# Stationary solutions to the Euler–Poisson equations in a perturbed half-space

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## 1 Introduction

The purpose of this survey paper is to discuss mathematically the formation of a plasma sheath near the surface of materials immersed in a plasma, and to study qualitative information of such a plasma sheath layer. In fact we summarize the results [8, 11, 13] investigating the stationary solutions to the Euler-Poisson equations in a half-space or perturbed half-space.

The plasma sheath appears when a material is immersed in a plasma and the plasma contacts with its surface. Since the thermal velocities of electrons are much higher than those of ions, more electrons tend to hit the surface of the material than ions do, which makes the material negatively charged with respect to the surrounding plasma. Then the material with a negative potential attracts and accelerates ions toward the surface, while repelling electrons away from it, and this results in the formation of non-neutral potential region near the surface, where a nontrivial equilibrium of the densities is achieved. Consequently, positive ions outnumber electrons in this region and this ion-rich layer near the

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boundary is referred to as the *plasma sheath.* This boundary layer shields the plasma from the negatively charged material. For the formation of sheath, Langmuir in [6] observed that positive ions must enter the sheath region with a sufficiently large kinetic energy. Bohm in [3] derived the original *Bohm criterion* for the plasma containing electrons and only one component of mono-valence ions, which states that the ion velocity $u$ at the plasma edge must exceed the ion acoustic speed for the case of planar wall. For more details of the sheath formation, we refer the readers to [4, 7, 9, 10].

The motion of positive ions in a plasma is governed by, after suitable nondimensionalization, the Euler-Poisson equations:

$$\rho_t + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + K\nabla \log \rho = \nabla \phi, \tag{1.1b}$$

$$\Delta \phi = \rho - e^{-\phi}, \tag{1.1c}$$

where the unknown functions $\rho$, $\boldsymbol{u} = (u_1, u_2, u_3)$ and $-\phi$ represent the density, velocity of the positive ion and the electrostatic potential, respectively. Moreover, $K > 0$ is a constant of temperature of the positive ion. The first equation describes the mass balance law, and the second one is the equation of momentum in which the pressure gradient and electrostatic potential gradient as well as the convection effect are taken into account. The third equation is the Poisson equation, which describes the relation between the potential and the density of charged particles. For this equation, the Boltzmann relation stating that the electron density is given by $\rho_e = e^{-\phi}$ is assumed.

Let us mention the mathematical results which studied the sheath formation by using the Euler–Poisson equation (1.1a)–(1.1c). The planar wall case have been well investigated. Ambroso, Méhats, and Raviart did a pioneering work [2] where the unique existence of monotone stationary solutions was proved in a bounded interval, provided that the Bohm criterion holds. Furthermore, Ambroso [1] numerically checked that solutions to initial–boundary value problems approach the stationary solutions constructed in [2] as the time variable becomes large. Suzuki [11] derived a necessary and sufficient condition, including the Bohm criterion, for the unique existence of monotone stationary solutions in a half-space. Furthermore, he showed the asymptotic stability of stationary solutions by assuming a condition slightly stronger than the criterion. After that, the stability theorem was shown under the Bohm criterion in [8]. For a multicomponent plasma containing electrons and several components of ions, similar results to [8, 11] were obtained in [12] under the generalized Bohm criterion derived by Riemann in [10]. These results validated mathematically the Bohm criterion and defined the fact that the sheath corresponds to the stationary solution.

For the planer wall cases, the formation of the sheath has been well-understood as above. It is of greater interest to study the cases that walls are nonplanar, and to know how the Bohm criterion depends on the shape of walls. For this direction, Jung, Kwon, and Suzuki in [5] studied the existence of stationary solutions in an annulus. They focused only on spherical symmetry solutions and then proposed *a Bohm criterion for the annulus*, which essentially differs from the original Bohm criterion. Recently, Suzuki and Takayama in [13] showed the existence and stability of stationary solutions in a perturbed half-space, and found out *a Bohm criterion for the perturbed half-space*.

