T^0_{\varphi,\boldsymbol{e}}$ is closable if and only if $T_{\varphi,\boldsymbol{e}}$ is closable if and only if $D(\varphi)$ is dense in $\mathcal{H}$. If this holds, then

$$\overline{T^0_{\varphi,\boldsymbol{e}}} = \bar{T}_{\varphi,\boldsymbol{e}} = (T_{\boldsymbol{e},\varphi})^*. \tag{2.1}$$

Furthermore, by Lemmas 2.3 and 2.4 in [10] we have

Lemma 2.3. *Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences in $\mathcal{H}$. Suppose that $D(\varphi)$ is dense in $\mathcal{H}$. Then we have the following*

(1) $\bar{T}_{\varphi,\boldsymbol{e}}$ has an inverse and $\bar{T}^{-1}_{\varphi,\boldsymbol{e}} \subseteq T_{\boldsymbol{e},\psi} = (T_{\psi,\boldsymbol{e}})^*$.

(2) The following (i), (ii) and (iii) are equivalent:
(i) D_φ is dense in $\mathcal{H}$.
(ii) $\bar{T}_{\varphi,\mathbf{e}}$ has a densely defined inverse.
(iii) $T^*_{\varphi,\mathbf{e}}(= T_{\mathbf{e},\varphi})$ has a densely defined inverse.
If this holds, then $T^{-1}_{\mathbf{e},\varphi} = (\bar{T}^{-1}_{\varphi,\mathbf{e}})^*$.

(3) For the operators $T_{\psi,\mathbf{e}}$ and $T_{\mathbf{e},\psi}$ the same results as in (1) and (2) hold.

By [10], Theorem 3.4, we also get

Theorem 2.4. *Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences in $\mathcal{H}$, and let $\mathcal{F}_e$ be an arbitrary ONB in $\mathcal{H}$. Then the following statements hold:*

(1) Suppose that both D_φ and D_ψ are dense in $\mathcal{H}$. Then $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$) is a generalized Riesz system with constructing pairs $(\mathcal{F}_e, \bar{T}_{\varphi,\mathbf{e}})$ and $(\mathcal{F}_e, T^{-1}_{\mathbf{e},\psi})$ (resp. $(\mathcal{F}_e, \bar{T}_{\psi,\mathbf{e}})$ and $(\mathcal{F}_e, T^{-1}_{\mathbf{e},\varphi})$), and $\bar{T}_{\varphi,\mathbf{e}}$ (resp. $\bar{T}_{\psi,\mathbf{e}}$) is the minimum among constructing operators of the generalized Riesz system $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$), and $T^{-1}_{\mathbf{e},\psi}$ (resp. $T^{-1}_{\mathbf{e},\varphi}$) is the maximum among constructing operators of $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$). Furthermore, any closed operator T (resp. K) satisfying $\bar{T}_{\varphi,\mathbf{e}} \subset T \subset T^{-1}_{\mathbf{e},\psi}$ (resp. $\bar{T}_{\psi,\mathbf{e}} \subset K \subset T^{-1}_{\mathbf{e},\varphi}$) is a constructing operator for $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$).

(2) Suppose that $D(\varphi)$ and D_φ are dense in $\mathcal{H}$. Then $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$) is a generalized Riesz system with a constructing pair $(\mathcal{F}_e, \bar{T}_{\varphi,\mathbf{e}})$ (resp. $(\mathcal{F}_e, T^{-1}_{\mathbf{e},\varphi})$) and the constructing operator $\bar{T}_{\varphi,\mathbf{e}}$ (resp. $T^{-1}_{\mathbf{e},\varphi}$) is the minimum (resp. the maximum) among constructing operators of $\mathcal{F}_\varphi$ (resp. $\mathcal{F}_\psi$).

(3) Suppose that $D(\psi)$ and D_ψ are dense in $\mathcal{H}$. Then $\mathcal{F}_\psi$ (resp. $\mathcal{F}_\varphi$) is a generalized Riesz system with a constructing pair $(\mathcal{F}_e, \bar{T}_{\psi,\mathbf{e}})$ (resp. $(\mathcal{F}_e, T^{-1}_{\mathbf{e},\psi})$) and the constructing operator $\bar{T}_{\psi,\mathbf{e}}$ (resp. $T^{-1}_{\mathbf{e},\psi}$) is the minimum (resp. the maximum) among constructing operators of $\mathcal{F}_\psi$ (resp. $\mathcal{F}_\varphi$).

