QUASI-PULLBACK OF CERTAIN SIEGEL MODULAR FORMS AND BORCHERDS PRODUCTS

KEN-ICHI YOSHIKAWA

Abstract. A holomorphic torsion invariant of $K3$ surfaces with involution was introduced by the author [19], and it has been proved in [14] that this invariant is expressed as the product of an explicit Borcherds product and an explicit Siegel modular form. In this note, we report that these automorphic forms are closed under the operation called quasi-pullback (Theorem 5.9). As a result, we obtain some Borcherds products as the quasi-pullback of certain Siegel modular form (Theorem 5.8).

Contents

1. Mirror symmetry at genus one : The BCOV conjecture

Let X be a Calabi-Yau threefold. The generating function of the genus one instanton numbers of X is defined as the following infinite product

$$F_1^{\mathrm{top}}(q) = q^{c_2^\vee/24} \prod_{\lambda\in H_2(X,\mathbb{Z})\setminus\{0\}} \left(1-q^\lambda\right)^{n_0(\lambda)/12} \prod_{k\geq 1}\left(1-q^{k\lambda}\right)^{n_1(\lambda)},$$

where $q^\lambda = e^{2\pi i\langle\lambda,t\rangle}$, $t\in H^2(X,\mathbb{C})$, $n_g(\lambda)$ is the genus-g Gopakumar-Vafa invariant of X for $\lambda\in H_2(X,\mathbb{Z})$, and $c_2^\vee$ is the Poincaré dual of $c_2(X)$. Hence $q^{c_2^\vee/24} = \exp\left(2\pi i\int_X c_2(X)\wedge t\right)$.

In their paper [1], by making use of holomorphic analytic torsions, physicists Bershadsky-Cecotti-Ooguri-Vafa introduced an invariant τ_{BCOV} of Calabi-Yau threefolds, which is the B-model counterpart of the generating function F_1^{top}. After Bershadsky-Cecotti-Ooguri-Vafa, this invariant of Calabi-Yau threefolds is called the *BCOV invariant*. (See Section 2.) Then the conjecture of Bershadsky-Cecotti-Ooguri-Vafa (mirror symmetry at genus one) is formulated as follows.

Conjecture 1.1 (Bershadsky-Cecotti-Ooguri-Vafa [1])**.** Let $f\colon \mathcal{X}^\vee\to(\Delta^*)^n$ be a mirror family of X over a punctured polydisc of dimension $n = h^{1,1}(X) = h^{1,2}(X^\vee)$, whose general fiber $X^\vee$ is a Calabi-Yau threefold such that $h^{1,1}(X) = h^{1,2}(X^\vee)$,

$h^{1,2}(X) = h^{1,1}(X^\vee)$ and $\chi(X^\vee) = -\chi(X)$. Let $q = (q_1, \ldots, q_n)$ be the system of canonical coordinates on $(\Delta^*)^n$ induced from the mirror family $f\colon \mathcal{X}^\vee \to (\Delta^*)^n$. Then in the canonical coordinates, the following equality of functions on $(\Delta^*)^n$ holds:

$$\tau_{\mathrm{BCOV}}(X_s^\vee) = C \left\| F_1^{\mathrm{top}}(q)^2 \left(\frac{\Xi_s}{\int_{A_0} \Xi_s} \right)^{3+n-\frac{\chi}{12}} \otimes \left(q_1 \frac{\partial}{\partial q_1} \wedge \cdots \wedge q_n \frac{\partial}{\partial q_n} \right) \right\|^2 .$$

Here $\chi = \chi(X) = -\chi(X^\vee)$, $\Xi_s \in H^0(X_s^\vee, K_{X_s^\vee})$ is a holomorphic family of nowhere vanishing holomorphic 3-forms, $A_0 \in H_3(X_s^\vee, \mathbb{Z})$ is the cycle invariant under the monodromy, $\|\cdot\|$ is the natural norm, and C is a constant.

The BCOV conjecture is still widely open. The only known case of the BCOV conjecture is the pair of quintic threefolds and mirror quintic threefolds ([26], [9]). Very recently, the BCOV invariant is extended to Calabi-Yau manifolds of arbitrary dimension by Eriksson-Freixas i Montplet-Mourougane ([7]). Combining the earlier result of Zinger ([26]), they establish the BCOV conjecture for the pair consisting of a Calabi-Yau hypersurface of $\mathbb{P}^n$ and its mirror family for all $n \geq 5$ ([8]). (In higher dimensions, $F_1^{\mathrm{top}}(q)$ should be formulated in terms of Gromov-Witten invariants.)

There is a class of Calabi-Yau threefolds called the Borcea-Voisin threefolds, whose BCOV invariant can be computed explicitly ([22]). The BCOV invariant of the Borcea-Voisin threefolds of fixed topological type is expressed as the product of the following three types of automorphic forms; a Borcherds product, a Siegel modular form (pulled back to a domain of type IV via the Torelli map), and the Dedekind η-function. From Conjecture 1.1 for the Borcea-Voisin threefolds, we have the following

Conjectural Observation *The pullback of a certain Siegel modular form to a domain of type IV via the Torelli map should be expressed as an infinite product of Borcherds type near the zero-dimensional cusp.*

In this note, we report a result related to this conjectural observation (Theorem 5.8). The details shall be given in the forthcoming paper ([23]).

2. BCOV invariants

2.1. **BCOV torsion.** Let (X, g_X) be a compact Kähler manifold. Let $\square_{p,q} = (\bar{\partial} + \bar{\partial}^*)^2$ be the Hodge-Kodaira Laplacian of (X, g_X) acting on the (p,q)-forms on X. Then the spectral zeta function of $\square_{p,q}$ is defined as

$$\zeta_{p,q}(s) := \sum_{\lambda \in \sigma(\square_{p,q}) \setminus \{0\}} \lambda^{-s} \dim E(\lambda; \square_{p,q}),$$

where $\sigma(\square_{p,q}) \subset \mathbb{R}_{\geq 0}$ is the set of eigenvalues of $\square_{p,q}$, $E(\lambda; \square_{p,q})$ is the eigenspace of $\square_{p,q}$ corresponding to the eigenvalue λ. Since $\square_{p,q}$ is an elliptic operator, it is classical that $\sigma(\square_{p,q})$ is a discrete set of $\mathbb{R}_{\geq 0}$ and $E(\lambda; \square_{p,q})$ is of finite dimensional. Moreover, the following is also classical:

Fact 2.1. *$\zeta_{p,q}(s)$ converges absolutely when $\mathrm{Re}\, s > \dim X$, admits a meromorphic continuation to $\mathbb{C}$, and is holomorphic at $s = 0$.*

The notion of analytic torsion was introduced by Ray-Singer ([17]) and was extended to the case of general twisting bundles by Bismut-Gillet-Soulé ([2]).