This paper provides an overview of the results in [8,11,13] which studied the stationary solutions in a half-space or perturbed half-space, and also observes how the Bohm criterion varies. We consider an initial–boundary value problem of (1.1a)–(1.1c) in a perturbed half-space

$$\Omega := \{x = (x_1, x_2, x_3) = (x_1, x') \in \mathbb{R}^3 \,|\, x_1 > M(x')\} \text{ with } M \in \bigcap_{k=1}^{\infty} H^k(\mathbb{R}^2).$$

We remark that $\Omega$ with $M \equiv 0$ is just a half-space $\mathbb{R}^3_+$. The initial and boundary data are prescribed as

$$(\rho, \boldsymbol{u})(0, x) = (\rho_0, \boldsymbol{u}_0)(x), \tag{1.1d}$$

$$\lim_{x_1 \to \infty} (\rho, u_1, u_2, u_3, \phi)(t, x_1, x') = (1, u_+, 0, 0, 0), \tag{1.1e}$$

$$\phi(t, M(x'), x') = \phi_b \quad \text{for } x' \in \mathbb{R}^2, \tag{1.1f}$$

where $u_+$ and $\phi_b$ are constants.

Note that $\lim_{x_1 \to \infty} \rho(x_1, x') = 1$ is necessary owing to (1.1c) and $\lim_{x_1 \to \infty} \phi(x_1, x') = 0$ in the construction of classical solutions to (1.1c). We seek solutions in the region, where the following two conditions hold:

$$\inf_{x \in \Omega} \rho(x) > 0, \tag{1.2}$$

$$\inf_{x \in \partial\Omega} \frac{\boldsymbol{u}(x) \cdot \nabla(M(x') - x_1)}{\sqrt{1 + |\nabla M(x')|^2}} - \sqrt{K} > 0, \tag{1.3}$$

by assuming the same conditions for the initial data $(\rho_0, \boldsymbol{u}_0)$:

$$\inf_{x \in \Omega} \rho_0(x) > 0, \quad \inf_{x \in \partial\Omega} \frac{\boldsymbol{u}_0(x) \cdot \nabla(M(x') - x_1)}{\sqrt{1 + |\nabla M(x')|^2}} - \sqrt{K} > 0. \tag{1.4}$$

The second condition in (1.4) is necessary for the well-posedness of the problem (1.1).

The stationary solutions $(\rho^s, \boldsymbol{u}^s, \phi^s)$ to (1.1) solves the equations

$$\nabla \cdot (\rho^s \boldsymbol{u}^s) = 0, \tag{1.5a}$$

$$(\boldsymbol{u}^s \cdot \nabla)\, \boldsymbol{u}^s + K \nabla(\log \rho^s) = \nabla \phi^s, \tag{1.5b}$$

$$\Delta \phi^s = \rho^s - e^{-\phi^s} \tag{1.5c}$$

with the conditions

$$\inf_{x\in\Omega} \rho^s(x) > 0, \tag{1.5d}$$

$$\lim_{x_1\to\infty} (\rho^s, u_1^s, u_2^s, u_3^s, \phi)(t, x_1, x') = (1, u_+, 0, 0, 0), \tag{1.5e}$$

$$\phi^s(t, M(x'), x') = \phi_b \quad \text{for } x' \in \mathbb{R}^2. \tag{1.5f}$$

Before closing the introduction, we present several notations to be used throughout this paper. For a nonnegative integer $k$ and $\Omega \subset \mathbb{R}^3$, $H^k(\Omega)$ is the $k$th order Sobolev space in the $L^2$ sense, equipped with the norm $\|\cdot\|_k$. We also define weighted Sobolev spaces $H^k_\alpha(\Omega)$ and $H^k_{\alpha,\lambda}(\Omega)$ for $\alpha > 0$ and $\lambda \geq 2$ by

$$H^k_\alpha(\Omega) := \left\{ f \in H^k(\Omega) \;\middle|\; \|f\|^2_{k,\alpha} = \sum_{j=0}^k \int_\Omega e^{\alpha x_1} |\nabla^j f|^2 \, dx < \infty \right\},$$

$$H^k_{\alpha,\lambda}(\Omega) := \left\{ f \in H^k(\Omega) \;\middle|\; \|f\|^2_{k,\alpha,\lambda} = \sum_{j=0}^k \int_\Omega w_{\alpha,\lambda} |\nabla^j f|^2 \, dx < \infty \right\},$$

where

$$w_{\alpha,\lambda}(x_1) := \left(1 + \min\left\{\alpha, (1 + |M|_{L^\infty(\mathbb{R}^2)})^{-1}\right\} x_1\right)^\lambda.$$

