The decay property of the multidimensional compressible flow in the exterior domain

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Abstract

This is a report of the recent work [13], which is a joint work with Yoshihiro Shibata from Waseda University. In [13], we established the L_p-L_q decay estimate of some model problem of the compressible flow with the free boundary condition in the exterior domain in $\mathbb{R}^N (N \geq 3)$. Furthermore, our proof in [13] followed the local energy approach.

1 Introduction

In this note, we consider the following model problem * in some smooth exterior domain $\Omega \subset \mathbb{R}^N (N \geq 3)$:

$$\begin{cases} \partial_t \rho + \gamma_1 \operatorname{div} \mathbf{v} = 0 & \text{in} \quad \Omega \times \mathbb{R}_+, \\ \gamma_1 \partial_t \mathbf{v} - \operatorname{Div} \big(\mathbf{S}(\mathbf{v}) - \gamma_2 \rho \mathbf{I}\big) = 0 & \text{in} \quad \Omega \times \mathbb{R}_+, \\ \big(\mathbf{S}(\mathbf{v}) - \gamma_2 \rho \mathbf{I}\big) \mathbf{n}_\Gamma = 0 & \text{on} \quad \Gamma \times \mathbb{R}_+, \\ (\rho, \mathbf{v})|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

In (1.1), the constants $\gamma_1, \gamma_2, \mu, \nu > 0$, $\mathbf{S}(\mathbf{v}) = \mu\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top\big) + (\nu - \mu) \operatorname{div} \mathbf{v} \mathbf{I}$, the (i, j)th entry of the matrix $\nabla \mathbf{v}$ is $\partial_i v_j$, and $\mathbf{I}$ is the $N \times N$ identity matrix. In addition, $\mathbf{M}^\top$ is the transposed matrix of $\mathbf{M} = [M_{ij}]$, $\operatorname{Div} \mathbf{M}$ denotes an N-vector of functions whose i-th component is $\sum_{j=1}^N \partial_j M_{ij}$, $\operatorname{div} \mathbf{v} = \sum_{j=1}^N \partial_j v_j$, and $\mathbf{v} \cdot \nabla = \sum_{j=1}^N v_j \partial_j$ with $\partial_j = \partial / \partial x_j$. Moreover, $\mathbf{n}_\Gamma$ stands for the unit normal vector to the boundary Γ of Ω.

The system (1.1) comes from the study of the motion the barotropic viscous gases in some moving exterior domain $\Omega_t \subset \mathbb{R}^N$ $(N \geq 3)$, described by the following compressible Navier-Stokes equations with the free boundary conditions:

$$\begin{cases} \partial_t \rho + \operatorname{div} \big((\rho_e + \rho) \mathbf{v}\big) = 0 & \text{in} \quad \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ (\rho_e + \rho)(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \operatorname{Div} \big(\mathbf{S}(\mathbf{v}) - P(\rho_e + \rho) \mathbf{I}\big) = 0 & \text{in} \quad \bigcup_{0<t<T} \Omega_t \times \{t\}, \\ \big(\mathbf{S}(\mathbf{v}) - P(\rho_e + \rho) \mathbf{I}\big) \mathbf{n}_{\Gamma_t} = -P(\rho_e) \mathbf{n}_{\Gamma_t}, \;\; V_{\Gamma_t} = \mathbf{v} \cdot \mathbf{n}_{\Gamma_t} & \text{on} \quad \bigcup_{0<t<T} \Gamma_t \times \{t\}, \\ (\rho, \mathbf{v}, \Omega_t)|_{t=0} = (\rho_0, \mathbf{v}_0, \Omega). \end{cases} \tag{1.2}$$

*In [13], we treated some model problem like (1.1) with variable coefficients.

In (1.2), the reference mass density $\rho_e > 0$, the unknown mass density is $\rho + \rho_e$, and the unknown velocity field is $\mathbf{v} = (v_1, \dots, v_N)^\top$. Moreover, $\mathbf{n}_{\Gamma_t}$ is the outer unit normal vector to the boundary Γ_t of Ω_t, and V_{Γ_t} stands for the normal velocity of the moving surface Γ_t. In the next section, we shall see that (1.1) can be regarded as the linearized model of (1.2) via the *partial* Lagrangian coordinates. Here, let us emphasize that the linear theory on (1.1) is fundamental to the (local or global) solvability of (1.2).