Theorem 2.4 shows how the problem stated in Introduction (under what conditions biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are generalized Riesz systems) can be solved in the case when either D_φ and $D(\psi)$ or D_ψ and $D(\varphi)$ are dense in $\mathcal{H}$. But, this problem has not been solved completely in case that both D_φ and D_ψ are not dense in $\mathcal{H}$, which is what is interesting for us here. We will see how the operators $T_{\varphi,\mathbf{e}}$, $T_{\mathbf{e},\varphi}$, $T_{\psi,\mathbf{e}}$ and $T_{\mathbf{e},\psi}$ will be relevant in our analysis, together with the $(\mathcal{D}, \mathcal{E})$-quasi bases we will define in the next section. This result is a generalization of the one obtained in [3].

3 $(\mathcal{D},\mathcal{E})$-quasi bases

In this section we extend the notion of $\mathcal{D}$-quasi bases by introducing a second dense subset $\mathcal{E}$ of the Hilbert space $\mathcal{H}$, and we relate these new families of vectors to generalized Riesz systems.

Definition 3.1. *Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences in $\mathcal{H}$ and let $\mathcal{D}$ and $\mathcal{E}$ be dense subspaces such that $D_\psi \subseteq \mathcal{D} \subseteq D(\varphi)$ and $D_\varphi \subseteq \mathcal{E} \subseteq D(\psi)$. Then $(\{\varphi_n\},\{\psi_n\})$ is said to be a $(\mathcal{D},\mathcal{E})$-quasi basis if*

$$\sum_{k=0}^{\infty} \langle x, \varphi_k\rangle \langle \psi_k, y\rangle = \langle x, y\rangle$$

for all $x \in \mathcal{D}$ and $y \in \mathcal{E}$.

It is clear that any $(\mathcal{D},\mathcal{D})$-quasi basis is a $\mathcal{D}$-quasi basis in the sense of [1].

Example 1:– A very simple example of a $(\mathcal{D},\mathcal{E})$-quasi basis can be constructed as follows. Let $\{e_n\}$ be an ONB for $\mathcal{H}$. Let α_n be an unbounded sequence of positive real numbers having 0 as limit point. To be more concrete, let us take

$$\alpha_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Let $Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, e_n\rangle e_n$ be defined on the domain

$$D(T) = \left\{ x \in \mathcal{H} : \sum_{k=0}^{\infty} (2k+1)^2 |(x, e_{2k+1})|^2 < \infty \right\}.$$

The operator T is unbounded, selfadjoint, invertible with inverse T^{-1} defined as $T^{-1}y = \sum_{n=1}^{\infty} \alpha_n^{-1} \langle x, e_n\rangle e_n$ on the domain

$$D(T^{-1}) = \left\{ y \in \mathcal{H} : \sum_{k=1}^{\infty} (2k)^2 |(y, e_{2k})|^2 < \infty \right\}.$$

Both $D(T)$ and $D(T^{-1})$ are dense subspaces of $\mathcal{H}$ and they are different as one can easily check. Let us set $\varphi_n = Te_n$ and $\psi_n = T^{-1}e_n$, $n \in \mathbb{N}$. The $\varphi_n = \alpha_n e_n$, while $\psi_n = T^{-1}e_n = \alpha_n^{-1} e_n$. Moreover $D(\varphi) = D(T)$, $D(\psi) = D(T^{-1})$. Then we have

$$\sum_{n=0}^{\infty} \langle x, \varphi_n\rangle \langle \psi_n, y\rangle = \sum_{n=0}^{\infty} \langle x, \alpha_n e_n\rangle \left\langle \alpha_n^{-1} e_n, y\right\rangle = \langle x, y\rangle .$$

Thus, $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(D(\varphi), D(\psi))$-quasi basis.