Furthermore, $C^k([0,T];\mathcal{H})$ denotes the space of $k$ times continuously differentiable functions on the interval $[0,T]$ with values in some Hilbert space $\mathcal{H}$.

# 2 Results on the half-space $\mathbb{R}^3_+$

In this section, we review the results on the existence and stability of stationary solutions in the half-space $\Omega = \mathbb{R}^3_+$. The paper [11] derived a necessary and sufficient condition for the existence of planar stationary solutions $(\rho^s, \boldsymbol{u}^s, \phi^s) = (\tilde{\rho}, \tilde{\boldsymbol{u}}, \tilde{\phi}) = (\tilde{\rho}, \tilde{u}, 0, 0, \tilde{\phi})$. They are solutions to (1.5) independent of the tangential variable $x'$, and thus satisfy the equations

$$(\tilde{\rho}\tilde{u})_{x_1} = 0, \tag{2.1a}$$

$$\left(\frac{1}{2}\tilde{u}^2 + K \log \tilde{\rho}\right)_{x_1} = \tilde{\phi}_{x_1}, \tag{2.1b}$$

$$\tilde{\phi}_{x_1 x_1} = \tilde{\rho} - e^{-\tilde{\phi}}, \tag{2.1c}$$

and the conditions

$$\inf_{x_1\in\mathbb{R}_+} \tilde{\rho}(x_1) > 0, \tag{2.1d}$$

$$\lim_{x_1\to\infty} (\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1) = (1, u_+, 0), \tag{2.1e}$$

$$\tilde{\phi}(0) = \phi_b. \tag{2.1f}$$

The existence is summarized in the following theorem.

**Theorem 2.1** ([11])**.** (i) *Let the end state* $u_+$ *satisfy* $K < u_+^2 < K+1$. *If* $\phi_b \neq 0$, *stationary problem* (2.1) *does not admit any solutions as*

$$\tilde{\rho}, \tilde{u}, \tilde{\phi} \in C(\overline{\mathbb{R}_+}) \quad \text{and} \quad \tilde{\rho}, \tilde{u}, \tilde{\phi}, \tilde{\phi}_{x_1} \in C^1(\mathbb{R}_+). \tag{2.2}$$

*If* $\phi_b = 0$, *the end state* $(\tilde{\rho}, \tilde{u}, \tilde{\phi}) = (1, u_+, 0)$ *is the unique solution.*

(ii) *Let the end state* $u_+$ *satisfy either* $u_+^2 \leq K$ *or* $K+1 \leq u_+^2$. *Then stationary problem* (2.1) *has a unique monotone solution* $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ *as* (2.2) *if and only if the boundary data* $\phi_b$ *satisfies conditions*

$$V(\phi_b) \geq 0 \quad \text{and} \quad \phi_b \geq f(|u_+|/\sqrt{K}), \tag{2.3}$$

*where the Sagdeev potential* $V$ *and the function* $f$ *is defined by*

$$V(\phi) := \int_0^\phi \left[ f^{-1}(\eta) - e^{-\eta} \right] d\eta, \quad f(\rho) := K \log \rho + \frac{u_+^2}{2\rho^2} - \frac{u_+^2}{2},$$

*and the inverse function* $f^{-1}$ *is defined by adopting the branch which contains the end state* $(\rho, \phi) = (1, 0)$. *Moreover, if* $K+1 < u_+^2$ *and* $\phi_b > f(|u_+|/\sqrt{K})$, *the solution* $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ *belongs to* $C^\infty(\mathbb{R}_+)$ *and satisfies*

$$|\partial_{x_1}^l (\tilde{\rho} - 1)(x_1)| + |\partial_{x_1}^l (\tilde{u} - u_+)(x_1)| + |\partial_{x_1}^l \tilde{\phi}(x_1)| \leq C |\phi_b| e^{-\alpha x_1}, \quad l = 0, 1, 2, \ldots,$$

*where* $\alpha$ *and* $C$ *are positive constants independent of* $\phi_b$.