In [13], we established the L_p-L_q decay property of (1.1), which originates from the theory of the parabolic equations. For simplicity, let us review the heat equation in the whole space $\mathbb{R}^N (N \geq 3)$:

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \\ v|_{t=0} = v_0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{1.3}$$

In view of the explicit solution formula of (1.3), namely,

$$v(x,t) = \int_{\mathbb{R}^N} G_t(x-y) v_0(y)\,dy, \quad G_t(x) = \frac{1}{(4\pi t)^{N/2}} \exp\Big(-\frac{|x|^2}{4t}\Big),$$

it is not hard to verify that v admits the L_p-L_q decay estimate

$$\|\partial_x^\alpha v(\cdot,t)\|_{L_p(\mathbb{R}^N)} \lesssim t^{-(N/q-N/p)/2-|\alpha|/2} \|v_0\|_{L_q(\mathbb{R}^N)}, \tag{1.4}$$

for any $1 \leq q \leq p \leq \infty$, $\alpha \in \mathbb{N}_0^N$, and $t > 0$. Here $\mathbb{N}_0$ denotes the set of all nonnegative integers, and $A \lesssim B$ stands for $A \leq CB$ for some harmless constant C.

The L_p-L_q decay theory plays a vital role in the solvability of the model in fluid dynamics. For example, the extension of (1.4) for the incompressible flow in the exterior domain was done in [7, 8]. Let us write A_S for the Stokes operator associated to the Dirichlet boundary condition in the smooth exterior domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$. Then the results in [7, 8] yield that

$$\begin{aligned} \|e^{tA_S}\mathbf{v}_0\|_{L_p(\Omega)} &\lesssim t^{-N(1/q-1/p)/2} \|\mathbf{v}_0\|_{L_q(\Omega)}, \\ \|\nabla e^{tA_S}\mathbf{v}_0\|_{L_p(\Omega)} &\lesssim t^{-\sigma_1(p,q,N)} \|\mathbf{v}_0\|_{L_q(\Omega)}, \end{aligned} \tag{1.5}$$

for $t > 1$, $1 < q \leq p < \infty$ and

$$\sigma_1(p,q,N) = \begin{cases} (N/q - N/p)/2 + 1/2 & \text{for} \ \ 1 < p \leq N, \\ N/(2q) & \text{for} \ \ N < p < \infty. \end{cases}$$

Moreover, the gradient estimate of e^{tA_S} in (1.5) is also sharp for $p > N$ (see [8]).

On the other hand, for the compressible Navier-Stokes equations, Matsumura and Nishida in [10] proved the global wellposedness whenever the initial data were give small in $H^3(\mathbb{R}^3)$. Moreover, the authors in [9] obtained the L_2-L_1 type decay property of the solutions near the equilibrium $(\rho_e, 0)$,

$$\|(\rho-\rho_e, \mathbf{v})\|_{L_2(\mathbb{R}^3)} \leq C_0\, t^{-3/4} \quad (t>1), \tag{1.6}$$

for some constant C_0 depending on the small quantity $\|(\rho_0-\rho_e, \mathbf{v}_0)\|_{L_1(\mathbb{R}^3)\cap H^3(\mathbb{R}^3)}$. For the further discussion in Besov regularity framework, one may refer to [1, 2, 3, 4, 6, 11].

To state our main result on the L_p-L_q decay estimate of (1.1), we introduce some notion. Let $\{T(t)\}_{t\geq 0}$ be the C_0-semigroup generated by the operator

$$\mathcal{A}_\Omega(\rho, \mathbf{v}) = \Big(\gamma_1 \mathrm{div}\, \mathbf{v}, -\gamma_1^{-1}\mathrm{Div}\, \big(\mathbf{S}(\mathbf{v}) - \gamma_2\rho\mathbf{I}\big)\Big)$$

in the space $H^{1,0}_p(\Omega) = H^1_p(\Omega)\times L_p(\Omega)^N$ for $1<p<\infty$ (see Theorem 4.2). Denote the solution of (1.1) by $(\rho, \mathbf{v}) = T(t)(\rho_0, \mathbf{v}_0)$ and $\mathbf{v} = \mathcal{P}_v T(t)(\rho_0, \mathbf{v}_0)$. Then our main result reads as follows.

Theorem 1.1. *(L_p-L_q decay estimate) Let Ω be a C^3 exterior domain in $\mathbb{R}^N$ with $N\geq 3$. Assume that $(\rho_0, \mathbf{v}_0)\in L_q(\Omega)^{1+N}\cap H^{1,0}_p(\Omega)$ with $H^{1,0}_p(\Omega) = H^1_p(\Omega)\times L_p(\Omega)^N$ for $1\leq q\leq 2\leq p<\infty$, and $\{T(t)\}_{t\geq 0}$ is the semigroup associated to (1.1) in $H^{1,0}_p(\Omega)$. For convenience, we set*

$$|||(\rho_0, \mathbf{v}_0)|||_{p,q} = \|(\rho_0, \mathbf{v}_0)\|_{L_q(\Omega)} + \|(\rho_0, \mathbf{v}_0)\|_{H^{1,0}_p(\Omega)}.$$

