Two zeta functions contained in the Poincaré series (ポアンカレ級数に含有される2種のゼータ関数)

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Abstract

Two zeta-functions associated with the classical Poincaré series attached to modular group are introduced. Integral representations, transformation formulas and some functional properties are given. As an application, we obtain two new proofs of the Fourier series expansion of the Poincaré series attached to $SL(2,\mathbb{Z})$.
This manuscript is a summarized version of [No1] and the forthcoming paper [No2].

1 Exponential type generating functions

Let $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half plane. Let $\Gamma(s)$ be the Gamma function, $F(\alpha;\gamma;z)$ be Kummer's confluent hypergeometric function of the first kind (cf. [Er1, 6.5.(1)]), $J_\nu(z)$ be the Bessel function of the first kind, and $I_\nu(z)$ and $K_\nu(z)$ be modified Bessel functions (cf. [Er2, 7.2.1 (2), 7.2.2. (12), (13)]). Throughout this manuscript, $\zeta(s)$ and $\zeta(s,\alpha)$ denote the Riemann and the Hurwitz zeta-function respectively, $\int_{-\infty}^{(0+)}$ denotes integration over a Hankel contour, starting at negative infinity on the real axis, encircling the origin with a small radius in the positive direction, and returning to the starting point. We write $\int_{\infty e^{i\theta}}^{(0+)}$ an integration taken along a rotated Hankel contour, starting at $\infty e^{i(-2\pi+\theta)}$, encircling the origin in the positive direction, and returning to the point $\infty e^{i\theta}$.

First, we introduce following Dirichlet series:

$$\zeta_{\exp.\mathrm{II}}(s;\lambda;z) := \sum_{n=0}^{\infty} \frac{\exp(-2\pi i\lambda/(n+z))}{(n+z)^s}. \tag{1}$$

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Here, let $z \in \mathbb{C} \setminus \mathbb{R}$, $\mathrm{Re}(2\pi i\lambda/z) > 0$ and $0 < \mathrm{Re}(z) \leq 1$ in (1). Then the power series expansion involving the Hurwitz zeta-function

$$\zeta_{\exp.\mathrm{II}}(s;\lambda;z) = \sum_{m=0}^{\infty} \frac{(-2\pi i\lambda)^m}{m!} \zeta(s+m,z) \tag{2}$$

holds for $s \in \mathbb{C} \setminus \{1,0,-1,-2,...\}$, and we have

Theorem 1 ([No2]) *Let $z \in H$, $0 < \mathrm{Re}(z) \leq 1$ and $\lambda > 0$, and assume $\pi < \theta < 3\pi/2$. Then an integral representation*

$$\zeta_{\exp.\mathrm{II}}(s;\lambda;z) = (2\pi i\lambda)^{1-s}\Gamma(s-1) + \frac{(2\pi i\lambda)^{-s}}{\pi i} \int_{\infty e^{i\theta}}^{(0+)} \frac{u^{(s-1)/2} e^{zu/(2\pi i\lambda)}}{1-e^{u/(2\pi i\lambda)}} K_{s-1}(2\sqrt{u})du \tag{3}$$

holds for $s \in \mathbb{C} \setminus \{1,0,-1,-2,\dots\}$, which provides a holomorphic continuation to the whole s-plane except on $s \in \{1,0,-1,-2,\dots\}$. Above integral representation gives the functional relation

$$\begin{aligned}\zeta_{\exp.\mathrm{II}}(s;\lambda;z) &+ (-1)^s \zeta_{\exp.\mathrm{II}}(s;-\lambda;1-z) \\ &= 2\pi i e^{-\pi i s} \sum_{n=1}^{\infty} (n/\lambda)^{(s-1)/2} e^{2\pi i z n} I_{s-1}(4\pi i\sqrt{n\lambda}),\end{aligned} \tag{4}$$

for $s \in \mathbb{C} \setminus \{1,0,-1,-2,\dots\}$.

2 *J*-Bessel zeta-function

Let $\theta > 0$, $\nu \in \mathbb{C}$ and s be a complex variable. Next, we define J-Bessel zeta-function of order $\nu - 1$ as follows:

$$\mathbf{J}_{\nu-1}(s;\theta) := \sum_{n=1}^{\infty} \frac{J_{\nu-1}(2\sqrt{\theta n})}{n^{s+\frac{\nu+1}{2}}}. \tag{5}$$

By the estimates of the J-Bessel function, the Dirichlet series above converges absolutely in the region $\mathrm{Re}(s) > 0$ when $\mathrm{Re}(\nu) > 1/2$, and converges absolutely in the region $\mathrm{Re}(s) > \lfloor 3/2 - \mathrm{Re}(\nu) \rfloor /2 - 1$ when $\mathrm{Re}(\nu) \leq 1/2$. For the specific case, ν is an integer, $\mathbf{J}_{\nu-1}(s;\theta)$ converges absolutely in the region $\mathrm{Re}(s) > (1-\nu)/2$.

