Remarks on conjugation and antilinear operators and their numerical range

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Abstract

In this paper, we investigate the numerical ranges of conjugations and antilinear operators on a Hilbert space, which will be shown to be annuli in general. This result proves that Toeplitz-Hausdorff Theorem, which says the convexity on the numerical ranges of linear operators, does not hold for the ones of antilinear operators. Moreover, we extend these results to a Banach space.

1 Introduction

The results in this paper will be appeared in other journals. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$.

For $T \in \mathcal{L}(\mathcal{H})$, its numerical range $W(T)$ is defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

where $\langle \cdot, \cdot \rangle$ is the standard sesquilinear form on $\mathcal{H}$ and $|| \cdot ||$ is its induced norm.

Theorem 1.1. (Toeplitz-Hausdorff Theorem, [8], [20])
For $T \in \mathcal{L}(\mathcal{H})$, its numerical range $W(T)$ is convex in $\mathbb{C}$.

Now, we give basic properties of the numerical range $W(T)$ of $T \in \mathcal{L}(\mathcal{H})$ which come from [7, 18, 19]. Let $T, S \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then the following properties hold.

(i) $W(T^*) = \overline{W(T)}$.
(ii) $W(T) = \{\lambda\}$ if and only if $T = \lambda I$.
(iii) $W(T)$ contains all of the eigenvalues of T.
(iv) $W(T)$ lies in the closed disk of radius $\|T\|$ centered at 0.
(v) $W(\alpha T + \beta I_{\mathcal{H}}) = \alpha W(T) + \beta I$ for $\alpha, \beta \in \mathbb{C}$.
(vi) $W(UTU^*) = W(T)$ for a unitary U.
(vii) T is self-adjoint, i.e., $T = T^*$ if and only if $W(T) \subset \mathbb{R}$.
(viii) $W(T)$ is closed (and compact) when $\mathcal{H}$ is finite dimensional.
(ix) $W(T + S) \subset W(T) + W(S)$.

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Example 1.2. (i) If $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$, then $W(T)$ is the closed unit interval.
(ii) If $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$, then $W(T)$ is the closed disc of radius $\dfrac{1}{2}$ centered at 0.
(iii) If $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ on $\mathbb{C}^2$, then $W(T)$ is the closed elliptical disc foci at 0 and 1, minor axis 1 and major axis $\sqrt{2}$.

Theorem 1.3. (i) *Let T be a 2×2 matrix with distinct eigenvalues α and β and corresponding normalized eigenvectors x and y. Then $W(T)$ is the closed elliptical disc foci at α and β, minor axis $\dfrac{\gamma|\alpha-\beta|}{\delta}$ and major axis $\dfrac{|\alpha-\beta|}{\delta}$ where $\gamma = |(x,y)|$ and $\delta = \sqrt{1-\gamma^2}$.*
(ii) *Let T have only one eigenvalue α. Then $W(T)$ is the closed disc of radius $\dfrac{1}{2}\|T-\alpha\|$ centered at α.*

Example 1.4. (i) If $T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then $W(T)$ is the equilateral triangle whose vertices are the three cubic roots of 1, i.e., $1, w$, and w^2.
(ii) If $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $\mathbb{C}^3$, then $W(T)$ is the union of all the closed segments that join the point 1 to all points of the closed disc with center 0 and radius $\dfrac{1}{2}$.

Example 1.5. Let T be defined on ℓ^2 by

$$T(x_0, x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, \cdots)$$

for $(x_1, x_2, x_3, \cdots) \in \ell^2$. Then $W(T) = \mathbb{D}$ where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

Theorem 1.6. *Let $T \in \mathcal{L}(\mathcal{H})$. Then $\sigma(T) \subset \overline{W(T)}$ where $\sigma(T)$ is the spectrum of T.*

Recall that two operators $T, S \in \mathcal{L}(\mathcal{H})$ are *approximately unitarily equivalent* if there exists a sequence $\{U_n\}_{n\geq 1}$ of unitaries such that $\lim_{n\to\infty} \|U_n S U_n{}^* - T\| = 0$.