Then for $t\geq 1$, there exists a positive constant C such that

$$\begin{aligned}
\|T(t)(\rho_0, \mathbf{v}_0)\|_{L_p(\Omega)} &\leq C t^{-(N/q-N/p)/2} |||(\rho_0, \mathbf{v}_0)|||_{p,q},\\
\|\nabla T(t)(\rho_0, \mathbf{v}_0)\|_{L_p(\Omega)} &\leq C t^{-\sigma_1(p,q,N)} |||(\rho_0, \mathbf{v}_0)|||_{p,q},\\
\|\nabla^2 \mathcal{P}_v T(t)(\rho_0, \mathbf{v}_0)\|_{L_p(\Omega)} &\leq C t^{-\sigma_2(p,q,N)} |||(\rho_0, \mathbf{v}_0)|||_{p,q},
\end{aligned}$$

where the indices $\sigma_1(p,q,N)$ and $\sigma_2(p,q,N)$ are given by

$$\sigma_1(p,q,N) = \begin{cases} (N/q-N/p)/2+1/2 & \text{for } 2\leq p\leq N,\\ N/(2q) & \text{for } N<p<\infty,\end{cases}$$

$$\sigma_2(p,q,N) = \begin{cases} 3/(2q) & \text{for } N=3,\\ (N/q-N/p)/2+1 & \text{for } N\geq 4 \text{ and } 2\leq p\leq N/2,\\ N/(2q) & \text{for } N\geq 4 \text{ and } N/2<p<\infty.\end{cases}$$

To establish the L_p-L_q estimates in Theorem 1.1, we use the so-called *local energy approach*. Assume that $\Omega\subset\mathbb{R}^N$ is an exterior domain such that $\mathbb{R}^N\backslash\Omega\subset B_R$, and B_R denotes the ball centred at origin with radius $R>1$. Then we can prove

Theorem 1.2. *(local energy estimate) Let Ω be a C^3 exterior domain in $\mathbb{R}^N$ for $N \geq 3$. Let $1 < p < \infty$ and $L > 2R$. Denote that* [†]

$$\Omega_L = \Omega \cap B_L, \quad H^{1,2}_p(\Omega_L) = H^1_p(\Omega_L) \times H^2_p(\Omega_L)^N,$$
$$X_{p,L}(\Omega) = \{(d, \mathbf{f}) \in H^{1,0}_p(\Omega) : \operatorname{supp} d, \operatorname{supp} \mathbf{f} \subset \overline{\Omega_L}\,\}.$$

Then for any $(\rho_0, \mathbf{v}_0) \in X_{p,L}(\Omega)$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there exists a positive constant $C_{p,k,L}$ such that

$$\|\partial_t^k T(t)(\rho_0, \mathbf{v}_0)\|_{H^{1,2}_p(\Omega_L)} \leq C_{p,k,L}\, t^{-N/2-k} \|(\rho_0, \mathbf{v}_0)\|_{H^{1,0}_p(\Omega)}, \ \ \forall\, t \geq 1.$$

We will have Theorem 1.1 so long as Theorem 1.2 is established. To prove Theorem 1.2, we consider the resolvent problem of (1.1):

$$\begin{cases} \lambda\eta + \gamma_1 \operatorname{div} \mathbf{u} = d & \text{in} \quad \Omega, \\ \gamma_1 \lambda \mathbf{u} - \operatorname{Div}\big(\mathbf{S}(\mathbf{u}) - \gamma_2 \eta \mathbf{I}\big) = \mathbf{f} & \text{in} \quad \Omega, \\ \big(\mathbf{S}(\mathbf{u}) - \gamma_2 \eta \mathbf{I}\big)\mathbf{n}_\Gamma = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.7}$$

The analysis of (1.7) is the main concern of this note. One difficulty is to describe the behaviour of the solution of (1.7) if λ locates near the origin. This is contained in the result of section 3. On the other hand, it is easy to study (1.7) whenever λ is sufficient large (see Theorem 4.1 in section 4). The case λ is uniformly bounded from above is more involved (see Theorem 4.3).

Notation

For convenience, we introduce some useful notation. For any domain G in $\mathbb{R}^N$, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $L_p(G)$ ($L_{p,\mathrm{loc}}(G)$) stands for the (local) Lebesgue space, and $H^k_p(G)$ ($H^k_{p,\mathrm{loc}}(G)$) for the (local) Sobolev space. Moreover, we write

$$H^{k,\ell}_p(G) = H^k_p(G) \times H^\ell_p(G)^N, \quad H^{k,\ell}_{p,\mathrm{loc}}(G) = H^k_{p,\mathrm{loc}}(G) \times H^\ell_{p,\mathrm{loc}}(G)^N.$$