Example 2:– Let $H_0 = p^2 + x^2$ be (twice) the self-adjoint Hamiltonian of a one-dimensional harmonic oscillator. We consider H_0 to be the closure of the operator acting in the same way on the Schwartz space $\mathcal{S}(\mathbb{R})$, and $T = \mathbb{1} + p^2$, which is an unbounded self-adjoint operator defined on $D(T) = W^{2,2}(\mathbb{R})$, the Sobolev space of functions having first and second order weak derivative in $L^2(\mathbb{R})$. The operator $T = H_0 + \mathbb{1} - x^2$ is unbounded, invertible with bounded inverse T^{-1}. The eigensystem of H_0 is well known:

$$H_0 e_n(x) = (2n+1)e_n(x), \qquad e_n(x) = \frac{1}{\sqrt{2^n n! \pi^{1/2}}}\, H_n(x)\, e^{-x^2/2}$$

$n \geq 0$, where $H_n(x)$ is the n-th Hermite polynomial. It is easy to see that $e_n(x) \in D(T)$ so that we can define $\varphi_n(x) = (Te_n)(x)$ and $\psi_n(x) = (T^{-1}e_n)(x)$. We get

$$\varphi_n(x) = (2 + 2n - x^2)e_n(x), \qquad \psi_n(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\, e_n(y)\, dy.$$

These functions are respectively eigenvectors of $H = TH_0T^{-1}$ and $H^\dagger$, with eigenvalue $2n+1$. Some computations show that, for instance,

$$H = H_0 - 2\left(\mathbb{1} + 2x\frac{d}{dx}\right) G \star .$$

Here $G(x)$ is the Green function of T, $G(x) = \frac{1}{2}e^{-|x|}$, and $(G \star f)(x) = \int_{\mathbb{R}} G(x-y)f(y)dy$, for all $f(x) \in L^2(\mathbb{R})$. Of course we can rewrite H as follows: $H = H_0 - 2(\mathbb{1} + 2ixp)\, G\star$, which is manifestly non self-adjoint.

The sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are biorthogonal and form a $(D(T), \mathcal{H})$-quasi basis, since

$$\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \langle \psi_k, g \rangle = \langle f, g \rangle ,$$

for all $f(x) \in D(T)$ and $g(x) \in L^2(\mathbb{R})$.

Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be biorthogonal sequences. Suppose that $\mathcal{F}_\varphi$ is a generalized Riesz system with constructing pair $(\mathcal{F}_{\boldsymbol{e}}, T)$. We put $\psi_n^T := (T^{-1})^* e_n$, $n = 0, 1, \cdots$. Then $\mathcal{F}_\psi$ and $\mathcal{F}_\psi^T := \{\psi_n^T\}$ are biorthogonal sequences, but $\mathcal{F}_\psi$ does not necessarily coincide with $\mathcal{F}_\psi^T$. For this reason we will call the constructing pair $(\mathcal{F}_{\boldsymbol{e}}, T)$ natural for the biorthogonal sequences $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ if $\mathcal{F}_\psi = \mathcal{F}_\psi^T$. If D_φ is dense in $\mathcal{H}$, then $(\mathcal{F}_e, T)$ is automatically natural for $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$.

The next theorem, which is the main result of this paper, shows that the notion of $(\mathcal{D}, \mathcal{E})$-quasi basis is intimately linked to that of generalized Riesz system.

Theorem 3.2. *Let $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ be a biorthogonal pair and $\mathcal{D}$ and $\mathcal{E}$ be dense subspaces in $\mathcal{H}$ such that $D_\psi \subseteq \mathcal{D} \subseteq D(\varphi)$ and $D_\varphi \subseteq \mathcal{E} \subseteq D(\psi)$. Then the following statements are equivalent:*

(i) $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis.

(ii) For any ONB $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$, $\mathcal{F}_\varphi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, T)$ satisfying $D(T^*) \supseteq \mathcal{D}$ and $D(T^{-1}) \supseteq \mathcal{E}$.

(iii) For any ONB $\mathcal{F}_e = \{e_n\}$ in $\mathcal{H}$, $\mathcal{F}_\psi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, K)$ satisfying $D(K^*) \supseteq \mathcal{E}$ and $D(K^{-1}) \supseteq \mathcal{D}$.

If the statement (i) holds, then we can take $(\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}})^{-1}$ and $(\overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}})^{-1}$ as T and K in (ii) and (iii), respectively. If D_φ is not dense in $\mathcal{H}$, then $T_{\mathbf{e},\psi}$ does not have an inverse, but $T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}$ has an inverse.