Definition 2.2. The analytic torsion of (X, g_X) with respect to the vector bundle $\Omega_X^p = \Lambda^p \Omega_X$ is the real number defined as

$$\tau(X, \Omega_X^p) = \exp[-\sum_{q \geq 0} (-1)^{p+q} q\, \zeta'_{p,q}(0)].$$

To express the B-model counter part of the generating function F_1^{top}, Bershadsky-Cecotti-Ooguri-Vafa ([1]) introduced the following combination of the Ray-Singer analytic torsions.

Definition 2.3. The BCOV torsion of (X, g_X) is the real number defined as

$$\mathcal{T}_{\mathrm{BCOV}}(X, g_X) = \exp[-\sum_{p,q \geq 0} (-1)^{p+q} pq\, \zeta'_{p,q}(0)] = \prod_{p \geq 0} \tau(X, \Omega_X^p)^{(-1)^p p}.$$

We remark that in general, the analytic torsions $\tau(X, \Omega_X^p)$, $p \geq 0$ and the BCOV torsion $\mathcal{T}_{\mathrm{BCOV}}(X, g_X)$ are NOT invariants of X, since they do depend on the choice of the Kähler metric g_X by the anomaly formula ([2]).

2.2. **Calabi-Yau manifolds.**

Definition 2.4. A smooth irreducible compact Kähler n-fold X is called a Calabi-Yau manifold if the following conditions are satisfied:

- $K_X = \Omega_X^n \cong \mathcal{O}_X$
- $H^q(X, \mathcal{O}_X) = 0 \quad (0 < q < n)$.

When $n = 1$, Calabi-Yau curves are elliptic curves. When $n = 2$, Calabi-Yau surfaces are $K3$ surfaces. In these low dimensional cases, the moduli space of *polarized* Calabi-Yau manifolds is a locally symmetric variety. In higher dimensions, this is no longer the case. If $n \geq 3$, the topological types of Calabi–Yau n-folds is not unique. It is not known that for a fixed $n \geq 3$, the number of deformation types of Calabi-Yau n-folds is finite or not. The global structure of the moduli space of polarized Calabi-Yau manifolds of dimension $n \geq 3$ is not understood in general.

2.3. **BCOV invariant for Calabi-Yau threefolds.** For Calabi-Yau threefolds, one can construct an invariant by making use of the BCOV torsion as follows.

Definition 2.5 (Bershadsky-Cecotti-Ooguri-Vafa [1], Fang-Lu-Yoshikawa [9])**.** The BCOV invariant of a Calabi–Yau threefold X is defined by

$$\begin{aligned}\tau_{\mathrm{BCOV}}(X) &= \mathrm{Vol}(X,\gamma)^{-3+\frac{\chi(X)}{12}} \mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma])^{-1} \mathcal{T}_{\mathrm{BCOV}}(X,\gamma) \\ &\quad \times \exp\left[-\frac{1}{12}\int_X \log\left(\frac{\sqrt{-1}\eta\wedge\overline{\eta}}{\gamma^3/3!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|_{L^2}^2}\right) c_3(X,\gamma)\right].\end{aligned}$$

Here γ is a Kähler form on X, $\eta \in H^0(X, \Omega_X^3) \setminus \{0\}$ is a nowhere vanishing holomorphic 3-form, $\chi(X)$ is the topological Euler number of X, $c_3(X, \gamma)$ is the Euler form of (X, γ), and

$$\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma]) = \mathrm{Vol}(H^2(X,\mathbb{R})/H^2(X,\mathbb{Z}), \langle\cdot,\cdot\rangle_{L^2,\gamma})$$

is the covolume of the lattice $H^2(X, \mathbb{Z})$ with respect to the metric on $H^2(X, \mathbb{R})$ induced from γ, i.e. the volume of the real torus $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$ with respect to γ. We remark that the cohomology group $H^2(X, \mathbb{R})$ is endowed with the inner product by identifying its vector with the corresponding harmonic 2-form with respect to γ.

Theorem 2.6 ([9]). *$\tau_{\rm BCOV}(X)$ is an invariant of X, i.e., it is independent of the choice of γ, η. In particular, $\tau_{\rm BCOV}$ gives rise to a function on the moduli space of Calabi-Yau threefolds.*

Very recently, the notion of BCOV invariant is extended to Calabi-Yau manifolds of arbitrary dimension by Eriksson-Freixas i Montplet-Mourougane ([7]) and to certain pairs consisting of a Kähler manifold and its divisor by Y. Zhang ([24], [25], [10]), who also proves the birational invariance of the BCOV invariants.

3. Calabi-Yau threefolds of Borcea-Voisin

There is a class of Calabi-Yau threefolds, called Borcea-Voisin threefolds, whose BCOV invariant is computed explicitly in terms of various automorphic forms. In this section, we recall Borcea-Voisin threefolds. For this purpose, we need the notion of 2-elementary $K3$ surfaces.

3.1. **Borcea-Voisin threefolds.**

Definition 3.1. A pair (S,θ) consisting of a $K3$ surface and its involution θ is called a *2-elementary $K3$ surface* if θ is anti-symplectic i.e., $\theta^* = -1$ on $H^0(S,\Omega^2_S)$.

Definition 3.2 ([3], [18]). For a 2-elementary $K3$ surface (S,θ) and an elliptic curve T, define a Calabi-Yau threefold $X_{(S,\theta,T)}$ as

$$X_{(S,\theta,T)} := \text{ crepant resolution of } \frac{S\times T}{\theta\times(-1_T)}.$$

The threefold $X_{(S,\theta,T)}$ is called a Borcea-Voisin threefold. The type of $X_{(S,\theta,T)}$ is defined as the isometry class of the anti-invariant lattice of θ

$$H^2(S,\mathbb{Z})_- := \{l\in H^2(S,\mathbb{Z});\ \theta^*(l) = -l\}.$$

In what follows, we often identify a lattice with its isometry class. By a lattice, we mean a free $\mathbb{Z}$-module of finite rank endowed with a non-degenerate, integral, symmetric bilinear form. For an even lattice L, its dual lattice $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ is denoted by $L^\vee$. The finite abelian group $A_L := L^\vee/L$ is called the discriminant group of L, which is equipped with the $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form q_L called the discriminant form. A lattice L is 2-elementary if there exists $l\in\mathbb{Z}_{\geq 0}$ with $A_L\cong(\mathbb{Z}/2\mathbb{Z})^{\oplus l}$. The parity $\delta(L)\in\{0,1\}$ of a 2-elementary lattice L is defined as follows. If $q_L(A_L)\subset\mathbb{Z}/2\mathbb{Z}$, then $\delta(L)=0$. Otherwise, $\delta(L)=1$.