For any Banach spaces X, Y, the total of the bounded linear transformations from X to Y is denoted by $\mathcal{L}(X;Y)$. We also write $\mathcal{L}(X)$ for short if $X = Y$. In addition, $\operatorname{Hol}(\Lambda; X)$ denotes the set of X-valued analytic mappings defined on the domain $\Lambda \subset \mathbb{C}$. To study the resolvent problem (1.7), we introduce that

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| \leq \pi - \varepsilon\}, \quad \Sigma_{\varepsilon,b} = \{\lambda \in \Sigma_\varepsilon : |\lambda| \geq b\},$$
$$K = \Big\{\lambda \in \mathbb{C} : \big(\Re\lambda + \frac{\gamma_1\gamma_2}{\mu+\nu}\big)^2 + \Im\lambda^2 > \big(\frac{\gamma_1\gamma_2}{\mu+\nu}\big)^2\Big\}, \tag{1.8}$$
$$V_{\varepsilon,b} = \Sigma_{\varepsilon,b} \cap K, \quad \dot{U}_b = \{\lambda \in \mathbb{C}\backslash(-\infty, 0] : |\lambda| < b\}$$

for any $0 < \varepsilon < \pi/2$ and $b > 0$.

[†] $\overline{E}$ stands for the closesure of E for any subset $E \subset \mathbb{R}^N$.

2 Formulation via partial Lagrangian coordinates

In this section, we will introduce the partial Lagrangian coordinates, and we will also see that the linearized form of (1.2) is (1.1). Let $\kappa = \kappa(x)$ be a smooth functions which equals to 1 for $x \in B_R$ and vanishes outside of B_{2R}. Define the partial Lagrangian transformation as follows:

$$x = X_{\mathbf{u}}(y,t) = y + \int_0^t \kappa(y)\mathbf{u}(y,s)\,ds \in \Omega_t \cup \Gamma_t, \ \ \forall\, y \in \Omega \cup \Gamma, \tag{2.1}$$

for some smooth vector $\mathbf{u} = \mathbf{u}(\cdot, s)$ and $0 \le t \le T$. By assuming the condition

$$\int_0^T \|\kappa(\cdot)\mathbf{u}(\cdot,s)\|_{H^1_\infty(\Omega)}\,ds \le \delta < 1/2 \tag{2.2}$$

for small constant $\delta > 0$, we denote $X_{\mathbf{u}}^{-1}(\cdot, t)$ for the inverse of $X_{\mathbf{u}}(\cdot, t)$ in (2.1). Suppose that

$$\rho(x,t) = \eta\big(X_{\mathbf{u}}^{-1}(x,t),t\big), \quad \mathbf{v}(x,t) = \mathbf{u}\big(X_{\mathbf{u}}^{-1}(x,t),t\big), \quad \Omega_t = \big\{x = X_{\mathbf{u}}(y,t) \mid y \in \Omega\big\},$$

solve (1.2) for some function η defined in Ω. We will derive the equations formally satisfied by $(\rho, \mathbf{u})$ in Ω in what follows.

Assume that Γ is a compact hypersurface of C^2 class. The kinematic (non-slip) condition $V_{\Gamma_t} = \mathbf{v} \cdot \mathbf{n}_t$ is automatically satisfied under the transformation $X_{\mathbf{u}}$, because $\kappa = 1$ near the boundary Γ. Denote that

$$\nabla_y X_{\mathbf{u}} = \mathbf{I} + \int_0^t \nabla_y(\kappa(y)\mathbf{u}(y,s))\,ds,$$

and $J_{\mathbf{u}} = \det(\nabla_y X_{\mathbf{u}})$. Then by the assumption (2.2), there exists the inverse of $\nabla_y X_{\mathbf{u}}$, that is,

$$\big(\nabla_y X_{\mathbf{u}}\big)^{-1} = \mathbf{I} + \mathbf{V}_0(\mathbf{k}), \quad \mathbf{k} = \int_0^t \nabla_y(\kappa(y)\mathbf{u}(y,s))\,ds,$$

where $\mathbf{V}_0(\mathbf{k}) = [V_{0ij}(\mathbf{k})]_{N\times N}$ is a matrix-valued function given by

$$\mathbf{V}_0(\mathbf{k}) = \sum_{j=1}^{\infty} (-\mathbf{k})^j.$$

In particular, $\mathbf{V}_0(0) = 0$. By the chain rule, we introduce the gradient, divergence and stress tensor operators with respect to the transformation (2.1),