Theorem 1.7. *Let $T, S \in \mathcal{L}(\mathcal{H})$. If T and S are approximately unitarily equivalent, then $\overline{W(T)} = \overline{W(S)}$.*

Definition 1.8. *An operator C is said to be a conjugation on $\mathcal{H}$ if the following conditions hold:*
(i) *C is antilinear; $C(ax+by) = \overline{a}Cx + \overline{b}Cy$ for all $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$,*
(ii) *C is isometric; $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$, and*
(iii) *C is involutive; $C^2 = I$.*

For any conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [10] for more details). We present the following examples for conjugations.

Example 1.9. Let us define an operator C as follows:
(i) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \cdots, \overline{x_n})$ on $\mathbb{C}^n$.
(ii) $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_n}, \overline{x_{n-1}}, \overline{x_{n-2}}, \cdots, \overline{x_1})$ on $\mathbb{C}^n$.
(iii) $[Cf](x) = \overline{f(x)}$ on $\mathcal{L}^2(\mathcal{X}, \mu)$.
(iv) $[Cf](x) = \overline{f(1-x)}$ on $L^2([0,1])$.
(v) $[Cf](x) = \overline{f(-x)}$ on $L^2(\mathbb{R}^n)$.
(vi) $Cf(z) = \overline{zf(z)}u(z) \in \mathcal{K}_u^2$ for all $f \in \mathcal{K}_u^2$ where u is an inner function and $\mathcal{K}_u^2 = H^2 \ominus uH^2$ is a Model space.
Then each C in (i)-(vi) is a conjugation.

Let $\mathcal{X}$ be a separable complex Banach space and $\mathcal{L}(\mathcal{X})$ denote the algebra of all bounded linear operators on $\mathcal{X}$. Let $\mathcal{X}^*$ be the dual space of a Banach space $\mathcal{X}$ and let T^* be the adjoint operator of $T \in \mathcal{L}(\mathcal{X})$. The set Π is defined by

$$\Pi = \{\, (x, f) \in \mathcal{X} \times \mathcal{X}^* \,:\, \|f\| = f(x) = \|x\| = 1 \,\}.$$

For $T \in \mathcal{L}(\mathcal{X})$, the *numerical range* $V(T)$ of T is defined by

$$V(T) = \{\, f(Tx) \,:\, (x, f) \in \Pi \,\}.$$

Let $\sigma(T)$ denote the spectrum of $T \in \mathcal{L}(\mathcal{X})$. For a subset M of $\mathbb{C}$, we denote the closure of M by $\overline{M}$. Note that for any $T \in \mathcal{L}(\mathcal{X})$, $\sigma(T) \subset \overline{V(T)}$ holds (see [W]) and $V(T)$ is connected (see [2] and [3, Corollary 5, page 102]). In general, $V(T)$ is ([3, Example 1, page 98]) and we denote the closed convex hull of $V(T)$ by $\overline{\mathrm{co}}\, V(T)$. An operator $T \in \mathcal{L}(\mathcal{X})$ is said to be *Hermitian* if $V(T) \subset \mathbb{R}$. If T is Hermitian on $\mathcal{X}$, then $V(T) = \mathrm{co}\, \sigma(T)$ ([3, Corollary 11, page 53]). If H is a Hermitian operator, then H^2 may not be Hermitian from [3, Example 1, Page 58]. In 2018, Chō and Tanahashi [6] introduce the concept of a conjugation on a Banach space. An operator $C : \mathcal{X} \to \mathcal{X}$ is called a *conjugation* on $\mathcal{X}$, if C satisfies

$$C^2 = I, \ \|C\| \leq 1, \ C(x+y) = Cx + Cy, \ C(\lambda x) = \overline{\lambda} Cx, \tag{1}$$

for $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{C}$. Note that (1) implies that $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$.

2 Main results

First, we consider the following questions:

(i) What is the numerical range $W(C)$ of a conjugation C on a Hilbert space $\mathcal{H}$?

(ii) What is the numerical range $W(A)$ of an antilinear operator A on a Hilbert space $\mathcal{H}$?

Theorem 2.1. (In 1965, Godic and Lucenko [14])
If U is a unitary operator on $\mathcal{H}$, then there exist conjugations C and J such that $U = CJ$ and $U^* = JC$.