Proof – Take arbitrary $x \in \mathcal{D}$ and $y \in \mathcal{E}$. Since $x \in D(T_{\mathbf{e},\varphi}) = D(\varphi)$ and $y \in D(T_{\mathbf{e},\psi}) = D(\psi)$, we have

$$\begin{aligned} \langle x, y\rangle &= \sum_{n=0}^{\infty} \langle x, \varphi_n\rangle \langle \psi_n, y\rangle = \sum_{n=0}^{\infty} \langle x, T_{\varphi,\mathbf{e}} e_n\rangle \langle T_{\psi,\mathbf{e}} e_n, y\rangle \\ &= \sum_{n=0}^{\infty} \langle T_{\mathbf{e},\varphi} x, e_n\rangle \langle e_n, T_{\mathbf{e},\psi} y\rangle = \langle T_{\mathbf{e},\varphi} x, T_{\mathbf{e},\psi} y\rangle , \end{aligned}$$

which implies that

$$(\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}})^{-1} \subseteq (T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}})^* \quad \text{and} \quad (\overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}})^{-1} \subseteq (T_{\mathbf{e},\psi}\lceil_{\mathcal{E}})^*. \tag{3.1}$$

Now we put $T := (\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}})^{-1}$. Since $D(T) = \overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}} D(\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}}) \supseteq \overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}}\mathcal{E} \supseteq \overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}} D_\varphi = D_{\boldsymbol{e}}$ and $D((T^{-1})^*) = D((\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}})^*) \supseteq D((\overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}})^{-1}) = \overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}} D(\overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}}) \supseteq \overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}} D_\psi = D_{\boldsymbol{e}}$, it follows that T is a densely defined closed operator in $\mathcal{H}$ with densely defined inverse such that $\boldsymbol{e} \subseteq D(T) \cap D((T^{-1})^*)$. Furthermore, we have

$$\begin{aligned} T e_n &= (\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}})^{-1} \overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}} \varphi_n = \varphi_n, \\ (T^{-1})^* e_n &= (\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}})^* e_n = T_{\psi,\mathbf{e}} e_n = \psi_n, \quad n = 0, 1, \cdots . \end{aligned}$$

Thus, $\mathcal{F}_\varphi$ is a generalized Riesz system with a natural constructing pair $(\mathcal{F}_e, T)$. Furthermore, we have $D(T^{-1}) = D(\overline{T_{\mathbf{e},\psi}\lceil_{\mathcal{E}}}) \supseteq \mathcal{E}$ and by (3.1) $D(T^*) \supseteq D(\overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}}) \supseteq \mathcal{D}$. Thus (i) $\Rightarrow$ (ii).
In a similar way, setting $K = (\overline{T_{\mathbf{e},\varphi}\lceil_{\mathcal{D}}})^{-1}$, we can show that $\mathcal{F}_\psi$ is a generalized Riesz system for a natural constructing pair $(\mathcal{F}_e, K)$ satisfying $D(K^*) \supseteq \mathcal{E}$ and $D(K^{-1}) \supseteq \mathcal{D}$. Thus (i) implies

(iii).
(ii) $\Rightarrow$ (i) Take arbitrary $x \in \mathcal{D}$ and $y \in \mathcal{E}$. Since

$$\begin{aligned}\sum_{k=0}^{\infty} \langle x, \varphi_k \rangle \langle \psi_k, y \rangle &= \sum_{k=0}^{\infty} \langle x, T e_n \rangle \left\langle (T^{-1})^* e_n, y \right\rangle \\ &= \sum_{k=0}^{\infty} \langle T^* x, e_n \rangle \left\langle e_n, T^{-1} y \right\rangle = \left\langle T^* x, T^{-1} y \right\rangle = \langle x, y \rangle ,\end{aligned}$$

it follows that $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ is a $(\mathcal{D}, \mathcal{E})$-quasi basis. Similarly we can show (iii) $\Rightarrow$ (i). This completes the proof. □

Suppose that a biorthogonal pair $(\{\varphi_n\}, \{\psi_n\})$ is $(\mathcal{D}, \mathcal{E})$-quasi basis. Then $(\mathcal{D}, \mathcal{E})$ is not unique, and so we define the following family $\mathfrak{M}_{\varphi,\psi}$ by

$$\mathfrak{M}_{\varphi,\psi} = \{(\mathcal{D}, \mathcal{E});\ \ (\{\varphi_n\}, \{\psi_n\}) \text{ is } (\mathcal{D}, \mathcal{E}) - \text{quasi basis}\}$$