$$\begin{aligned}
&\nabla_{\mathbf{u}} = \big(\mathbf{I} + \mathbf{V}_0(\mathbf{k})\big)\nabla_y, \quad \operatorname{div}_{\mathbf{u}}\mathbf{u} = \big(\mathbf{I} + \mathbf{V}_0(\mathbf{k})\big) : \nabla_y\mathbf{u} = J^{-1}\operatorname{div}_y\Big(J\big(\mathbf{I} + \mathbf{V}_0(\mathbf{k})\big)^{\top}\mathbf{u}\Big),\\
&\mathbf{D}_{\mathbf{u}}(\mathbf{u}) = \big(\mathbf{I} + \mathbf{V}_0(\mathbf{k})\big)\nabla\mathbf{u} + (\nabla\mathbf{u})^{\top}\big(\mathbf{I} + \mathbf{V}_0(\mathbf{k})\big)^{\top} = \mathbf{D}(\mathbf{u}) + \mathbf{V}_0(\mathbf{k})\nabla\mathbf{u} + \big(\mathbf{V}_0(\mathbf{k})\nabla\mathbf{u}\big)^{\top},
\end{aligned} \tag{2.3}$$

$$\mathbf{S}_{\mathbf{u}}(\mathbf{u}) = \mu\mathbf{D}_{\mathbf{u}}(\mathbf{u}) + (\nu - \mu)(\operatorname{div}_{\mathbf{u}}\mathbf{u})\mathbf{I}, \quad \operatorname{Div}_{\mathbf{u}}\mathbf{A} = J_{\mathbf{u}}^{-1}\operatorname{Div}_y\Big(J_{\mathbf{u}}\mathbf{A}\big(\mathbf{I} + \mathbf{V}_0(\mathbf{k})\big)\Big).$$

In addition, the ith component of $\operatorname{Div}_{\mathbf{u}}\mathbf{A}$ can be also written via

$$(\operatorname{Div}_{\mathbf{u}}\mathbf{A})_i = \sum_{j,k=1}^{N} \big[\mathbf{I} + \mathbf{V}_0(\mathbf{k})\big]_{jk}\partial_k A_{ij}, \quad \forall\, i = 1, \dots, N. \tag{2.4}$$

In particular, $\operatorname{Div}_{\mathbf{u}}\mathbf{A} = 0$ if $\mathbf{A}$ is a constant matrix. Then according to (2.3), $(\rho, \mathbf{u})$ fulfils

$$\begin{cases} \partial_t\eta + (1-\kappa)\mathbf{u}\cdot\nabla_{\mathbf{u}}\eta + (\rho_e + \eta)\operatorname{div}_{\mathbf{u}}\mathbf{u} = 0 & \text{in } \Omega\times(0,T), \\ (\rho_e+\eta)\big(\partial_t\mathbf{u} + (1-\kappa)\mathbf{u}\cdot\nabla_{\mathbf{u}}\mathbf{u}\big) - \operatorname{Div}_{\mathbf{u}}\big(\mathbf{S}_{\mathbf{u}}(\mathbf{u}) - P(\rho_e+\eta)\mathbf{I}\big) = 0 & \text{in } \Omega\times(0,T), \\ \big(\mathbf{S}_{\mathbf{u}}(\mathbf{u}) - P(\rho_e+\eta)\mathbf{I}\big)\mathbf{n}_{\mathbf{u}} = -P(\rho_e)\mathbf{n}_{\mathbf{u}} & \text{on } \Gamma\times(0,T), \\ (\eta,\mathbf{u})|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in } \Omega, \end{cases} \tag{2.5}$$

where $\mathbf{n}_\Gamma$ denotes for the unit normal vector to Γ, and $\mathbf{n}_{\mathbf{u}}$ is defined by

$$\mathbf{n}_{\mathbf{u}} = \frac{\big(\mathbf{I}+\mathbf{V}_0(\mathbf{k})\big)\mathbf{n}_\Gamma}{\big|\big(\mathbf{I}+\mathbf{V}_0(\mathbf{k})\big)\mathbf{n}_\Gamma\big|}.$$

It is clear that the boundary condition in (2.5) is equivalent to

$$\Big(\mathbf{S}_{\mathbf{u}}(\mathbf{u}) - \big(P(\rho_e+\eta) - P(\rho_e)\big)\mathbf{I}\Big)\big(\mathbf{I}+\mathbf{V}_0(\mathbf{k})\big)\mathbf{n}_\Gamma = 0. \tag{2.6}$$

On the other hand, as $P(\cdot)$ is smooth, we infer from Taylor's theorem that

$$P(\rho_e+\eta) - P(\rho_e) = P'(\rho_e)\eta + \frac{\eta^2}{2}\int_0^1 P''(\rho_e+\theta\eta)(1-\theta)\,d\theta. \tag{2.7}$$

Thus (2.6) and (2.7) yield that the principal terms of (2.5) are given as in the left-hand side of (1.1) by setting $(\gamma_1,\gamma_2) = \big(\rho_e, P'(\rho_e)\big)$.

3 Resolvent problem for λ near zero

In this section, we will give the behaviour of the solution of the system (1.7) whenever λ lies near the origin. This situation is the most significant part of this work.