Theorem 2.1 ensures that the planer stationary solution exists under the original Bohm criterion:

$$u_+^2 > K + 1, \quad u_+ < 0. \tag{2.4}$$

We remark that the conditions in (2.3) are valid for $\phi_b$ with $|\phi_b| \ll 1$ provided that the criterion holds.

From now on we study the stability of monotone stationary solutions assuming (2.4). First let us mention the reason that we require (2.4) in the stability analysis. We introduce

$$\psi := \rho - 1, \quad \boldsymbol{\eta} := \boldsymbol{u} - (u_+, 0, 0), \quad \sigma := \phi$$

and linearize (1.1a)–(1.1c) around the end state $(1, u_+, 0, 0, 0)$ as

$$\psi_t + u_+ \psi_{x_1} + \mathrm{div} \eta = 0, \tag{2.5a}$$

$$\eta_t + u_+ \eta_{x_1} + K \nabla \psi = \nabla \sigma, \tag{2.5b}$$

$$\Delta \sigma = \psi + \sigma. \tag{2.5c}$$

Then we consider (2.5) in the whole space $\mathbb{R}^3$, and calculate the spectrums:

$$\mu(i\xi) = i\left(-\xi_1 u_+ \pm |\xi|\sqrt{K + \frac{1}{1+|\xi|^2}}\right), \quad -i\xi_1 u_+ \quad \text{for } \xi \in \mathbb{R}^3.$$

We see that the real part of all the spectra are zero, and thus there is no dissipative structure in this framework. To make a dissipative structure, we introduce new unknown functions

$$(\Psi, \boldsymbol{H}, \Sigma) := (e^{\beta x_1/2}\psi, e^{\beta x_1/2}\boldsymbol{\eta}, e^{\beta x_1/2}\sigma) \text{ for } \beta \in (0, \sqrt{2}),$$

and rewrite the equations (2.5) for $(\Psi, \boldsymbol{H}, \Sigma)$. The spectrums are given by

$$\mu(i\xi) = \frac{\beta u_+}{2} + i\left(-\xi_1 u_+ \pm \sqrt{K\zeta - \frac{1}{\zeta} + 1 - K}\right), \quad \frac{\beta u_+}{2} - i\xi_1 u_+ \quad \text{for } \xi \in \mathbb{R}^3,$$

where

$$\zeta := 1 + |\xi|^2 - \frac{\beta^2}{4} + i\beta\xi_1.$$

It is shown in [8, Proposition 1.2] that

$$\sup_{\xi\in\mathbb{R}^N} \mathrm{Re}\,(\mu(i\xi)) = \max\{\mathrm{Re}\,(\mu(0))\} = \frac{\beta}{2}\left(u_+ + \sqrt{K + (1-\beta^2/4)^{-1}}\right). \tag{2.6}$$

The equations for $(\Psi, \boldsymbol{H}, \Sigma)$ is linearly stable if and only if the rightmost of (2.6) is negative. Furthermore, the negativity holds if and only if (2.4) holds and $\beta$ is suitably small. Consequently, the original Bohm criterion (2.4) is a reasonable assumption for the stability analysis.

We are now in a position to mention the stability theorems.

**Theorem 2.2** ([8]). *Let $M \equiv 0$ and $u_+$ satisfy* (2.4). *There exist positive constants $\beta$ and $\delta$ such that if $\|(\rho_0 - \tilde{\rho}, \boldsymbol{u}_0 - \tilde{\boldsymbol{u}})\|_{3,\beta} + |\phi_b| \leq \delta$, then initial–boundary value problem* (1.1) *has a unique time-global solution $(\rho, \boldsymbol{u}, \phi)$ with* (1.2) *and* (1.3) *in the following space:*

$$(\rho - \tilde{\rho}, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \phi - \tilde{\phi}) \in \left[\bigcap_{i=0}^{1} C^i([0,T]; H^{3-i}_\beta(\mathbb{R}^3_+))\right]^4 \times C([0,T]; H^5_\beta(\mathbb{R}^3_+)).$$