Fact 3.3 ([3], [18], [16]). *The following hold:*

1. *The deformation type of $X_{(S,\theta,T)}$ is determined by its type $H^2(S,\mathbb{Z})_-$.*
2. *$H^2(S,\mathbb{Z})_-$ is a primitive 2-elementary sublattices of the $K3$-lattice*

$$\mathbb{L}_{K3} := \mathbb{U}\oplus\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{E}_8\oplus\mathbb{E}_8$$

with signature $(2,b^-)$, $0\leq b^-\leq 19$. Here $\mathbb{U}$ is an even unimodular lattice of signature $(1,1)$ and $\mathbb{E}_8$ is a negative-definite E_8-lattice, i.e., a negative-definite even unimodular lattice of rank 8.

Here a sublattice $\Lambda\subset\mathbb{L}_{K3}$ is primitive if $\mathbb{L}_{K3}/\Lambda$ is a torsion free abelian group. In what follows, for the root systems A_k, D_k, E_k, the corresponding *negative-definite* root lattices are denoted by $\mathbb{A}_k$, $\mathbb{D}_k$, $\mathbb{E}_k$, respectively. We set $\mathbb{A}_1^+ := \mathbb{A}_1(-1)$ etc. For a lattice $L=(\mathbb{Z}^r,\langle\cdot,\cdot\rangle)$ and an integer $k\in\mathbb{Z}\setminus\{0\}$, we define $L(k) := (\mathbb{Z}^r, k\langle\cdot,\cdot\rangle)$. Then the isometry classes of primitive 2-elementary sublattices of $\mathbb{L}_{K3}$

with signature $b^+ = 2$ was determined by Nikulin ([15]), which consists of 75 isometry classes and whose list is given in the following table.

TABLE 1. Primitive 2-elementary sublattices of $\mathbb{L}_{K3}$ with $b^+ = 2$

g	$\delta = 1$		$\delta = 0$
0	$(\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1^{\oplus t}$	$(0 \le t \le 9)$	$\mathbb{U}(2)^{\oplus 2}$
1	$\mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus t}$	$(0 \le t \le 9)$	$\mathbb{U} \oplus \mathbb{U}(2)$, $\mathbb{U}(2)^{\oplus 2} \oplus \mathbb{D}_4$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$
2	$\mathbb{U}^{\oplus 2} \oplus \mathbb{A}_1^{\oplus t}$	$(1 \le t \le 9)$	$\mathbb{U}^{\oplus 2}$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4$, $\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8(2)$
3	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^{\oplus t}$	$(1 \le t \le 6)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4^{\oplus 2}$
4	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_6 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \le t \le 5)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4^{\oplus 2}$
5	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_7 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \le t \le 5)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_8$
6	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(1 \le t \le 5)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8$, $\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{D}_8$
7	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(1 \le t \le 2)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{E}_8$
8	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_6 \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \le t \le 1)$	
9	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_7 \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \le t \le 1)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_8 \oplus \mathbb{E}_8$
10	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8^{\oplus 2} \oplus \mathbb{A}_1$		$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8^{\oplus 2}$

3.2. **Period domain for 2-elementary $K3$ surfaces.** Let (S, θ) be a 2-elementary $K3$ surface. Fix a lattice $\Lambda \subset \mathbb{L}_{K3} = \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$ such that

$$\Lambda \cong H^2(S, \mathbb{Z})_-.$$

Since Λ is a 2-elementary lattice with $\operatorname{sign}(\Lambda) = (2, r-2)$, $r = \operatorname{rk}\Lambda$, one can attach a domain of type IV to the lattice Λ as follows.

Definition 3.4. For a lattice Λ with $\operatorname{sign}(\Lambda) = (2, r-2) = (2, b^-)$, define

$$\begin{aligned}\Omega_\Lambda = \Omega_\Lambda^+ \amalg \Omega_\Lambda^- &:= \{[\eta] \in \mathbb{P}(\Lambda \otimes \mathbb{C});\ \langle \eta, \eta \rangle_\Lambda = 0,\ \langle \eta, \bar{\eta} \rangle_\Lambda > 0\} \\ &\cong \frac{SO_0(2, b^-)}{SO(2) \times SO(b^-)}\end{aligned}$$

Here $\Omega_\Lambda^\pm$ is a bounded symmetric domain of type IV of $\dim \Omega_\Lambda = r - 2$.

3.3. **Period of 2-elementary $K3$ surfaces.** Let (S, θ) be a 2-elementary $K3$ surface of type Λ. Then there exists an isometry $\alpha \colon H^2(S, \mathbb{Z}) \cong \mathbb{L}_{K3}$ such that

$$\alpha(H^2(S, \mathbb{Z})_-) = \Lambda.$$

Let $\omega \in H^0(S, \Omega_S^2)$ be a nowhere vanishing holomorphic 2-form on S. Since $H^0(S, \Omega_S^2) \subset H^2(S, \mathbb{C})_-$ by the condition $\theta^* = -1$ on $H^0(S, \Omega_S^2)$, one has $\alpha(\omega) \in \Lambda \otimes \mathbb{C}$.

Definition 3.5. The period of a 2-elementary $K3$ surface (S, θ) of type Λ is defined by

$$\varpi(S, \theta) := [\alpha(\omega)] = \left[\left(\cdots, \int_{\lambda_i} \omega, \cdots\right)\right] \in O^+(\Lambda) \backslash \Omega_\Lambda^+,$$

where $\{\lambda_i\}$ is a $\mathbb{Z}$-basis of $H_2(S, \mathbb{Z})_-$ and $O^+(\Lambda)$ is the automorphism group of Λ preserving Ω_Λ^+.