Theorem 3.1. *Let $(d,\mathbf{f}) \in X_{p,L}(\Omega)$ for $1 < p \le r$ and $L > 2R > 0$. Then there exist a constant $\lambda_1 > 0$ and two families of the operators $(\mathbb{M}_\lambda, \mathbb{V}_\lambda)$ for any $\lambda \in \dot U_{\lambda_1} = \{\lambda \in \mathbb{C}\backslash(-\infty,0] : |\lambda| < \lambda_1\}$ with*

$$\mathbb{M}_\lambda \in \operatorname{Hol}\Big(\dot U_{\lambda_1}; \mathcal{L}\big(X_{p,L}(\Omega); H^1_{p,\mathrm{loc}}(\Omega)\big)\Big), \quad \mathbb{V}_\lambda \in \operatorname{Hol}\Big(\dot U_{\lambda_1}; \mathcal{L}\big(X_{p,L}(\Omega); H^2_{p,\mathrm{loc}}(\Omega)^N\big)\Big),$$

so that $(\eta,\mathbf{u}) = (\mathbb{M}_\lambda, \mathbb{V}_\lambda)(d,\mathbf{f})$ solves (1.7). Moreover, there exist families of the operators

$$\mathbb{M}^i_\lambda \in \operatorname{Hol}\Big(\dot U_{\lambda_1}; \mathcal{L}\big(X_{p,L}(\Omega); H^1_{p,\mathrm{loc}}(\Omega)\big)\Big) \ (i = 1,2),$$

$$\mathbb{V}^j_\lambda \in \operatorname{Hol}\Big(\dot U_{\lambda_1}; \mathcal{L}\big(X_{p,L}(\Omega); H^2_{p,\mathrm{loc}}(\Omega)^N\big)\Big) \ (j = 0,1,2),$$

such that

$$\mathbb{M}_\lambda = (\lambda^{N-2}\log\lambda)\mathbb{M}^1_\lambda + \mathbb{M}^2_\lambda,$$
$$\mathbb{V}_\lambda = \big(\lambda^{N/2-1}(\log\lambda)^{\sigma(N)}\big)\mathbb{V}^0_\lambda + (\lambda^{N-2}\log\lambda)\mathbb{V}^1_\lambda + \mathbb{V}^2_\lambda,$$

for any $\lambda \in \dot{U}_{\lambda_1}$ *and* $\sigma(N) = \big((-1)^N + 1\big)/2$.

In the following, we outline the main strategy of the proof of Theorem 3.1. Without loss of generality, we shall prove Theorem 3.1 for $L = 5R$. To construct the solution mapping of (1.7), we consider the auxiliary problem:

$$\begin{cases} \gamma_1 \mathrm{div}\,\mathbf{u} = d & \text{in } \Omega_{5R}, \\ -\mathrm{Div}\,\big(\mathbf{S}(\mathbf{u}) - \gamma_2\eta\mathbf{I}\big) = \mathbf{f} & \text{in } \Omega_{5R}, \\ \big(\mathbf{S}(\mathbf{u}) - \gamma_2\eta\mathbf{I}\big)\mathbf{n}_\Gamma = 0 & \text{on } \Gamma, \\ \big(\mathbf{S}(\mathbf{u}) - \gamma_2\eta\mathbf{I}\big)\mathbf{n}_{S_{5R}} = 0 & \text{on } S_{5R}, \end{cases} \tag{3.1}$$

Here, $\mathbf{n}_{S_{5R}}$ denotes the unit outer normal to $S_{5R} = \{x \in \mathbb{R}^N \mid |x| = 5R\}$.

The homogeneous system (3.1) lacks of the uniqueness in general. So we need some trick to fix it. Let $3R < b_0 < b_1 < b_2 < b_3 < 4R$ and set

$$D_{b_1,b_2} = \{x \in \mathbb{R}^N \mid b_1 < |x| < b_2\}, \quad D^+_{b_1,b_2} = \{x \in D_{b_1,b_2} \mid x_j > 0 \ \ (j = 1, \dots, N)\}.$$

Now, we introduce the vectors of the rigid motion. Set

$$\mathbf{r}_j(x) = \begin{cases} \mathbf{e}_j = (0, \dots, \underbrace{1}_{\text{jth component}}, \dots, 0) & \text{for } j = 1, \dots, N, \\ x_k\mathbf{e}_\ell - x_\ell\mathbf{e}_k \ \ (k, \ell = 1, \dots, N) & \text{for } j = N+1, \dots, M. \end{cases} \tag{3.2}$$