In order to find a better constructing pair for $(\{\varphi_n\}, \{\psi_n\})$ we have investigated the ordered set $\mathfrak{M}_{\varphi,\psi}$ with the following order $\preceq$:

For $(\mathcal{D}_1, \mathcal{E}_1)$, $(\mathcal{D}_2, \mathcal{E}_2) \in \mathfrak{M}_{\varphi,\psi}$, $(\mathcal{D}_1, \mathcal{E}_1) \preceq (\mathcal{D}_2, \mathcal{E}_2)$ if and only if $\mathcal{D}_2 \subset \mathcal{D}_1$ and $\mathcal{E}_1 \subset \mathcal{E}_2$. For details refer to [11, 12].

4 Physical Examples

In this section we investigate connections between generalized Riesz systems and $(\mathcal{D}, \mathcal{E})$-quasi bases using the same physical examples as [4, 11]. Let $\{f_n\}$, $n = 0, 1, \cdots$ be an ONB in $L^2(\mathbb{R})$ consisting of the Hermite functions which is contained in the Schwartz space $\mathcal{S}(\mathbb{R})$ of all infinitely differential rapidly decreasing functions on $\mathbb{R}$. We define the momentum operator p and the position operator q by

$$\begin{cases} D(p) & := \text{the set of all differentiable functions } f \text{ on } \mathbb{R} \\ & \quad\ \text{such that } \frac{df}{dx} \in L^2(\mathbb{R}), \\ (pf)(x) & = -i\frac{df}{dx}, \quad f \in D(p) \end{cases}$$

and

$$\begin{cases} D(q) & = \{f \in L^2(\mathbb{R});\ \int_{-\infty}^{+\infty} |xf(x)|^2 dx < \infty\}, \\ (qf)(x) & = xf(x),\ \ f \in D(q). \end{cases}$$

Then p and q are self-adjoint operators in $L^2(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ is a core for p and q, and furthermore $p\mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and $q\mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, and $[p, q] := pq - qp = -i\mathbb{1}$ on $\mathcal{S}(\mathbb{R})$. Next we define the standard bosonic operators a, $a^\dagger$ by

$$a = \frac{1}{\sqrt{2}}(q + ip) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}(q - ip).$$

Then,

$$\begin{aligned} af_n &= \begin{cases} 0, & n = 0 \\ \sqrt{n} f_{n-1} & n = 1, 2, \cdots, \end{cases} \\ a^\dagger f_n &= \sqrt{n+1} f_{n+1}, \quad n = 0, 1, \cdots \end{aligned}$$

and $[a, a^\dagger] = \mathbb{1}$ on $\mathcal{S}(\mathbb{R})$.

The following Examples 3, 4 and 5 are extensions of the standard quantum harmonic oscillator.

Example 3:- The Hamiltonian of this model is the unbounded self-adjoint operator with bounded inverse, introduced in [4]. Let $T = \mathbb{1} + q^2$. Then $\{f_n\} \subset D(T)$. Let us define two biorthogonal sequences

$$\varphi_n = Tf_n \quad \text{and} \quad \psi_n = T^{-1} f_n, \quad n \in \mathbb{N}_0.$$

Then we can show that

$$\begin{aligned} \varphi_n(x) &= \frac{1}{\sqrt{2^n n! \pi^{\frac{1}{2}}}} (1 + x^2) H_n(x) e^{-\frac{x^2}{2}}, \\ \psi_n(x) &= \frac{1}{\sqrt{2^n n! \pi^{\frac{1}{2}}}} \frac{H_n(x)}{1 + x^2} e^{-\frac{x^2}{2}}, \quad n \in \mathbb{N}_0 \end{aligned}$$

and the biorthogonal pair $(\{\varphi_n\}, \{\psi_n\})$ is regular. Hence it follows from Corollary 4 and Theorem 6 in [12] that $T = \bar{T}_{\varphi,\boldsymbol{e}} = T^{-1}_{\boldsymbol{e},\psi}$, $(D_\psi, L^2(\mathbb{R}))$ is the largest element of $\mathfrak{M}_{\varphi,\psi}$ and $(D_\psi, L^2(\mathbb{R}))^\sim$ is a unique element of $\mathfrak{M}^\sim_{\varphi,\psi}$. We see this from a physical point of view.