*Moreover, it holds that*

$$\sup_{x\in\mathbb{R}^3_+} |(\rho - \tilde{\rho}, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \phi - \tilde{\phi})(t,x)| \leq Ce^{-\gamma t} \quad \textit{for } t \in [0,\infty),$$

*where $C$ and $\gamma$ are positive constants independent of $\phi_b$ and $t$.*

**Theorem 2.3** ([8]). *Let $M \equiv 0$, $\lambda \geq 2$, $\nu \in (0, \lambda]$, and $u_+$ satisfy* (2.4). *There exist positive constants $\beta$ and $\delta$ such that if $\|(\rho_0 - \tilde{\rho}, \boldsymbol{u}_0 - \tilde{\boldsymbol{u}})\|_{3,\beta,\lambda} + |\phi_b| \leq \delta$, then initial–boundary value problem* (1.1) *has a unique time-global solution $(\rho, \boldsymbol{u}, \phi)$ with* (1.2) *and* (1.3) *in the following space:*

$$(\rho - \rho^s, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \phi - \tilde{\phi}) \in \left[ \bigcap_{i=0}^{1} C^i([0,T]; H^{3-i}_{\beta,\lambda}(\mathbb{R}^3_+)) \right]^4 \times C([0,T]; H^{5-i}_{\beta,\lambda}(\mathbb{R}^3_+)).$$

*Moreover, it holds that*

$$\sup_{x \in \mathbb{R}^3_+} |(\rho - \tilde{\rho}, \boldsymbol{u} - \tilde{\boldsymbol{u}}, \phi - \tilde{\phi})(t,x)| \leq C(1+t)^{-\lambda+\nu} \quad \text{for } t \in [0, \infty),$$

*where $C$ is a positive constant independent of $\phi_b$ and $t$.*

# 3 Results on the perturbed half-space $\Omega$

This section is devoted to the study of the stationary solutions in the perturbed half-space $\Omega$. Besides the original Bohm criterion (2.4), we require the supersonic outflow condition for the end state $u_+$:

$$\inf_{x \in \partial\Omega} \frac{-u_+}{\sqrt{1 + |\nabla M(x')|^2}} - \sqrt{K} > 0. \tag{3.1}$$

This condition follows from replacing $\boldsymbol{u_0}$ by $(u_+, 0, 0)$ into the well-posedness condition in (1.4). Therefore, it is necessary if we seek the solutions to problem (1.1) in a neighborhood of the end state $(\rho, u_1, u_2, u_3, \phi) = (1, u_+, 0, 0, 0)$. We remark that (2.4) ensures (3.1) for the case $M \equiv 0$ i.e. $\Omega = \mathbb{R}^3_+$.

The stationary solution $(\rho^s, \boldsymbol{u}^s, \phi^s)$ is constructed by regarding it as a perturbation of $(\tilde{\rho}, \tilde{\boldsymbol{u}}, \tilde{\phi})(\tilde{M}(x)) = (\tilde{\rho}, \tilde{u}, 0, 0, \tilde{\phi})(\tilde{M}(x))$, where $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ is the planer stationary solution in $\mathbb{R}^3_+$ and

$$\tilde{M}(x) := x_1 - M(x').$$

The result is summarized in the following theorem. It is worth pointing out that we do not require any smallness assumptions for the function $M$ representing the boundary $\partial\Omega$.