Theorem 2 ([No1], Theorem 1.1) *Let $\nu \in \mathbb{C}$ and $\theta > 0$. The J-Bessel zeta-function has an integral representation*

$$\mathbf{J}_{\nu-1}(s;\theta) = \frac{\theta^{s+\frac{\nu+1}{2}}\Gamma(-s)}{2\pi i\Gamma(\nu)} \int_{-\infty}^{(0+)} \frac{u^s e^{\theta u}}{1-e^{\theta u}} F(-s;\nu;-u^{-1})du, \tag{6}$$

which provides a meromorphic continuation to the whole s-plane. Further, the transformation formula

$$\mathbf{J}_{\nu-1}(s;\theta) = \frac{\theta^{\frac{\nu-1}{2}}\Gamma(-s)}{\Gamma(\nu)} \sum_{n=-\infty, n\neq 0}^{\infty} (2\pi i n)^s F\left(-s;\nu;\frac{-\theta}{2\pi i n}\right) \tag{7}$$

holds for $\mathrm{Re}(s) < -1$, *and the power series expansion*

$$\mathbf{J}_{\nu-1}(s;\theta) = \sum_{m=0}^{\infty} \frac{\theta^{\frac{\nu-1}{2}}}{\Gamma(\nu+m)\, m!} \zeta(s+1-m)(-\theta)^m, \tag{8}$$

holds for $s \in \mathbb{C} \setminus \{0,1,2,\dots\}$. *The J-Bessel zeta-function also satisfies the following recurrence formula:*

$$\mathbf{J}_{\nu-1}(s;\theta) + \mathbf{J}_{\nu+1}(s-1;\theta) = \frac{\nu}{\sqrt{\theta}} \mathbf{J}_{\nu}(s;\theta). \tag{9}$$

Remark 1. The power series expressions (2) and (8) are one of the generalizations of Ramanujan's formula [Ra] (the binomial type power series):

$$\zeta(s,1+x) = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{\Gamma(s)\, m!} \zeta(s+m)(-x)^m, \tag{10}$$

for $|x| < 1$ and $s \in \mathbb{C} \setminus \{1\}$. An exponential type series was initially studied by Chowla and Hawkins [CH], and Gauss' hypergeometric type and Kummer's confluent hypergeometric type series were introduced by Katsurada [Kt]. For related results and generalizations of Ramanujan's formula (10), refer to [SC].

Remark 2. More general zeta functions twisted by hypergeometric or Bessel functions are treated by Kaczorowski and Perelli [KP4]. They derived meromorphic continuations of these zeta-functions via the properties of the nonlinear twists obtained in [KP1]–[KP3].

3 Relation to the Poincaré series

Let $m \in \mathbb{Z}_{>0}$ and $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half-plane. We denote $\gamma(z) = (az+b)/(cz+d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and use the notation $\mathrm{e}(z) = \exp(2\pi i z)$. Let $k \geq 4$ be an integer, and define the m-th *Poincaré series* attached to $SL_2(\mathbb{Z})$ of weight k by

$$P_m^k(z) := (-1)^k \sum_{\{c,d\}} \frac{\mathrm{e}(m\gamma(z))}{(cz+d)^k}. \tag{11}$$

Here the summation is taken over $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$, a complete system of representation of $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in SL_2(\mathbb{Z})\right\} \backslash SL_2(\mathbb{Z}) \cong \{(c,d) \in \mathbb{Z}^2 \mid \gcd(c,d) = 1,\ c > 0 \text{ or } c = 0,\, d = 1\}$. The functional relation of $\zeta_{\exp.\mathrm{II}}(s;\lambda;z)$ or transformation formula of $\mathbf{J}_{\nu-1}(s;\theta)$ induce the following proposition which leads to the Fourier expansion of $P_m^k(z)$.

Proposition 1 *Let* $\mu > 0$*. The equality*

$$2\pi(-1)^{\frac{k}{2}} \sum_{n=1}^{\infty} \left(\frac{n}{\mu}\right)^{\frac{k-1}{2}} J_{k-1}(4\pi\sqrt{\mu n})\mathrm{e}(nz) = (-1)^k \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}\left(-\mu/(z+n)\right)}{(z+n)^k} \tag{12}$$

holds for positive integer k*.*

Remark 3. It is easy to see the functional relation (4) is equivalent to (12). To show (12) via the transformation formula (7) is rather complicated (see [No1, Proposition 4.1]). As is well-known, the equality (12), from right side to left side, can be shown by using Fourier transform. Our procedure in the proofs of Proposition 1 differs from these existing methods.

By the definition of $P_m^k(z)$, we see

$$P_m^k(z) = (-1)^k \mathrm{e}(mz) + (-1)^k \sum_{\substack{(c,d)\in\mathbb{Z}^2,\, c>0 \\ \gcd(c,d)=1}} \frac{\mathrm{e}(m\gamma(z))}{(cz+d)^k}. \tag{13}$$

According to ordinary steps, we rearrange the d-sum into n-sum with finite d-sum modulo c, and apply Proposition 1 with taking $\mu = m/c^2$ and replacing z by $z + d/c$. Thus, we achieve the following;

Theorem 3 (Fourier series expansion of the Poincaré series)

$$P_m^k(z) = (-1)^k \mathrm{e}(mz) + (-1)^{\frac{k}{2}} 2\pi \sum_{n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{k-1}{2}}$$
$$\sum_{c=1}^{\infty} \frac{1}{c} K_c(m,n) J_{k-1}\left(\frac{4\pi}{c}\sqrt{mn}\right) \mathrm{e}(nz).$$

Here, the Kloosterman sum is defined as follows:

$$K_c(m,n) = \sum_{\substack{d \mod c \\ \gcd(c,d)=1}} \mathrm{e}\left(\frac{m\bar{d}+nd}{c}\right), \qquad (\bar{d}d \equiv 1 \mod c).$$

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