$$\begin{aligned} H_0 &:= p^2 + q^2 \\ &= \sum_{n=0}^{\infty} (2n + 1) f_n \otimes \bar{f}_n, \end{aligned}$$

where H_0 is the standard quantum harmonic oscillator. We consider the non-self-adjoint Hamiltonians for $\{\varphi_n\}$ and $\{\psi_n\}$ as follows:

$$\begin{aligned} H_\varphi &:= TH_0T^{-1} = p^2 + V_\varphi(q) + 4iq(\mathbb{1} + q^2)^{-1} p, \\ H_\psi &:= T^{-1}H_0T = p^2 + V_\psi(q) - 4iq(\mathbb{1} + q^2)^{-1} p, \end{aligned}$$

where

$$V_\varphi(q) = q^2 + 2(\mathbb{1} - 3q^2)(\mathbb{1} + q^2)^{-2},$$
$$V_\psi(q) = q^2 - 2(\mathbb{1} + q^2)^{-1}$$

and they can be seen as a modified version of the harmonic oscillator where an extra potential is added, going to zero as q^{-2}, and the manifestly non-self-adjoint terms $\pm 4iq(\mathbb{1}+q^2)^{-1}p$ appear. The raising operators A_φ, A_ψ and B_φ, B_ψ for $\{\varphi_n\}$ and $\{\psi_n\}$ are as follows:

$$A_\varphi = \frac{1}{\sqrt{2}}\left(q - 2q(\mathbb{1}+q^2)^{-1} + ip\right), \quad B_\varphi = \frac{1}{\sqrt{2}}\left(q + 2q(\mathbb{1}+q^2)^{-1} - ip\right),$$
$$A_\psi = \frac{1}{\sqrt{2}}\left(q + 2q(\mathbb{1}+q^2)^{-1} + ip\right), \quad B_\psi = \frac{1}{\sqrt{2}}\left(q - 2q(\mathbb{1}+q^2)^{-1} - ip\right).$$

Then

$$H_\varphi = 2B_\varphi A_\varphi + \mathbb{1} \quad \text{and} \quad H_\psi = 2B_\psi A_\psi + \mathbb{1}.$$

Example 4:- The Hamiltonian of this model is the non-self-adjoint operator, introduced in [14, 6]. Let $T := e^{a^\dagger + a}$, then $T^{-1} = e^{-(a^\dagger + a)}$. We put

$$\varphi_n := Tf_n = \frac{1}{\sqrt{n!}}(a^\dagger + \mathbb{1})^n \varphi_0$$
$$\psi_n := T^{-1} f_n = \frac{1}{\sqrt{n!}}(a^\dagger - \mathbb{1})^n \psi_0, \quad n \in \mathbb{N}_0,$$

where

$$\varphi_0 := e f_0(x - \sqrt{2}) = \frac{e}{\pi^{\frac{1}{4}}} e^{-\frac{1}{2}(x-\sqrt{2})^2}.$$
$$\psi_0 := e f_0(x + \sqrt{2}).$$

Then $\{\varphi_n\}$ and $\{\psi_n\}$ are regular biorthogonal sequences in $L^2(\mathbb{R})$ which are generalized Riesz systems with constructing pairs $(\{f_n\}, e^{a^\dagger + a})$ and $(\{f_n\}, e^{-(a^\dagger + a)})$, respectively, and

$$\begin{aligned} H_\varphi &= TH_0T^{-1} \\ &= \frac{1}{2}H_0 + i\sqrt{2}p \\ &= a^\dagger a + (a - a^\dagger) + \frac{1}{2}\mathbb{1}, \end{aligned}$$

$$A_\varphi = TaT^{-1} = a - \mathbb{1},$$
$$B_\varphi = Ta^\dagger T^{-1} = a^\dagger + \mathbb{1}.$$