Above, M is a constant only depending of the dimension N. For any vector $\mathbf{u}$ satisfying $\mathbf{D}(\mathbf{u}) = 0$, $\mathbf{u}$ is represented by a linear combination of $\{\mathbf{r}_j\}_{j=1}^M$, namely $\mathbf{u} = \sum_{j=1}^M a_j\mathbf{r}_j$ with some $a_j \in \mathbb{R}$. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\mathrm{supp}\,\psi \subset D_{b_1,b_2}$, and $\psi = 1$ on some ball $B \subset D^+_{b_1,b_2}$. We introduce a family of vectors $\mathfrak{Q}_\psi = \{\mathbf{q}_j\}_{j=1}^M$, the normalization of $\{\mathbf{r}_j\}_{j=1}^M$ in such a way that

$$(\mathbf{q}_j, \mathbf{q}_k)_\psi = (\psi\mathbf{q}_j, \mathbf{q}_k)_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \psi(x)\,\mathbf{q}_j(x)\cdot\mathbf{q}_k(x)\,dx = \delta_{jk}. \tag{3.3}$$

Moreover, for simplicity we write

- $\mathbf{f} \perp \mathfrak{Q}_R$ if $(\mathbf{f}, \mathbf{q}_j)_{\Omega_{5R}} = 0$ for any $\mathbf{q}_j \in \mathfrak{Q}_\psi$;
- $\mathbf{f} \perp \mathfrak{Q}_\psi$ if $(\mathbf{f}, \mathbf{q}_j)_\psi = 0$ for any $\mathbf{q}_j \in \mathfrak{Q}_\psi$.

With the notations above, we can prove the following elliptic estimates for (3.1).

Theorem 3.2. *Let $1 < p \leq r$. Let $(d, \mathbf{f}) \in H^{1,0}_p(\Omega_{5R})$ with $\mathbf{f} \perp \mathfrak{Q}_R$. Then there exist operators*

$$(\mathcal{J}, \mathcal{W}) \in \mathcal{L}(H^{1,0}_p\big(\Omega_{5R}), H^{1,2}_p(\Omega_{5R})\big)$$

such that $(\eta, \mathbf{u}) = (\mathcal{J}, \mathcal{W})(d, \mathbf{f})$ is a unique solution of (3.1) with $\mathbf{u} \perp \mathfrak{Q}_R$. Moreover, the following estimate holds,

$$\|\eta\|_{H^1_p(\Omega_{5R})} + \|\mathbf{u}\|_{H^2_p(\Omega_{5R})} \leq C\big(\|d\|_{H^1_p(\Omega_{5R})} + \|\mathbf{f}\|_{L_p(\Omega_{5R})}\big),$$

for some constant $C > 0$.

The proof of Theorem 3.2 is one core but technical result in our work [13]. Here, we admit such result and proceed with the proof of Theorem 3.1. Let φ, ψ_0, and ψ_∞ be the cut-off functions such that $0 \leq \varphi, \psi_0, \psi_\infty \leq 1$, φ, ψ_0, $\psi_\infty \in C^\infty(\mathbb{R}^N)$, and

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \leq b_1, \\ 0 & \text{for } |x| \geq b_2, \end{cases} \quad \psi_0(x) = \begin{cases} 1 & \text{for } |x| \leq b_2, \\ 0 & \text{for } |x| \geq b_3, \end{cases} \quad \psi_\infty(x) = \begin{cases} 1 & \text{for } |x| \geq b_1, \\ 0 & \text{for } |x| \leq b_0. \end{cases} \tag{3.4}$$

For any $(d, \mathbf{f}) \in H^{1,0}_p(\Omega_{5R})$, we have

$$\|\psi_\infty d\|_{H^1_p(\mathbb{R}^N)} + \|\psi_\infty \mathbf{f}\|_{L_p(\mathbb{R}^N)} \leq C(\|d\|_{H^1_p(\Omega)} + \|\mathbf{f}\|_{L_p(\Omega)}). \tag{3.5}$$

Then, by the theory in [12, Subsection 3.1] and (3.5), there exists a $\lambda_0 > 0$ such that $(\eta_\lambda, \mathbf{u}_\lambda) = (\mathcal{M}_\lambda, \mathcal{V}_\lambda)(\psi_\infty d, \psi_\infty \mathbf{f})$ solves the following equations:

$$\begin{cases} \lambda\eta_\lambda + \gamma_1 \operatorname{div} \mathbf{u}_\lambda = \psi_\infty d & \text{in} \quad \mathbb{R}^N, \\ \gamma_1 \lambda \mathbf{u}_\lambda - \operatorname{Div}\big(\mathbf{S}(\mathbf{u}_\lambda) - \gamma_2 \eta_\lambda \mathbf{I}\big) = \psi_\infty \mathbf{f} & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{3.6}$$

and satisfies the estimate:

$$\|\eta_\lambda\|_{H^1_p(B_{6R})} + \|\mathbf{u}_\lambda\|_{H^2_p(B_{6R})} \leq C(\|d\|_{H^1_p(\Omega)} + \|\mathbf{f}\|_{L_p(\Omega)}). \tag{3.7}$$