Theorem 3.6 ([19]). *Set*

$$\mathcal{M}_\Lambda^0 := O^+(\Lambda)\backslash(\Omega_\Lambda^+ - \mathcal{D}_\Lambda),$$

where

$$\mathcal{D}_\Lambda = \bigcup_{d\in\Delta_\Lambda} H_d, \qquad H_d = \Omega_\Lambda \cap d^\perp, \qquad \Delta_\Lambda = \{d \in \Lambda;\, \langle d, d\rangle = -2\}$$

is the discriminant divisor of Ω_Λ^+*, i.e., the Heegner divisor of norm* -2*-vectors of* Λ*. Then* $\mathcal{M}_\Lambda^0$ *is a coarse moduli space of* 2*-elementary* $K3$ *surfaces of type* Λ *via the period mapping. Namely, one has the following:*

(1) *There is a one-to-one correspondence between the set of isomorphism classes of* 2*-elementary* $K3$ *surfaces of type* Λ *and* $\mathcal{M}_\Lambda^0$ *via the period mapping.*
(2) *For every family* $f\colon (\mathcal{X}, \iota) \to T$ *of* 2*-elementary* $K3$ *surfaces of type* Λ*, the period mapping* $\varpi\colon T \ni t \to \varpi(X_t, \iota_t) \in \mathcal{M}_\Lambda^0$ *is holomorphic.*

3.4. **Moduli space of Borcea-Voisin threefolds.** A Borcea-Voisin threefold $X_{(S,\theta,T)}$ admits double fibrations:

$$\pi_1\colon X_{(S,\theta,T)} \to S/\theta, \qquad \pi_2\colon X_{(S,\theta,T)} \to T/\pm 1 = \mathbb{P}^1.$$

Here S/θ is either a rational surface or an Enriques surface by Nikulin ([16]).

Theorem 3.7 ([20]). *The coarse moduli space of Borcea-Voisin threefolds of type* Λ *endowed with the double fibration structures is isomorphic to the modular variety*

$$\mathcal{M}_\Lambda^0 \times X(1) = O^+(\Lambda)\backslash(\Omega_\Lambda^+ - \mathcal{D}_\Lambda) \times (\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H})$$

via the period mapping

$$X_{(S,\theta,T)} \mapsto (\varpi(S,\theta), \varpi(T)),$$

where $\varpi(S,\theta)$ *and* $\varpi(T)$ *are the periods of* (S,θ) *and* T*, respectively.*

4. Borcherds products and a formula for τ_{BCOV}

To give a formula for the BCOV invariant of the Borcea-Voisin threefolds, we recall Borcherds products. For this sake, recall that the Dedekind η-function and the Jacobi θ-series are defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad \vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}, \qquad q = e^{2\pi i\tau}.$$

Definition 4.1. For a 2-elementary lattice Λ, define a modular form ϕ_Λ for $\Gamma_0(4)$ by

$$\phi_\Lambda(\tau) := \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}\,\vartheta(\tau)^{12-r(\Lambda)}.$$

To construct a Borcherds product for Λ, we lift $\phi_\Lambda(\tau)$ to a vector-valued modular form for $\mathrm{Mp}_2(\mathbb{Z})$ using the Weil representation $\rho_\Lambda\colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\Lambda^\vee/\Lambda])$.

4.1. **Weil representation for 2-elementary lattices.** Let Λ be a 2-elementary lattice, i.e., $A_\Lambda = (\mathbb{Z}/2\mathbb{Z})^{\ell(\Lambda)}$. Let $\mathbb{C}[A_\Lambda] = \mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^{\ell(\Lambda)}] = \mathbb{C}^{2^{\ell(\Lambda)}}$ be the group ring of $A_\Lambda = \Lambda^\vee/\Lambda$. Let $\{\mathfrak{e}_\gamma\}_{\gamma\in A_\Lambda}$ be the standard basis of the group ring $\mathbb{C}[A_\Lambda]$. Let $\mathrm{Mp}_2(\mathbb{Z}) := \{((\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}), \sqrt{c\tau+d});\ (\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}) \in \mathrm{SL}_2(\mathbb{Z})\}$ be the metaplectic double covering of $\mathrm{SL}_2(\mathbb{Z})$. Let S, T be the standard generator of $\mathrm{Mp}_2(\mathbb{Z})$ given by

$$S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right), \qquad T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right).$$

Definition 4.2. The Weil representation $\rho_\Lambda \colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[A_\Lambda])$ is defined by

$$\rho_\Lambda(T)\,\mathfrak{e}_\gamma := e^{\pi i \gamma^2}\mathfrak{e}_\gamma, \qquad \rho_\Lambda(S)\,\mathfrak{e}_\gamma := \frac{i^{-\sigma(\Lambda)/2}}{\sqrt{|A_\Lambda|}} \sum_{\delta\in A_\Lambda} e^{-2\pi i \gamma\cdot\delta}\,\mathfrak{e}_\delta,$$

where $\sigma(\Lambda) = b^+(\Lambda) - b^-(\Lambda)$.

4.2. **Vector-valued modular forms for 2-elementary lattices.**

Definition 4.3. For an even 2-elementary lattice Λ, define a vector-valued modular form F_Λ by

$$F_\Lambda(\tau) := \sum_{\gamma\in\widetilde{\Gamma}_0(4)\backslash \mathrm{Mp}_2(\mathbb{Z})} (\phi_\Lambda|_\gamma)(\tau)\,\rho_\Lambda(\gamma^{-1})\,\mathfrak{e}_0,$$

where $|_\gamma$ is the Petersson slash operator, i.e., $(f|_\gamma)(\tau) := (c\tau+d)^{-wt(f)} f(\frac{a\tau+b}{c\tau+d})$ for $\gamma = (\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}) \in \mathrm{SL}_2(\mathbb{Z})$.

Fact 4.4 ([21])**.** *$F_\Lambda(\tau)$ is a modular form of type ρ_Λ of weight $1 - b^-(\Lambda)/2$, i.e.,*

$$F_\Lambda\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{1-\frac{b^-(\Lambda)}{2}} \rho_\Lambda\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \sqrt{c\tau+d}\right)\cdot F_\Lambda(\tau)$$

for all $((\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}), \sqrt{c\tau+d}) \in \mathrm{Mp}_2(\mathbb{Z})$.

4.3. **Borcherds products.** Let

$$F_\Lambda(\tau) = \sum_{\gamma\in\Lambda^\vee/\Lambda} \mathfrak{e}_\gamma \sum_{m\in\mathbb{Z}+\gamma^2/2} c_\gamma(m) q^m$$

be the Fourier series expansion of $F_\Lambda(\tau)$ at $+i\infty$. For simplicity, assume the following splitting:

$$\Lambda = \mathbb{U}(-1) \oplus L,$$

where L is a 2-elementary Lorentzian lattice. Let

$$\mathcal{C}_L = \mathcal{C}_L^+ \amalg \mathcal{C}_L^- = \{x \in L\otimes\mathbb{R};\ x^2 > 0\}$$

be the positive cone of L. The vector valued modular form F_L induces the following chamber structure on $\mathcal{C}_L$

$$\mathcal{C}_L \setminus \bigcup_{l\in L^\vee,\, l^2<0,\, c_{[l]}(l)\neq 0} l^\perp.$$

Its connected component is called a Weyl chamber of (L, F_L).