Lemma 2.2. (In 2014, S. R. Garcia, E. Prodan, and M. Putinar [12]) *If C and J are conjugations on $\mathcal{H}$, then $U := CJ$ is a unitary operator. Moreover, U is both C-symmetric and J-symmetric.*

A vector $x \in \mathcal{H}$ is called *isotropic* with respect to C if $\langle Cx, x\rangle = 0$ (see [12]).

Lemma 2.3. (Garcia, Prodan and Putinar, [12, Lemma 4.11])
If $C : \mathcal{H} \to \mathcal{H}$ is a conjugation, then every subspace of dimension ≥ 2 contains isotropic vectors for the bilinear form $\langle \cdot, C\cdot\rangle$.

Theorem 2.4. (In 2018, Hur and Lee [9]) *Let C be a conjugation on $\mathcal{H}$. Then its the numerical range $W(C)$ is the following:*

(i) $W(C) = \{z : |z| = 1\}$, *when* $\dim \mathcal{H} = 1$ (*equivalently,* $\mathcal{H} = \mathbb{C}$).

(ii) $W(C) = \{z : |z| \leq 1\}$ *for* $\dim \mathcal{H} \geq 2$.

A bounded *antilinear* operator A on a Hilbert space $\mathcal{H}$ is defined by taking complex conjugation on the coefficients on a linear one, i.e., for $x, y \in \mathcal{H}$ and for $\alpha, \beta \in \mathbb{C}$

$$A(\alpha x + \beta y) = \overline{\alpha} A(x) + \overline{\beta} A(y).$$

Crucial observation For any antilinear operator A and $x \in \mathcal{H}$,

$$\langle Ae^{i\theta}x, e^{i\theta}x\rangle = \langle e^{-i\theta}Ax, e^{i\theta}x\rangle = e^{-2i\theta}\langle Ax, x\rangle \quad \text{for real } \theta, \tag{2}$$

which means that, if any complex number λ is in $W(A)$, then the circle $\{z \in \mathbb{C} : |z| = |\lambda|\}$ is contained in $W(A)$.

In other words, (2) shows why the numerical ranges of any antilinear operators should be *circular* regions, which would be much easier than the numerical ranges of linear operators. For a linear operator T, the quantity

$$\langle Te^{i\theta}x, e^{i\theta}x\rangle = \langle Tx, x\rangle$$

is independent of θ, so a similar computation (2) for linear operators does not give further information on $W(T)$.

Theorem 2.5. (In 2018, Hur and Lee [9]) *Let A be a bounded antilinear operator on $\mathcal{H}$. Put $a =: \inf\{|\langle Ax, x\rangle| : ||x|| = 1\}$ and $b =: \sup\{|\langle Ax, x\rangle| : ||x|| = 1\}$. Then its numerical range $W(A)$ of A is the following:*

(i') *When* $\dim \mathcal{H} = 1$ (*equivalently,* $\mathcal{H} = \mathbb{C}$), $a = b$ *and* $W(A) = \{z : |z| = a\}$.

(ii') *For* $\dim \mathcal{H} \geq 2$, *$W(A)$ is contained in the annulus whose boundaries are two circles $\{z : |z| = a\}$ and $\{z : |z| = b\}$. Inner or outer boundary circle is in $W(A)$ if and only if the infimum or supremum becomes the minimum or maximum, respectively.*

Note that if T is a linear operator and A is an antilinear operator, then TA and AT are an antilinear operators.

Example 2.6. (In 2018, Hur and Lee [9]) Consider $A_1 := C$ diag $\{2-1/n\}_{n=1}^{\infty}$ on $\ell^2(\mathbb{N})$, where C is the canonical conjugation on $\ell^2(\mathbb{N})$ and diag $\{2-1/n\}_{n=1}^{\infty}$ is the (infinite-sized) diagonal matrix (which is linear). Then

$$W(A_1) = \{z : 1 \leq |z| < 2\}.$$

Similarly put $A_2 := C$ diag $\{1/n\}_{n=1}^{\infty}$ on $\ell^2(\mathbb{N})$ and $A_3 := A_1 \oplus A_2$, where $\oplus$ is the direct sum of two antilinear operators. Hence

$$W(A_2) = \{z : 0 < |z| \leq 1\} \text{ and } W(A_3) = \{z : 0 < |z| < 2\}.$$

Next, we consider the following questions:
(i) What is the numerical range $V(C)$ of a conjugation C on a Hilbert space $\mathcal{X}$?
(ii) What is the numerical range $V(A)$ of an antilinear operator A on a Hilbert space $\mathcal{X}$?

