

# Moduli stacks of $(\varphi, \Gamma)$ -modules

Toby Gee (joint with Matthew Emerton)

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 884596)



European Research Council  
Established by the European Commission

## Galois representations

$K/\mathbf{Q}_p$  finite extension.

$G_K := \text{Gal}(\overline{K}/K)$ .

$R$  a topological ring.

Is there a moduli space of representations  $\rho : G_K \rightarrow \text{GL}_d(R)$ ?

## Moduli spaces of Galois representations

There is a good theory over  $\mathbf{Z}_l$ ,  $l \neq p$ , or even  $\mathbf{Z}[1/p]$ .

Basic point: the action of wild inertia doesn't deform, and tame inertia is "easy". Can work with Weil–Deligne representations and get finite type lci moduli spaces (Helm, ...)

Over  $\mathbf{Z}_p$ : doesn't work.

Mazur: fix  $\bar{\rho} : G_K \rightarrow \mathrm{GL}_d(\mathbf{F})$ ,  $\mathbf{F}/\mathbf{F}_p$  finite, and consider lifts of  $\bar{\rho}$  to  $\rho : G_K \rightarrow \mathrm{GL}_d(R)$ , where  $R$  is Artin local with residue field  $\mathbf{F}$ .  
Moduli space:  $\mathrm{Spf} R_{\bar{\rho}}$ .

How can we let  $\bar{\rho}$  vary?

## Existence of the stack

$K/\mathbf{Q}_p$  finite extension,  $G_K := \mathrm{Gal}(\bar{K}/K)$ .

### Theorem

*There is a Noetherian formal algebraic stack  $\mathcal{X}_d$  over  $\mathrm{Spf} \mathbf{Z}_p$ , such that  $\mathcal{X}_d(\mathrm{Spf} \bar{\mathbf{Z}}_p)$  is naturally equivalent to the groupoid of continuous representations  $G_K \rightarrow \mathrm{GL}_d(\bar{\mathbf{Z}}_p)$ .*

*The underlying reduced substack  $\mathcal{X}_{d,\mathrm{red}}$  is an algebraic stack of finite type over  $\mathbf{F}_p$ , and is equidimensional of dimension  $[K : \mathbf{Q}_p]d(d-1)/2$ .*

Similarly for  $\mathcal{X}_d(\bar{\mathbf{F}}_p)$ . Not true for general  $p$ -adically complete  $\mathbf{Z}_p$ -algebras, e.g.  $\mathbf{F}_p[x]$ .

n.b.  $\mathcal{X}_d$  is not a  $p$ -adic formal algebraic stack. It is probably  $[K : \mathbf{Q}_p]d^2$ -dimensional.

## The 1-dimensional case

For simplicity from now, unless otherwise stated:  $K = \mathbf{Q}_p$ .

Characters  $G_{\mathbf{Q}_p} \rightarrow \overline{\mathbf{F}}_p^\times$  are of the form  $\lambda_a \bar{\varepsilon}^i$ ,  $0 \leq i < p - 1$ , where  $\lambda_a$  is the unramified character taking  $\text{Frob} \mapsto a$ .

$\mathcal{X}_1$  has  $(p - 1)$  irreducible components, indexed by  $i$ .

$\mathcal{X}_{1,\text{red}}$  is 0-dimensional: one dimension for  $a$ , but automorphisms are  $\mathbf{G}_m$ .

## Definition of the stack

$A$  a finite type  $\mathbf{Z}/p^a\mathbf{Z}$ -algebra for some  $a \geq 1$ .

### Definition

$\mathcal{X}_d(A) :=$  rank  $d$  projective étale  $(\varphi, \Gamma)$ -modules with  $A$ -coefficients.

Extend to general  $p$ -adically complete  $A$  by taking limits.

Identification of  $\mathcal{X}_d(\overline{\mathbf{Z}}_p)$  with  $G_K \rightarrow \text{GL}_d(\overline{\mathbf{Z}}_p)$  is due to Fontaine.