Let us recall a realization of Ω_Λ^+ as a tube domain. The tube domain $L\otimes\mathbb{R} + i\,\mathcal{C}_L^+$ is isomorphic to Ω_Λ^+ via the exponential map

$$L\otimes\mathbb{R} + i\,\mathcal{C}_L^+ \ni z \to \exp(z) := \left[\left(1, z, \frac{\langle z, z\rangle}{2}\right)\right] \in \Omega_\Lambda^+,$$

where $(1, \langle z, z\rangle/2) \in \mathbb{U}(-1)\otimes\mathbb{C}$ and $z \in L\otimes\mathbb{C}$. Through this identification, $O^+(\Lambda)$ acts on $L\otimes\mathbb{R} + i\,\mathcal{C}_L^+$.

Definition 4.5 (Borcherds [5])**.** Let $\mathcal{W} \subset \mathcal{C}_L^+$ be a Weyl chamber of (L, F_L). For $z \in L\otimes\mathbb{R} + i\,\mathcal{W}$, define the Borcherds product $\Psi_\Lambda(z, F_\Lambda)$ as

$$\Psi_\Lambda(z, F_\Lambda) := e^{2\pi i\langle \varrho, z\rangle} \prod_{\gamma\in L^\vee/L} \prod_{\lambda\in L+\gamma,\, \lambda\cdot\mathcal{W}>0} \left(1 - e^{2\pi i\langle\lambda, z\rangle}\right)^{c_\gamma(\lambda^2/2)}.$$

We refer to [5] for more about the Borcherds products.

By the identification $L\otimes\mathbb{R} + i\,\mathcal{C}_L^+ \cong \Omega_\Lambda^+$ as above, the Borcherds product $\Psi_\Lambda(z, F_\Lambda)$ is viewed as a formal function on Ω_Λ^+.

Theorem 4.6 (Borcherds [5])**.** *The infinite product $\Psi_\Lambda(z, F_\Lambda)$ extends to a (possibly meromorphic) automorphic form on $L\otimes\mathbb{R}+i\,\mathcal{C}_L^+ \cong \Omega_\Lambda^+$ for $O^+(\Lambda)$ of weight $c_0(0)/2$.*

It is possible to give the weight and the divisor of $\Psi_\Lambda(z, F_\Lambda)$ in terms of the invariants of the lattice Λ and the Fourier coefficients of F_Λ. See [21], [14].

4.4. **The structure of the invariant.** Recall that Λ is the type of $X_{(S,\theta,T)}$, $r = \mathrm{rk}\,\Lambda$ is the rank of Λ, and $\delta \in \{0, 1\}$ is the parity of the discriminant form q_Λ. Let $g = g(\Lambda)$ be the total genus of the fixed curve

$$S^\theta = \{x \in S;\, \theta(x) = x\}.$$

Namely, if $S^\theta = \sum_i C_i$ is the decomposition into the connected components, then $g := \sum_i g(C_i)$, where $g(C_i)$ is the genus of the compact Riemann surface C_i. Let χ_g^8 be the Siegel modular form on $\mathfrak{S}_g$ of weight $2^{g+1}(2^g + 1)$ defined as

$$\chi_g(T)^8 := \prod_{(a,b)\ \text{even}} \theta_{a,b}(T)^8, \qquad T \in \mathfrak{S}_g,$$

where

$$\theta_{a,b}(T) := \sum_{m\in\mathbb{Z}^g} \exp\left[\pi i^t(m + a)T(m + a) + 2\pi i^t(m + a)b\right]$$

is the Riemann theta constant. Let $\varpi(S^\theta) \in \mathcal{A}_g$ be the period of the fixed-curve S^θ.

Theorem 4.7 ([20], [22], [21], [14])**.** *Let $(r, \delta) \neq (12, 0), (20, 0)$. Then there exists a constant C_Λ depending only on Λ such that*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)})^{2^{g-1}(2^g+1)} = C_\Lambda \left\|\varPsi_\Lambda\left(\varpi(S, \theta), 2^{g-1}F_\Lambda\right)\right\|^2 \times \left\|\chi_g\left(\varpi(S^\theta)\right)^8\right\|^2 \left\|\eta\left(\varpi(T)\right)^{24}\right\|^{2^g(2^g+1)}.$$

Here $\|\cdot\|$ is the Petersson norm.

If $(r, \delta) = (12, 0)$ or $(20, 0)$, then χ_g vanishes identically on the periods of the fixed curves $\varpi(S^\theta)$ ([14]), so that the above formula does not hold in these cases. To express the BCOV invariant in the remaining cases, we introduce another Siegel modular form. Let $\varUpsilon_g$ be the Siegel modular form on $\mathfrak{S}_g$ and weight $2(2^g-1)(2^g+2)$ defined as

$$\varUpsilon_g(T) := \chi_g(T)^8 \sum_{(a,b)\ \text{even}} \theta_{a,b}(T)^{-8}.$$

Then $\varUpsilon_g$ is the elementary symmetric polynomial of degree $(2^g - 1)(2^{g-1} + 1)$ in the even theta constants $\theta_{a,b}(T)^8$.

Theorem 4.8 ([22], [14]). *Let $(r,\delta)=(12,0)$. Then there exists a constant C_Λ depending only on Λ such that*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)})^{(2^{g-1}+1)(2^g-1)} = C_\Lambda \left\|\Psi_\Lambda\left(\varpi(S,\theta),(2^{g-1}+1)F_\Lambda\right)\right\|^2 \times \left\|\Upsilon_g\left(\varpi(S^\theta)\right)\right\|^2 \left\|\eta\left(\varpi(T)\right)^{24}\right\|^{2(2^{g-1}+1)(2^g-1)}.$$