A topological space X is called *connected* if there are two open subsets A and B in X such that $X = A \cup B$ and $A \cap B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$.

Lemma 2.7. (Bonsall and Duncan [3, Theorem11.4])
Let $\mathcal{X}$ be a complex Banach space. Then Π is a connected subset of $\mathcal{X} \times \mathcal{X}^*$ with the norm $\times$ weak topology.*

We define the numerical range of C by

$$V(C) = \{\, f(Cx) : (x, f) \in \Pi \,\}.$$

Lemma 2.8. *If $\dim \mathcal{X} \geq 2$, then both 0 and 1 are in $V(C)$.*

Theorem 2.9. *Let $\mathcal{X}$ be a complex Banach space and let C be a conjugation on $\mathcal{X}$. Then $V(C)$ is in the complex plane $\mathbb{C}$.*

Theorem 2.10. *Let $\mathcal{X}$ be a Banach space and let C be a conjugation on $\mathcal{X}$. Then the numerical range $V(C)$ of C is the following:*

(i) $V(C) = \{z : |z| = 1\}$, *when* $\dim \mathcal{X} = 1$ (*equivalently,* $\mathcal{X} = \mathbb{C}$).

(ii) $V(C) = \{z : |z| \leq 1\}$ *for* $\dim \mathcal{X} \geq 2$.

In general, $V(T) \subset V(T^*)$ for $T \in \mathcal{L}(\mathcal{X})$ and its adjoint operator T^* on $\mathcal{X}^*$. For a conjugation C on $\mathcal{X}$, we define the dual conjugation C^* on $\mathcal{X}^*$ of C by

$$(C^* f)(x) = \overline{f(Cx)} \quad (x \in \mathcal{X}),$$

where $\overline{f(Cx)}$ is the complex conjugation of the complex number $f(Cx)$.

The numerical range $V(C^*)$ of C^* is given by

$$V(C^*) = \{ \mathcal{F}(C^*f) : \|\mathcal{F}\| = \mathcal{F}(f) = \|f\| = 1, \ \mathcal{F} \in \mathcal{X}^{**}, \ f \in \mathcal{X}^* \}.$$

For $(x, f) \in \Pi$, let $\hat{x}$ be the Gelfand transformation of x. Then since $\|\hat{x}\| = \hat{x}(f) = \|f\| = 1$ and by the definition of C^* it holds

$$\hat{x}(C^*f) = (C^*f)(x) = \overline{f(Cx)},$$

we have $\{\overline{z} : z \in V(C)\} \subset V(C^*)$.

Corollary 2.11. *Let $\mathcal{X}$ be a complex Banach space and let C be a conjugation on $\mathcal{X}$. Then $V(C) = V(C^*)$.*

A space $\mathcal{X}$ is called *path-connected* if for any two points x and y in $\mathcal{X}$ there exists a continuous path f from $[0, 1]$ to $\mathcal{X}$ such that $f(0) = x$ and $f(1) = y$.

Remark In general, there is no relation between connectedness and path-connectedness. For example, topologist's sine curve, i.e.,

$$\{ x + i \sin \frac{1}{x} : 0 < x \leq 1 \} \cup \{ iy : -1 \leq y \leq 1 \} \subset \mathbb{C}$$

is connected but not path-connected (even though $\mathbb{C}$ is path-connected). Path-connectedness is not hereditary either, i.e., even though a total space X is path-connected, we do not know if every subset of X is path-connected.

Recall that $\mathcal{X}$ is a reflexive Banach space if $\mathcal{X}^{**} = \{\hat{x} : x \in \mathcal{X}\}$.