## étale $(\varphi, \Gamma)$ -modules

$A$  a finite type  $\mathbf{Z}/p^a\mathbf{Z}$ -algebra for some  $a \geq 1$ .

A rank  $d$  projective  $(\varphi, \Gamma)$ -module with  $A$ -coefficients is: a rank  $d$  projective  $A((T))$ -module  $M$  with commuting semilinear actions of  $\varphi$  and  $\Gamma$ .

$\varphi : A((T)) \rightarrow A((T))$  is  $A$ -linear,  $\varphi(1 + T) = (1 + T)^p$ .

$\Gamma = \text{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$ ,  $\varepsilon : \Gamma \rightarrow \mathbf{Z}_p^\times$  cyclotomic character.

$\gamma : A((T)) \rightarrow A((T))$  is  $A$ -linear,  $\gamma(1 + T) = (1 + T)^{\varepsilon(\gamma)}$ ,

étale:  $M = A((T)) \cdot \varphi(M)$ .

## Closed points and specializations

Irreducible representations  $G_{\mathbf{Q}_p} \rightarrow \text{GL}_2(\overline{\mathbf{F}}_p)$  are again discrete up to unramified twist, and give 0-dimensional substacks of the 1-dimensional algebraic stack  $\mathcal{X}_{2,\text{red}}$ .

Reducible indecomposable representations  $\begin{pmatrix} \lambda_a \bar{\varepsilon}^i & * \\ 0 & \lambda_b \bar{\varepsilon}^j \end{pmatrix}$  with  $* \neq 0$  can specialize to  $* = 0$ , so these are not closed points.

Fact: closed points of  $\mathcal{X}_d = \text{semisimple } G_K \rightarrow \text{GL}_d(\overline{\mathbf{F}}_p)$ .

## Closed points and specializations

Are there other specializations?

Consider the family of étale  $\varphi$ -modules over  $\overline{\mathbf{F}}_p$  given by

$$\varphi = \begin{pmatrix} a_p & -1 \\ T^i & 0 \end{pmatrix}$$

with  $1 \leq i \leq p-2$  and  $a_p \in \overline{\mathbf{F}}_p$ . (This can be equipped with an action of  $\Gamma$ ).

If  $a_p \neq 0$ , corresponding  $G_{\mathbf{Q}_p} \rightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_p)$  is reducible, but if  $a_p = 0$  it is irreducible.

This cannot happen for a literal family of representations!

## Irreducible components

Example from the previous slide is a feature, not a bug.

For each  $0 \leq i, j < p-1$  consider  $\begin{pmatrix} \lambda_a \bar{\varepsilon}^i & * \\ 0 & \lambda_b \bar{\varepsilon}^j \end{pmatrix}$  with  $a, b$  varying.

The closure of this is a 1-dimensional substack.

In fact if  $i \equiv j+1 \pmod{p-1}$  we have an extra irreducible component with  $a = b$ .

### Theorem

$(K = \mathbf{Q}_p)$   $\mathcal{X}_2$  has  $p(p-1)$  irreducible components, indexed by  $i, j$  as above.

More generally, the irreducible components of  $\mathcal{X}_d$  are indexed by  $(k_1, \dots, k_n)$  with  $0 \leq k_i - k_{i+1} \leq p-1$ ,  $0 \leq k_n < p-1$ .