In the case $(r,\delta)=(20,0)$, either $\Lambda=\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{E}_8\oplus\mathbb{E}_8$ or $\mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{E}_8\oplus\mathbb{E}_8$. We define a $\mathbb{C}[A_\Lambda]$-valued elliptic modular form f_Λ by

$$f_\Lambda(\tau) := \frac{\theta_{\mathbb{E}_8}(\tau)}{\eta(\tau)^{24}} \qquad (\Lambda=\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{E}_8\oplus\mathbb{E}_8)$$

and

$$f_\Lambda(\tau) := 8\sum_{\gamma\in\Lambda^\vee/\Lambda}\left\{\eta\left(\frac{\tau}{2}\right)^{-8}\eta(\tau)^{-8}+(-1)^{\gamma^2}\eta\left(\frac{\tau+1}{2}\right)^{-8}\eta(\tau+1)^{-8}\right\}\mathfrak{e}_\gamma + \eta(\tau)^{-8}\eta(2\tau)^{-8}\,\mathfrak{e}_0 \qquad (\Lambda=\mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{E}_8\oplus\mathbb{E}_8).$$

Theorem 4.9 ([22], [14]). *Let $(r,\delta)=(20,0)$. Then there exists a constant C_Λ depending only on Λ such that*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)})^{(2^{g-1}+1)(2^g-1)} = C_\Lambda \left\|\Psi_\Lambda\left(\varpi(S,\theta),2^{g-1}F_\Lambda+f_\Lambda\right)\right\|^2 \times \left\|\Upsilon_g\left(\varpi(S^\theta)\right)\right\|^2 \left\|\eta\left(\varpi(T)\right)^{24}\right\|^{2(2^{g-1}+1)(2^g-1)}.$$

By Theorems 4.7, 4.8, 4.9, the BCOV invariant of Borcea-Voisin threefolds is expressed as the Petersson norm of an automorphic form on the period domain.

5. Transition of Borcea-Voisin threefolds and quasi-pullback of some Siegel modular forms

5.1. The Hodge line bundle on the Siegel modular variety.

Definition 5.1. The Hodge line bundle on the Siegel modular variety

$$\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{S}_g$$

is the line bundle $\mathcal{F}_g=\mathrm{Sp}_{2g}(\mathbb{Z})\backslash(\mathfrak{S}_g\times\mathbb{C})$ in the sense of orbifolds associated to the automorphic factor

$$\mathrm{Sp}_{2g}(\mathbb{Z})\ni\begin{pmatrix}A\,B\\C\,D\end{pmatrix}\mapsto\det(C\tau+D)\in\mathcal{O}(\mathfrak{S}_g).$$

A section of $\mathcal{F}_g^{\otimes q}$ is identified with a Siegel modular form of weight q.

It is classical that $\mathcal{F}_g$ extends to an ample $\mathbb{Q}$-line bundle on the Satake compactification $\mathcal{A}_g^*$. Namely, there exists $N\in\mathbb{N}$ such that $\mathcal{F}_g^{\otimes q}$ is a very ample line bundle on $\mathcal{A}_g^*$ in the ordinary sense.

The line bundle $\mathcal{F}_g^{\otimes q}$ is equipped with the Hermitian metric $\|\cdot\|_{\mathcal{F}_g^{\otimes q}}$ called the Petersson norm: For any Siegel modular form S of weight q, we define

$$\|S(\tau)\|^2_{\mathcal{F}_g^{\otimes q}} := (\det\Im\tau)^q|S(\tau)|^2.$$

In what follows, let $q\in\mathbb{Z}_{>0}$ be such that $\mathcal{F}_g^{\otimes q}$ extends to a very ample line bundle on $\mathcal{A}_g^*$, the Satake compactification.

5.2. **The Torelli map.** Recall that $\mathcal{M}_\Lambda^0 = O^+(\Lambda)\backslash(\Omega_\Lambda^+ - \mathcal{D}_\Lambda)$ and that its points are identified with the isomorphism classes of 2-elementary $K3$ surfaces of type Λ via the period mapping. Hence a point of $\mathcal{M}_\Lambda^0$ can be expressed as $[(S,\theta)] = \varpi(S,\theta)$, where (S,θ) is a 2-elementary $K3$ surface of type Λ.

Recall that S^θ is the fixed curve of the involution $\theta\colon S \to S$. Let $[\mathrm{Jac}(S^\theta)]$ be the isomorphism class of the Jacobian variety of S^θ. By the Torelli theorem, $[\mathrm{Jac}(S^\theta)]$ is identified with its period $\varpi(S^\theta) \in \mathcal{A}_g$.

Definition 5.2. The Torelli map $J_\Lambda\colon \mathcal{M}_\Lambda^0 \to \mathcal{A}_g$ is defined by

$$J_\Lambda(S,\theta) := [\mathrm{Jac}(S^\theta)] = [\varpi(S^\theta)].$$

The Torelli map is identified with the $O^+(\Lambda)$-equivariant map

$$J_\Lambda\colon \Omega_\Lambda^+ \setminus \mathcal{D}_\Lambda \to \mathcal{A}_g.$$

Since $\mathcal{A}_g$ is a locally symmetric variety, it follows from the Borel-Kobayashi-Ochiai extension theorem that the Torelli map extends to an $O^+(\Lambda)$-equivariant holomorphic map

$$J_\Lambda\colon \Omega_\Lambda^+ \setminus \mathrm{Sing}\,\mathcal{D}_\Lambda \to \mathcal{A}_g^*.$$

In particular, J_Λ is a rational map from $\mathcal{M}_\Lambda$ to $\mathcal{A}_g^*$.

Let $\Delta_\Lambda = \{d \in \Lambda,\, d^2 = -2\}$ be the set of roots of Λ. For $d \in \Delta_\Lambda$, the corresponding hyperplane

$$H_d = \Omega_\Lambda^+ \cap d^\perp = \{[\eta] \in \Omega_\Lambda^+;\, \langle \eta, d\rangle = 0\}$$

is identified with $\Omega_{\Lambda\cap d^\perp}^+$. Since $\Lambda \cap d^\perp$ is again a primitive 2-elementary sublattice of signature $(2, r(\Lambda) - 3)$, one has the moduli space of 2-elementary $K3$ surfaces of type $\Lambda \cap d^\perp$. Hence the Torelli map

$$J_{\Lambda\cap d^\perp}\colon \Omega_{\Lambda\cap d^\perp}^+ \setminus \mathrm{Sing}\,\mathcal{D}_{\Lambda\cap d^\perp} \to \mathcal{A}_{g'}^*$$

is well defined, where $g' = g(\Lambda \cap d^\perp)$. Then we have the following compatibility of the Torelli maps.