Lemma 2.12. (Luna, [16, Corollary 7]) *Let $\mathcal{X}$ be a complex reflexive Banach space with $\dim \mathcal{X} \geq 2$ and let $T \in \mathcal{L}(\mathcal{X})$. Then $V(T)$ is path-connected.*

Theorem 2.13. *With same hypothsis as in provious theorem, if $\mathcal{X}$ is reflexive, then the numerical range $V(C)$ of C is*

$$V(C) = \{z : |z| \leq 1\}.$$

For an antilinear operator A on $\mathcal{X}$, we define the numerical range $V(A)$ by $V(A) = \{ f(Ax) : (x, f) \in \Pi \}$.

Theorem 2.14. *Let $\mathcal{X}$ be a Banach space and let A be a bounded antilinear operator on $\mathcal{X}$. Put $a := \inf\{|f(Ax)| : \|f\| = ||x|| = 1\}$ and $b := \sup\{|f(Ax)| : \|f\| = ||x|| = 1\}$. Then its numerical range $V(A)$ of A is the following:*

(i) *When $\dim \mathcal{X} = 1$ (equivalently, $\mathcal{X} = \mathbb{C}$), $a = b$ and $V(A) = \{z : |z| = a\}$.*

(ii) *For $\dim \mathcal{X} \geq 2$, $V(A)$ is contained in the annulus whose boundaries are two circles $\{z : |z| = a\}$ and $\{z : |z| = b\}$. Inner or outer boundary circle is in $V(A)$ if and only if the infimum or supremum becomes the minimum or maximum, respectively.*

For an antilinear operator A on $\mathcal{X}$, we define the adjoint operator A^* of A by

$$(A^*f)(x) = \overline{f(Ax)}, \quad (x \in \mathcal{X},\ f \in \mathcal{X}^*),$$

where $\overline{f(Ax)}$ is the complex conjugation of the complex number $f(Ax)$. Then A^* is an antilinear operator on $\mathcal{X}^*$.

Corollary 2.15. *Let $\mathcal{X}$ be a Banach space and let A be an antilinear operator on $\mathcal{X}$. Then $V(A) \subseteq V(A^*)$ and the equality holds when $\mathcal{X}$ is reflexive.*

Remark If $\mathcal{X}$ is non-reflexive, then $\Pi(\mathcal{X})$ is strictly smaller than $\Pi(\mathcal{X}^*)$ in the sense that there exists $f \in \mathcal{X}^*$ such that it does not have $x \in \mathcal{X}$ such that $(x, f) \in \Pi(\mathcal{X})$. Due to this, it is possible that, even though $V(A)$ does not contain a in (ii) on Theorem (for example), $V(A^*)$ may contain a.

It does not occur when we consider conjugation C, since $V(C)$ was closed.

Finally, we focus on the single-valued extension property of operators on a Banach space $\mathcal{X}$.

An operator $T \in \mathcal{L}(\mathcal{X})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{X}$-valued analytic function φ on G such that

$$(T - \lambda)\varphi(\lambda) \equiv 0$$

on G, then we have $\varphi(\lambda) \equiv 0$ on G (see [1]).

Definition 2.16. (i) *$T \in \mathcal{L}(\mathcal{H})$ has the property* (I) *if $\lambda \in \sigma_a(T)$ and $\{x_n\}$ is a sequence of unit vectors of $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$ as $n \to \infty$, then $\|(T-\lambda)^* x_n\| \to 0$ as $n \to \infty$.*
(ii) *$T \in \mathcal{L}(\mathcal{H})$ has the property* (I') *if $\lambda \in \sigma_a(T) \setminus \{0\}$ and $\{x_n\}$ is a sequence of unit vectors of $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$ as $n \to \infty$, then $\|(T-\lambda)^* x_n\| \to 0$ as $n \to \infty$.*
(iii) *$T \in \mathcal{L}(\mathcal{H})$ has the property* (II) *if $\lambda, \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ and $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors of $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$ and $\|(T-\mu)y_n\| \to 0$ as $n \to \infty$, then $\langle x_n, y_n \rangle \to 0$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$.*

Proposition 2.17. (Uchiyama and Tanahashi, [21, Proposition 3.1])
If $T \in \mathcal{L}(\mathcal{H})$ has the property (II)*, then T also has the single-valued extension property.*

Let $\mathcal{X}^*$ be the dual space of a Banach space $\mathcal{X}$ and let T^* be the adjoint operator of $T \in \mathcal{L}(\mathcal{X})$. The set Π is defined by

$$\Pi = \{\, (x, f) \in \mathcal{X} \times \mathcal{X}^* \ :\ \|f\| = f(x) = \|x\| = 1 \,\}.$$

Lemma 2.18. *Let $x \in \mathcal{X}$ be nonzero. Then there exists a functional $f \in \mathcal{X}^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Hence, for every unit vector $x \in \mathcal{X}$, there exists $f \in \mathcal{X}^*$ such that $(x, f) \in \Pi$.