By Proposition 8 in [11], $T_{\boldsymbol{\varphi},\boldsymbol{f}}$ is the smallest constructing operator and $T^{-1}_{\boldsymbol{f},\boldsymbol{\psi}}$ is the largest constructing operator for $\{\varphi_n\}$ and $T_{\boldsymbol{\varphi},\boldsymbol{f}} \subset e^{a^\dagger+a} \subset T^{-1}_{\boldsymbol{f},\boldsymbol{\psi}}$. Similarly, $T_{\boldsymbol{\psi},\boldsymbol{f}}$ is the smallest constructing operator and $T^{-1}_{\boldsymbol{f},\boldsymbol{\varphi}}$ is the largest constructing operator for $\{\psi_n\}$ and $T_{\boldsymbol{\psi},\boldsymbol{f}} \subset e^{-(a^\dagger+a)} \subset T^{-1}_{\boldsymbol{f},\boldsymbol{\varphi}}$. Furthermore, in [12] $(D(\boldsymbol{\varphi}), D_{\boldsymbol{\varphi}})$ is the smallest element of $\mathfrak{M}_{\boldsymbol{\varphi},\boldsymbol{\psi}}$, $(D_{\boldsymbol{\psi}}, D(\boldsymbol{\psi}))$ is the largest element of $\mathfrak{M}_{\boldsymbol{\varphi},\boldsymbol{\psi}}$ and $(D(\boldsymbol{\varphi}), D_{\boldsymbol{\varphi}}) \preceq (D(e^{a+a^\dagger}), D(e^{-(a+a^\dagger)}) \preceq (D_{\boldsymbol{\psi}}, D(\boldsymbol{\psi}))$.

The following example is a modification of the non-self-adjoint Hamiltonian H_φ in Example 3 exchanging the moment operator p for the position operator q.

Example 5:- The Hamiltonian of this model is the non-self-adjoint operator, introduced in [11], Example 10. Let $T := \overline{e^{-ia^\dagger}e^{ia}}$, then $T^{-1} = \overline{e^{-ia}e^{ia^\dagger}}$. We put

$$\begin{aligned} \varphi'_n &= \frac{1}{\sqrt{n!}}(a^\dagger + i\mathbb{1})^n \varphi'_0, \quad n = 1, 2, \cdots, \\ \psi'_n &= \frac{1}{\sqrt{n!}}(a^\dagger - i\mathbb{1})^n \psi'_0, \quad n = 1, 2, \cdots, \end{aligned}$$

where

$$\begin{aligned} \varphi'_0 &= e^{-ia^\dagger} f_0, \\ \psi'_0 &= e^{\mathbb{1}+ia^\dagger} f_0. \end{aligned}$$

Then $\{\varphi'_n\}$ and $\{\psi'_n\}$ are regular biorthogonal sequences in $L^2(\mathbb{R})$ which are generalized Riesz systems with constructing pairs $(\{f_n\}, \overline{e^{-ia^\dagger}e^{ia}})$ and $(\{f_n\}, \overline{e^{-ia}e^{ia^\dagger}})$, respectively, and

$$\begin{aligned} H' &= TH_0T^{-1} \\ &= \frac{1}{2}(p^2 + q^2) + \sqrt{2}iq \\ &= a^\dagger a + i(a + a^\dagger) + \frac{1}{2}\mathbb{1}, \\ A'_\varphi &= \overline{TaT^{-1}} = a + i\mathbb{1}, \quad B_\varphi = \overline{Ta^\dagger T^{-1}} = a^\dagger + i\mathbb{1}, \\ A_\psi &= \overline{T^{-1}aT} = a - i\mathbb{1}, \quad B_\psi = \overline{T^{-1}a^\dagger T} = a^\dagger - i\mathbb{1}. \end{aligned}$$

Furthermore, in [12] $(D(\boldsymbol{\varphi}'), D_{\boldsymbol{\varphi}'})$ is the smallest element of $\mathfrak{M}_{\boldsymbol{\varphi}',\boldsymbol{\psi}'}$, $(D_{\boldsymbol{\psi}'}, D(\boldsymbol{\psi}'))$ is the largest element of $\mathfrak{M}_{\boldsymbol{\varphi}',\boldsymbol{\psi}'}$ and $(D(\boldsymbol{\varphi}'), D_{\boldsymbol{\varphi}'}) \preceq (D(T^*), D(T^{-1})) \preceq (D_{\boldsymbol{\psi}'}, D(\boldsymbol{\psi}'))$.