Fact 5.3 ([21]). *The followin identity of maps hold:*

$$J_\Lambda|_{H_d \setminus \mathrm{Sing}\,\mathcal{D}_\Lambda} = J_{\Lambda\cap d^\perp}.$$

5.3. **The line bundle λ_Λ^q.** Let $q \in \mathbb{N}$ be such that $\mathcal{F}_g^{\otimes q}$ is an ample line bundle on $\mathcal{A}_g^*$ in the ordinary sense. Then $J_\Lambda^* \mathcal{F}_g^{\otimes q}$ is an $O^+(\Lambda)$-equivariant holomorphic line bundle on $\Omega_\Lambda^+ \setminus \mathrm{Sing}\,\mathcal{D}_\Lambda$. Let

$$\iota_\Lambda\colon \Omega_\Lambda^+ \setminus \mathrm{Sing}\,\mathcal{D}_\Lambda \hookrightarrow \Omega_\Lambda^+$$

be the inclusion.

Definition 5.4. Define λ_Λ^q as the trivial extension of $J_\Lambda^* \mathcal{F}_g^{\otimes q}$ from $\Omega_\Lambda^+ \setminus \mathrm{Sing}\,\mathcal{D}_\Lambda$ to Ω_Λ^+

$$\lambda_\Lambda^q := (\iota_\Lambda)_* \mathcal{O}_{\Omega_\Lambda^+ \setminus \mathrm{Sing}\,\mathcal{D}_\Lambda}\left(J_\Lambda^* \mathcal{F}_g^{\otimes q}\right).$$

Since $J_\Lambda\colon \Omega_\Lambda^+ \dashrightarrow \mathcal{A}_g^*$ is a rational map, one has the following:

Fact 5.5 ([19]). *λ_Λ^q is an $O^+(\Lambda)$-equivariant invertible sheaf on Ω_Λ^+.*

On $\Omega_\Lambda \setminus \mathcal{D}_\Lambda$, λ_Λ^q is endowed with the Hermitian metric

$$\|\cdot\|_{\lambda_\Lambda^q} := J_\Lambda^* \|\cdot\|_{\mathcal{F}_g^{\otimes q}}.$$

5.4. **Automorphic forms on Ω_Λ^+.** Fix a non-zero isotropic vector $\ell_\Lambda \in \Lambda$. Via the exponential map, one has the isomorphism of domains

$$\Omega_\Lambda^+ \cong L \otimes \mathbb{R} + i\mathcal{C}_L^+,$$

where $L = \ell_\Lambda^\perp/\mathbb{Z}\ell_\Lambda$ is a Lorentzian lattice. Define the automorphic factor $j_\Lambda(\gamma, \cdot) \in \mathcal{O}^*(\Omega_\Lambda^+)$, $\gamma \in O(\Lambda)$ by

$$j_\Lambda(\gamma, [z]) := \frac{\langle \gamma(z), \ell_\Lambda\rangle}{\langle z, \ell_\Lambda\rangle} \qquad [z] \in \Omega_\Lambda^+.$$

Definition 5.6. A section $F \in H^0(\Omega_\Lambda^+, \lambda_\Lambda^q)$ is called an automorphic form of weight (p, q) if

$$F(\gamma \cdot [z]) = j_\Lambda(\gamma, [z])^p \gamma(F([z]))$$

for all $\gamma \in O^+(\Lambda)$ and $[z] \in \Omega_\Lambda^+$

5.5. **Quasi-pullback of automorphic forms.** For a root $d \in \Delta_\Lambda$,

$$\frac{\langle z, d\rangle}{\langle z, \ell_\Lambda\rangle}, \qquad [z] \in \Omega_\Lambda^+$$

is a holomorphic function on Ω_Λ^+ with zero divisor $H_d = \Omega_\Lambda^+ \cap d^\perp$.

Definition 5.7. Let $F \in H^0(\Omega_\Lambda, \lambda_\Lambda^q)$ be a section vanishing on H_d of order k. Then

$$\rho_{\Lambda\cap d^\perp}^\Lambda(F) := \left.\left\{ \left(\frac{\langle \cdot, \ell_\Lambda\rangle}{\langle \cdot, d\rangle}\right)^k \cdot F \right\}\right|_{H_d} \in H^0(\Omega_{\Lambda\cap d^\perp}^+, \lambda_{\Lambda\cap d^\perp}^q)$$

is called the *quasi-pullback* of F to $H_d = \Omega_{\Lambda\cap d^\perp}^+$. The case $q = 0$ is the quasi-pullback in the classical sense ([4], [6], [12], [11], [13]).

In the classical case, the quasi-pullbacks of a Borcherds products is again a Borcherds product and its explicit formula is obtained by Ma ([13]).

5.6. **The quasi-pullback of the Siegel modular form χ_g^8.** Recall that χ_g is the Siegel modular form on the Siegel upper half space $\mathfrak{S}_g$ defined as

$$\chi_g = \prod_{(a,b)\ \text{even}} \theta_{a,b}.$$

As mentioned in the introduction, the quasi-pullback of the Siegel modular form $J_\Lambda^* \chi_g^{8(2^{g'}+1)}$ pulled back to Ω_Λ^+ produces a Borcherds product as follows.

Theorem 5.8 ([23]). *Let $d \in \Delta_\Lambda$. Set $\Lambda' := \Lambda \cap d^\perp$ and $g' := g(\Lambda')$. Then either $g' = g$ or $g' = g - 1$ ([21]). Assume that $(r, \delta), (r(\Lambda'), \delta(\Lambda')) \neq (12, 0), (20, 0)$. Then there exists a constant $C_{\Lambda,d}$ such that*

$$\frac{\rho_{\Lambda'}^\Lambda(J_\Lambda^*\chi_g^{8(2^{g'}+1)})}{J_{\Lambda'}^*\chi_{g'}^{16(2^g+1)}} = C_{\Lambda,d}\left(\frac{\rho_{\Lambda'}^\Lambda\left(\Psi_\Lambda(\cdot, (2^{g'}+1)2^{g-1}F_\Lambda)\right)}{\Psi_{\Lambda'}(\cdot, 2(2^g+1)2^{g'-1}F_{\Lambda'})}\right)^{-1}.$$

In particular, $\rho_{\Lambda'}^\Lambda(J_\Lambda^*\chi_g^{8(2^{g'}+1)})/J_{\Lambda'}^*\chi_{g'}^{16(2^g+1)}$ is a Borcherds product, since the quasi-pullback $\rho_{\Lambda'}^\Lambda\left(\Psi_\Lambda(\cdot, (2^{g'}+1)2^{g-1}F_\Lambda)\right)$ is a Borcherds product by Ma ([13]).

We remark that Theorem 5.8 reduces to a result of Ma ([13]) when $g' = g$.