**Theorem 3.1** ([13]). *Let $m \geq 3$ and $u_+$ satisfy* (2.4) *and* (3.1). *There exist positive constants $\beta \leq \alpha/2$, where $\alpha$ is being in Theorem 2.1, and $\delta$ such that if $|\phi_b| \leq \delta$, then stationary problem* (1.5) *has a unique solution $(\rho^s, \boldsymbol{u}^s, \phi^s)$ as*

$$(\rho^s, \boldsymbol{u}^s, \phi^s) - (\tilde{\rho} \circ \tilde{M}, \tilde{\boldsymbol{u}} \circ \tilde{M}, \tilde{\phi} \circ \tilde{M}) \in [H^m_\beta(\Omega)]^4 \times H^{m+1}_\beta(\Omega),$$
$$\|(\rho^s - \tilde{\rho} \circ \tilde{M}, \boldsymbol{u}^s - \tilde{\boldsymbol{u}} \circ \tilde{M})\|^2_{m,\beta} + \|\phi^s - \tilde{\phi} \circ \tilde{M}\|^2_{m+1,\beta} \leq C|\phi_b|,$$

*where $C$ is a positive constant independent of $\phi_b$.*

We also discuss the stability of stationary solutions in both exponential and algebraic weighted Sobolev spaces. Note that the smallness of $M$ is not assumed in the exponential weight case.

**Theorem 3.2** ([13])**.** *Let $u_+$ satisfy* (2.4) *and* (3.1)*. There exist positive constants $\beta \le \alpha/2$, where $\alpha$ is being in Theorem 2.1, and $\delta$ such that if $\|(\rho_0 - \rho^s, \boldsymbol{u}_0 - \boldsymbol{u}^s)\|_{3,\beta} + |\phi_b| \le \delta$, then initial–boundary value problem* (1.1) *has a unique time-global solution $(\rho, \boldsymbol{u}, \phi)$ with* (1.2) *and* (1.3) *in the following space:*

$$(\rho - \rho^s, \boldsymbol{u} - \boldsymbol{u}^s, \phi - \phi^s) \in \left[ \bigcap_{i=0}^{1} C^i([0,T]; H^{3-i}_{\beta}(\Omega)) \right]^4 \times C([0,T]; H^5_{\beta}(\Omega)).$$

*Moreover, it holds that*

$$\sup_{x\in\Omega} |(\rho - \rho^s, \boldsymbol{u} - \boldsymbol{u}^s, \phi - \phi^s)(t,x)| \le Ce^{-\gamma t} \quad \text{for } t \in [0,\infty),$$

*where $C$ and $\gamma$ are positive constants independent of $\phi_b$ and $t$.*

**Theorem 3.3** ([13])**.** *Let $\lambda \ge 2$, $\nu \in (0, \lambda]$, and $u_+$ satisfy* (2.4) *and* (3.1)*. There exist positive constants $\beta_0 \le \beta$, where $\beta$ is being in Theorem 3.1, and $\delta$ such that if $\|M\|_5 + \|(\rho_0 - \rho^s, \boldsymbol{u}_0 - \boldsymbol{u}^s)\|_{3,\beta_0,\lambda} + |\phi_b| \le \delta$, then initial–boundary value problem* (1.1) *has a unique time-global solution $(\rho, \boldsymbol{u}, \phi)$ with* (1.2) *and* (1.3) *in the following space:*

$$(\rho - \rho^s, \boldsymbol{u} - \boldsymbol{u}^s, \phi - \phi^s) \in \left[ \bigcap_{i=0}^{1} C^i([0,T]; H^{3-i}_{\beta_0,\lambda}(\Omega)) \right]^4 \times C([0,T]; H^{5-i}_{\beta_0,\lambda}(\Omega)).$$

*Moreover, it holds that*

$$\sup_{x\in\Omega} |(\rho - \rho^s, \boldsymbol{u} - \boldsymbol{u}^s, \phi - \phi^s)(t,x)| \le C(1+t)^{-\lambda+\nu} \quad \text{for } t \in [0,\infty),$$

*where $C$ is a positive constant independent of $\phi_b$ and $t$.*

Bohm originally derived the criterion (2.4) for the formation of sheaths only in the planer wall case. What most interests us in Theorems 3.1 and 3.2 is that his criterion with the supersonic outflow condition (3.1) also guarantees the formation of sheaths in any case that the shape of walls is drawn by a graph. Mathematically speaking, (2.4) and (3.1) are sharp conditions for the existence and stability of stationary solutions, since they are almost necessary conditions. Hence, it is reasonable to conclude that (2.4) and (3.1) are the Bohm criterion for the perturbed half-space.

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