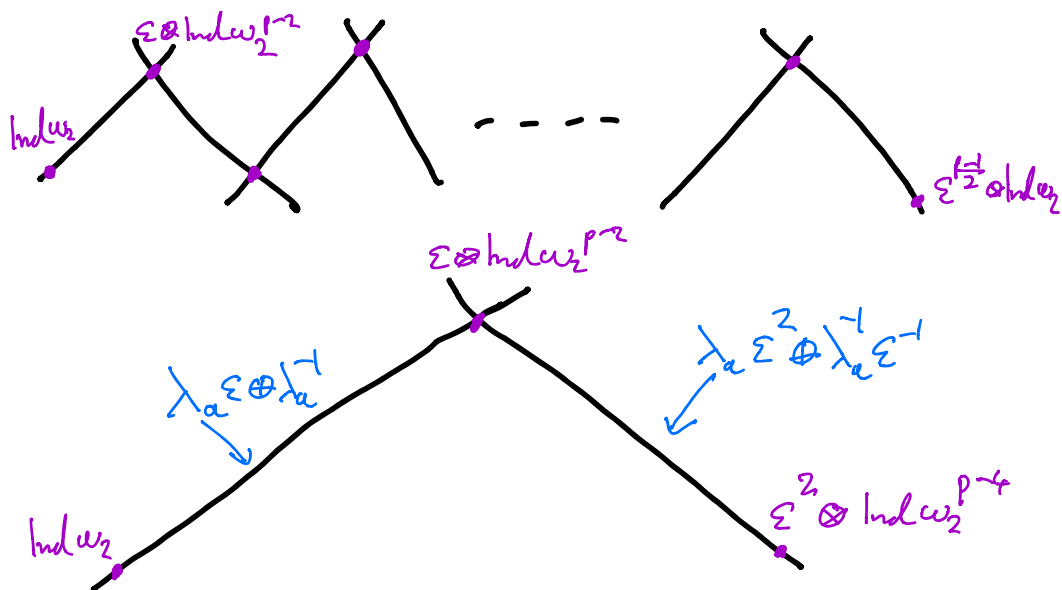
## A coarse moduli space for $GL_2(\mathbb{Q}_p)$

In general the specialisation relations are complicated and no obvious coarse moduli space.

For  $GL_2(\mathbb{Q}_p)$ : the only interesting specialisations are those we wrote down before.

Fixing determinants, have a 1-dimensional scheme, in fact a chain of  $\mathbf{P}^1$ s.

## A coarse moduli space for $GL_2(\mathbb{Q}_p)$



## The crystalline/potentially semistable substacks

Fix  $\underline{\lambda} = (\lambda_1 \geq \dots \geq \lambda_d)$ .

### Theorem

There is a unique closed substack  $\mathcal{X}_d^{\text{crys}, \underline{\lambda}}$  of  $\mathcal{X}_d$  which is flat over  $\text{Spf } \mathbf{Z}_p$ , and such that if  $A/\mathbf{Z}_p$  is finite flat, then

$$\mathcal{X}_d^{\text{crys}, \underline{\lambda}} = \{ \rho : G_{\mathbf{Q}_p} \rightarrow \text{GL}_d(A) \mid \rho \otimes \mathbf{Q}_p \text{ is crystalline with Hodge-Tate weights } \underline{\lambda} \}.$$

$\mathcal{X}_d^{\text{crys}, \underline{\lambda}}$  is a  $p$ -adic formal algebraic stack.

If  $\lambda_1 > \dots > \lambda_d$  then  $(\mathcal{X}_d^{\text{crys}, \underline{\lambda}})_{\mathbf{F}_p}$  is equidimensional of dimension equal to  $\dim \mathcal{X}_{d, \text{red}}$ .

Analogous result holds for (potentially) semistable representations, for any  $K$ .

## The geometric Breuil–Mézard conjecture I

If  $\lambda_1 > \dots > \lambda_d$  then  $(\mathcal{X}_d^{\text{crys}, \underline{\lambda}})_{\mathbf{F}_p}$  is equidimensional of dimension  $\dim \mathcal{X}_{d, \text{red}}$ .

Not necessarily reduced: write  $\mathcal{Z}(\underline{\lambda}) = \mathcal{Z}((\mathcal{X}_d^{\text{crys}, \underline{\lambda}})_{\mathbf{F}_p})$ , a formal sum of irreducible components of  $\mathcal{X}_{d, \text{red}}$ .

Question (Breuil–Mézard): Which components? What are the multiplicities?

## $p$ -adic local Langlands

Expectation/hope: there is a sheaf  $\mathcal{M}$  of  $\mathrm{GL}_d(K)$ -representations on  $\mathcal{X}_d$  which:

satisfies local-global compatibility for completed cohomology of locally symmetric spaces.

encodes the weight part of Serre's conjecture.

answers the question of Breuil–Mézard.