5.7. **Idea of the proof.** Set

$$\Phi_\Lambda := \Psi_\Lambda(\cdot, 2^{g-1}F_\Lambda) \otimes J_\Lambda^* \chi_g^8.$$

By Theorem 4.7, for Borcea-Voisin threefolds of type Λ with $(r,\delta) \neq (12,0), (20,0)$, one has the following equality of functions on the moduli space

$$\tau_{\mathrm{BCOV}} = C_\Lambda \left\| \Phi_\Lambda^{\frac{1}{2^{g-1}(2^g+1)}} \right\|^2 \|\eta^{24}\|^2.$$

A general Borcea-Voisin threefold of type Λ' is obtained as the crepant resolution of the Calabi-Yau orbifold, which is a degeneration of Borcea-Voisin threefolds of type Λ. (This follows from Definition 3.2 and [21, Th. 2.3].) In such a situation, the Borcea-Voisin threefold of type Λ' is called the *transition* of Borcea-Voisin threefolds of type Λ. One can define the transition of Calabi-Yau manifolds in a similar way. Namely, given a singular Calabi-Yau variety X_0 of dimension 3, one has two possible ways of obtaining smooth Calabi-Yau threefolds; one is a crepant resolution $\widetilde{X}_0$ of X_0 and the other is a smoothing X of X_0. Then $\widetilde{X}_0$ is called the transition of X. In general, $\widetilde{X}_0$ and X are not homeomorphic. A very natural problem to be studied is to understand the behavior of the BCOV invariant under transitions. For the Borcea-Voisin case, we have the following answer.

Theorem 5.9 ([23])**.** *There exists a constant $C_{\Lambda,d}$ such that*

$$\rho_{\Lambda'}^{\Lambda}\left(\Phi_\Lambda^{2^{g'}+1}\right) = C_{\Lambda,d}\, \Phi_{\Lambda'}^{2(2^g+1)}.$$

In particular, the automorphic forms corresponding to the BCOV invariant of the Borcea-Voisin threefolds are closed under the operation of quasi-pullbacks, and the effect of transition on the automorphic form corresponding to the BCOV invariant is given by the quasi-pullback.

Theorem 5.8 follows from Theorem 5.9 and Theorem 4.7.

References

[1] Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C. *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994), 311–427.

[2] Bismut, J.-M., Gillet, H., Soulé, C. *Analytic torsion and holomorphic determinant bundles I, II, III*, Commun. Math. Phys. **115** (1988), 49–78, 79–126, 301–351.

[3] Borcea, C. *K3 surfaces with involution and mirror pairs of Calabi–Yau manifolds*, Mirror Symmetry II (B. Green and S.-T. Yau eds) AMS/International Press (1997), 717–743.

[4] Borcherds, R.E. *Automorphic forms on $O_{s+2,s}(\mathbf{R})$ and infinite products*, Invent. Math. **120** (1995), 161–213.

[5] Borcherds, R.E. *Automorphic forms with singularities on Grassmanians*, Invent. Math. **132** (1998), 491-562.

[6] Borcherds, R.E., Katzarkov, L., Pantev, T., Shepherd-Barron, N.I. *Families of K3 surfaces.* J. Algebraic Geom. **7**, (1998), 183–193.

[7] Eriksson, D., Freixas i Montplet, G., Mourougane, C. *BCOV invariants of Calabi-Yau manifolds and degenerations of Hodge structures*, Duke Math. J. **170** (2021), 379–454.

[8] Eriksson, D., Freixas i Montplet, G., Mourougane, C. *On genus one mirror symmetry in higher dimensions and the BCOV conjectures*, arXiv:1911.06734 (2019).

[9] Fang, H., Lu, Z., Yoshikawa, K.-I. *Analytic torsion for Calabi–Yau threefolds*, J. Differential Geom. **80** (2008), 175–250.

[10] Fu, L., Zhang, Y. *From motivic integration to the birational invariance of BCOV invariants*, preprint, arXiv:2007.04835.

[11] Gritsenko, V.A., Hulek, K., Sankaran, G.K. *The Kodaira dimension of the moduli of K3 surfaces*, Invent. Math. **169**, (2007), 519–567.

[12] Kondo, S. *On the Kodaira dimension of the moduli space of K3 surfaces. II*, Compositio Math. **116**, (1999), 111–117.
[13] Ma, S. *Quasi-pullback of Borcherds products*, Bull. London Math. Soc. **51** (2019), 1061–1078.
[14] Ma, S., Yoshikawa, K.-I. *$K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space IV: The structure of the invariant*, Compositio Math. **156** (2020), 1965–2019.
[15] Nikulin, V.V. *Integral Symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14** (1980), 103–167.
[16] Nikulin, V.V. *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by* 2*-reflections*, J. Soviet Math. **22** (1983), 1401–1476.
[17] Ray, D.B., Singer, I.M. *Analytic torsion for complex manifolds*, Ann. of Math. **98** (1973), 154–177.
[18] Voisin, C. *Miroirs et involutions sur les surfaces $K3$*, Journée de Géométrie Algébrique d'Orsay (Orsay, 1992), Astérisque **218** (1993), 273–323.
[19] Yoshikawa, K.-I. *$K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space*, Invent. Math. **156** (2004), 53–117.
[20] Yoshikawa, K.-I. *Calabi–Yau threefolds of Borcea–Voisin, analytic torsion, and Borcherds products*, From Probability to Geometry (II), Volume in honor of Jean-Michel BISMUT, ed. by X. Dai, R. Léandre, X. Ma, W. Zhang, Astérisque **328** (2009), 351–389.
[21] Yoshikawa, K.-I. *$K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II*, J. reine angew. Math. **677** (2013), 15–70.
[22] Yoshikawa, K.-I. *Analytic torsion for Borcea-Voisin threefolds*, "Geometry, Analysis and Probability" in Honor of Jean-Michel Bismut, Bost, J.-B., Hofer, H., Labourie, F., Le Jan, Y., Ma, X., Zhang, W. (Eds), Progress in Math. **310** (2017), 279–361
[23] Yoshikawa, K.-I. *Transitions and BCOV invariants: Borcea-Voisin case (tentative title)*, in preparation
[24] Zhang, Y. An extension of BCOV invariant, IMRN (to appear), arXiv:1905.03964 (2019).
[25] Zhang, Y. BCOV invariant and blow-up, arXiv:2003.03805 (2020).
[26] Zinger, A. *The reduced genus* 1 *Gromov-Witten invariants of Calabi-Yau hypersurfaces*, J. Amer. Math. Soc. **22** (2009), 691–737.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, JAPAN
Email address: yosikawa@math.kyoto-u.ac.jp