Definition 2.19. (Banach space version)
(i) $T \in \mathcal{L}(\mathcal{X})$ *has the property* (I) *if* $\lambda \in \sigma_a(T)$ *and* $\{x_n\}$ *is a sequence of unit vectors of* $\mathcal{X}$ *such that* $\|(T-\lambda)x_n\| \to 0$ *as* $n \to \infty$*, then* $\|(T-\lambda)^* f_n\| \to 0$ *as* $n \to \infty$*, where* $f_n \in \mathcal{X}^*$ *such that* $(x_n, f_n) \in \Pi$*.*
(ii) $T \in \mathcal{L}(\mathcal{X})$ *has the property* (I') *if* $\lambda \in \sigma_a(T) \setminus \{0\}$ *and* $\{x_n\}$ *is a sequence of unit vectors of* $\mathcal{X}$ *such that* $\|(T-\lambda)x_n\| \to 0$ *as* $n \to \infty$*, then* $\|(T-\lambda)^* f_n\| \to 0$ *as* $n \to \infty$*, where* $f_n \in \mathcal{X}^*$ *such that* $(x_n, f_n) \in \Pi$*.*
(iii) $T \in \mathcal{L}(\mathcal{X})$ *has the property* (II) *if* $\lambda, \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ *and* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of unit vectors of* $\mathcal{X}$ *such that* $\|(T-\lambda)x_n\| \to 0$ *and* $\|(T-\mu)y_n\| \to 0$ *as* $n \to \infty$*, then* $f_n(y_n) \to 0$ *and* $g_n(x_n) \to 0$*, where* (x_n, f_n) *and* (y_n, g_n) *are in* Π*.*

Theorem 2.20. (Mattila, [17, Theorem 3.11]) *If* $\mathcal{X}^*$ *is uniformly convex and* $T \in \mathcal{L}(\mathcal{X})$ *is normal, then* T *has the property* (I)*.*

Theorem 2.21. *If* $T \in \mathcal{L}(\mathcal{X})$ *has the property* (I)*, then* T *has the property* (II)*.*

Corollary 2.22. *Let* $T \in \mathcal{L}(\mathcal{X})$ *have the property* (I)*. If* $\lambda, \mu \in \sigma_p(T)$ $(\lambda \neq \mu)$ *and* x, y *are the corresponding eigenvectors of* $\mathcal{X}$ *where* $\|x\| = \|y\| = 1$*, then for* $(x, f), (y, g) \in \Pi$*, it holds* $f(Ty) = g(Tx) = 0$*.*

Theorem 2.23. (Banach space version)
If $T \in \mathcal{L}(\mathcal{X})$ *has the property* (II) *or the property* (I')*, then* T *also has the single-valued extension property.*

For an operator $T \in \mathcal{L}(\mathcal{X})$ and for a vector $x \in \mathcal{X}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $\varphi : G \to \mathcal{X}$ such that $(T-\lambda)\varphi(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{X})$ by $X_T(F) = \{x \in \mathcal{X} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$ (see [1]).

Corollary 2.24. *If* $T \in \mathcal{L}(\mathcal{X})$ *has the property* (I') *or the property* (II)*, then the following statements hold.*
(i) *For any analytic function on some open neighborhood of* $\sigma(T)$*,* $f(T)$ *has the single-valued extension property.*
(ii) *If* $S \in \mathcal{L}(\mathcal{X})$ *and* $YS = TY$ *where* Y *has trivial kernel and dense range, then* S *has the single-valued extension property and* $YX_S(F) \subset X_T(F)$ *for any subset* F *of* $\mathbb{C}$*.*

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