Example 6:- (The Swanson model)
The Swanson Hamiltonian is a non-self-adjoint Hamiltonian introduced in [5, 14]. Let $T_\theta :=$

$e^{i(\theta/2)(a^2-(a^\dagger)^2)}$. We put

$$\varphi_n^{(\theta)} = \frac{1}{\sqrt{n!}}(\cos\theta a^\dagger + i\sin\theta a)^n \varphi_0^{(\theta)},$$
$$\psi_n^{(\theta)} = \frac{1}{\sqrt{n!}}(\cos\theta a^\dagger - i\sin\theta a)^n \psi_0^{(\theta)},$$

where

$$\begin{aligned}\varphi_0^{(\theta)} &= c_0\sum_{k=0}^{\infty} e^{-i\frac{\tan\theta}{2}(a^\dagger)^2} f_0 = c_0\sum_{k=0}^{\infty}(-i\tan\theta)^k\sqrt{\frac{(2k-1)!!}{(2k)!!}}f_{2k},\\ \psi_0^{(\theta)} &= d_0\sum_{k=0}^{\infty} e^{i\frac{\tan\theta}{2}(a^\dagger)^2} f_0 = d_0\sum_{k=0}^{\infty}(i\tan\theta)^k\sqrt{\frac{(2k-1)!!}{(2k)!!}}f_{2k}, \qquad (4.1)\end{aligned}$$

and where $(2k)!! = 2k(2k-2)\cdots 4\cdot 2$, $(2k-1)!! = (2k-1)(2k-3)\cdots 3\cdot 1$, and c_0 and d_0 are constants satisfying $\langle\varphi_0^{(\theta)}, \psi_0^{(\theta)}\rangle = 1$. Then $\boldsymbol{\varphi}_\theta$ and $\boldsymbol{\psi}_\theta$ are regular biorthogonal sequences in $L^2(\mathbb{R})$ contained in $\mathcal{S}(\mathbb{R})$ which are generalized Riesz systems with constructing pairs $(\{f_n\}, e^{i(\theta/2)(a^2-(a^\dagger)^2)})$ and $(\{f_n\}, e^{i(\theta/2)((a^\dagger)^2-a^2)})$, respectively, and

$$\begin{aligned}H_\theta &= TH_0T^{-1}\\ &= \frac{1}{2}(p^2+q^2) - \frac{i}{2}\tan 2\theta(p^2-q^2)\\ &= a^\dagger a + \frac{i}{2}\tan 2\theta(a^2+(a^\dagger)^2) + \frac{1}{2}\mathbb{1}, \quad \theta \neq 0 \in (-\frac{\pi}{4}, \frac{\pi}{4}).\\ A_\theta &= T_\theta a T_\theta^{-1} = (\cos\theta)a + i(\sin\theta)a^\dagger,\\ B_\theta &= T_\theta a^\dagger T_\theta^{-1} = (\cos\theta)a^\dagger + i(\sin\theta)a.\end{aligned}$$

By Proposition 8 in [11], $T_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}}$ (resp. $T_{\boldsymbol{\psi}_\theta,\boldsymbol{f}}$) is the smallest constructing operator and $T^{-1}_{\boldsymbol{\psi}_\theta,\boldsymbol{f}}$ (resp. $T^{-1}_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}}$) is the largest constructing operator for $\boldsymbol{\varphi}_\theta$ (resp. $\boldsymbol{\psi}_\theta$) and every closed operator T (resp. K) in $L^2(\mathbb{R})$ satisfying $T_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}} \subset T \subset T^{-1}_{\boldsymbol{\psi}_\theta,\boldsymbol{f}}$ (resp. $T_{\boldsymbol{\psi}_\theta,\boldsymbol{f}} \subset K \subset T^{-1}_{\boldsymbol{\varphi}_\theta,\boldsymbol{f}}$) is a constructing operator for $\boldsymbol{\varphi}_\theta$ (resp. $\boldsymbol{\psi}_\theta$). Furthermore, in [12] $(D(\boldsymbol{\varphi}_\theta), D_{\boldsymbol{\varphi}_\theta})$ is the smallest element of $\mathfrak{M}_{\boldsymbol{\varphi}_\theta,\boldsymbol{\psi}_\theta}$, $(D_{\boldsymbol{\psi}_\theta}, D(\boldsymbol{\psi}_\theta))$ is the largest element of $\mathfrak{M}_{\boldsymbol{\varphi}_\theta,\boldsymbol{\psi}_\theta}$ and $(D(\boldsymbol{\varphi}_\theta), D_{\boldsymbol{\varphi}_\theta}) \preceq (D(T^*), D(T^{-1})) \preceq (D_{\boldsymbol{\psi}_\theta}, D(\boldsymbol{\psi}_\theta))$.

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