...

( $p$ -adic analogue of Hellmann/Ben-Zvi–Chen–Helm–Nadler/Zhu.)

Dotto–Emerton–G. (in progress): holds for  $\mathrm{GL}_2(\mathbf{Q}_p)$ .

Idea: Colmez's construction of  $(D \boxtimes \mathbf{P}^1)/(D^\natural \boxtimes \mathbf{P}^1)$  makes sense on  $\mathcal{X}_2$ .

## The weight part of Serre's conjecture

$K = \mathbf{Q}_p$ . Let  $\sigma$  be an irreducible  $\overline{\mathbf{F}}_p$ -representation of  $\mathrm{GL}_d(\mathbf{F}_p)$ .

$$\mathcal{M}(\sigma) := \mathrm{Hom}_{\mathrm{GL}_d(\mathbf{Z}_p)}(\mathcal{M}, \sigma^\vee)^\vee.$$

Expectation: support of  $\mathcal{M}(\sigma)$  is a union of irreducible components of  $\mathcal{X}_{d,\mathrm{red}}$ .

$$\mathcal{Z}(\sigma) := \mathcal{Z}(\mathcal{M}(\sigma)).$$

If local-global compatibility holds, the support of  $\mathcal{Z}(\sigma)$  exactly determines the weight part of Serre's conjecture.

True for modular curves.



## The geometric Breuil–Mézard conjecture II

For  $\underline{\lambda} = (\lambda_1 > \dots > \lambda_d)$ , set  $\mathcal{M}(\underline{\lambda}) := \text{Hom}_{\text{GL}_d(\mathbf{Z}_p)}(\mathcal{M}, \pi_{\underline{\lambda}}^\vee)^\vee$ , where  $\pi_{\underline{\lambda}}$  = irreducible algebraic  $\text{GL}_d(\mathbf{Z}_p)$ -representation of highest weight  $(\lambda_1 - (d - 1), \dots, \lambda_{d-1} - 1, \lambda_d)$ .

Expectations imply:  $\mathcal{Z}(\mathcal{M}(\underline{\lambda})) = \mathcal{Z}(\underline{\lambda}) = \mathcal{Z}((\mathcal{X}_d^{\text{crys}, \underline{\lambda}})_{\mathbf{F}_p})$ .

Then we have the *geometric Breuil–Mézard conjecture*  $\mathcal{Z}(\underline{\lambda}) = \sum_{\sigma} n_{\sigma}(\underline{\lambda}) \mathcal{Z}(\sigma)$ , where  $n_{\sigma}(\underline{\lambda})$  = multiplicity of  $\sigma$  in  $\pi_{\underline{\lambda}} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ .

Known for  $\text{GL}_2(\mathbf{Q}_p)$  (Kisin, Paškūnas, ...)

Extends to potentially crystalline/semistable case, using inertial local Langlands correspondence.

This version can sometimes be proved if  $\underline{\lambda}$  is small, e.g. G.–Kisin, Le–Le Hung–Levin–Morra.

## Patched modules and deformation rings

No construction is known of  $\mathcal{M}$  other than for  $\text{GL}_1$  or  $\text{GL}_2(\mathbf{Q}_p)$ .

However there is a candidate after pulling back to the versal ring at a fixed  $\bar{\rho} : G_K \rightarrow \text{GL}_d(\bar{\mathbf{F}}_p)$ : the *patched module*  $M_{\infty}$  of Caraiani–Emerton–G.–Geraghty–Paškūnas–Shin.

Construction via globalisation and Taylor–Wiles patching of cohomology of unitary Shimura varieties.

The pullback of the geometric Breuil–Mézard conjecture  $\mathcal{Z}(\underline{\lambda}) = \sum_{\sigma} n_{\sigma}(\underline{\lambda}) \mathcal{Z}(\sigma)$  to the versal ring is equivalent to automorphy lifting theorems (Kisin).

e.g. can use solvable base change to reduce the potentially crystalline case to the crystalline case (G.–Kisin).