Moreover, we set $(\eta_0, \mathbf{u}_0) = (\mathcal{M}_0, \mathcal{V}_0)(\psi_\infty d, \psi_\infty \mathbf{f}) \in H^{1,2}_{p,\mathrm{loc}}(\mathbb{R}^N)$ fulfilling that

$$\begin{cases} \gamma_1 \operatorname{div} \mathbf{u}_0 = \psi_\infty d & \text{in} \quad \mathbb{R}^N, \\ -\operatorname{Div}\big(\mathbf{S}(\mathbf{u}_0) - \gamma_2 \eta_0 \mathbf{I}\big) = \psi_\infty \mathbf{f} & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{3.8}$$

and

$$\lim_{\substack{\lambda \in \dot{U}_{\lambda_0} \\ |\lambda| \to 0}} (\|\eta_\lambda - \eta_0\|_{H^1_p(B_{6R})} + \|\mathbf{u}_\lambda - \mathbf{u}_0\|_{H^2_p(B_{6R})}) = 0. \tag{3.9}$$

On the other hand, let us set

$$\mathbf{f}_{\mathcal{R}_d} = \sum_{j=1}^M (\psi_0 \mathbf{f}, \mathbf{q}_j)_{\Omega_{5R}} \psi \mathbf{q}_j, \quad \mathbf{f}_\perp = \psi_0 \mathbf{f} - \mathbf{f}_{\mathcal{R}_d} \in L_p(\Omega_{5R})^N.$$

Obviously, $\mathbf{f}_\perp \perp \mathfrak{Q}_R$. Then, Theorem 3.2 yields that there exists a (unique) solution $(\eta_\sharp, \mathbf{u}_\sharp) \in H^{1,2}_p(\Omega_{5R})$ with $\mathbf{u}_\sharp \perp \mathfrak{Q}_R$ of the following equations:

$$\begin{cases} \gamma_1 \operatorname{div} \mathbf{u}_\sharp = \psi_0 d & \text{in} \quad \Omega_{5R}, \\ -\operatorname{Div}\big(\mathbf{S}(\mathbf{u}_\sharp) - \gamma_2 \eta_\sharp \mathbf{I}\big) = \mathbf{f}_\perp & \text{in} \quad \Omega_{5R}, \\ \big(\mathbf{S}(\mathbf{u}_\sharp) - \gamma_2 \eta_\sharp \mathbf{I}\big)\mathbf{n}_\Gamma = 0 & \text{on} \quad \Gamma, \\ \big(\mathbf{S}(\mathbf{u}_\sharp) - \gamma_2 \eta_\sharp \mathbf{I}\big)\mathbf{n}_{S_{5R}} = 0 & \text{on} \quad S_{5R}, \end{cases} \tag{3.10}$$

possessing the estimate

$$\|\eta_\sharp\|_{H^1_p(\Omega_{5R})} + \|\mathbf{u}_\sharp\|_{H^2_p(\Omega_{5R})} \le C(\|d\|_{H^1_p(\Omega)} + \|\mathbf{f}\|_{L_p(\Omega)}). \tag{3.11}$$

We now introduce parametrices:

$$\widetilde{\eta}_\lambda = \Phi_\lambda(d, \mathbf{f}) = (1-\varphi)\eta_\lambda + \varphi\eta_\sharp, \quad \widetilde{\mathbf{u}}_\lambda = \Psi_\lambda(d, \mathbf{f}) = (1-\varphi)\mathbf{u}_\lambda + \varphi\mathbf{u}_\sharp$$

for $\lambda \in \dot{U}_{\lambda_0} \cup \{0\}$. Then the couple $(\widetilde{\eta}_\lambda, \widetilde{\mathbf{u}}_\lambda)$ plays a vital role in constructing the solution mapping of (1.7) whenever λ is near the zero. For more details, see [13].

4 Resolvent problem for λ away from zero

According to Theorem 3.1, it suffices to study (1.7) whenever λ is uniformly bounded from below. In this section, we first give the result when λ is far away from the origin. Then we consider (1.7) whenever λ lies in some ring-shaped region.

4.1 Resolvent problem for large λ

Recall the notion in (1.8). The following result can be regarded as the simplified version of [5, Theorem 2.4]:

Theorem 4.1. *Let $1 < p \le r < \infty$, and $0 < \varepsilon < \pi/2$. Assume that Ω is a C^2 exterior domain in $\mathbb{R}^N$ for $N \ge 3$. Then there exist $\lambda_2 > 0$ and two families of operators*

$$\big(\mathcal{P}_\infty(\lambda), \mathcal{V}_\infty(\lambda)\big) \in \operatorname{Hol}\Big(V_{\varepsilon,\lambda_2}; \mathcal{L}\big(H^{1,0}_p(\Omega); H^{1,2}_p(\Omega)\big)\Big),$$

such that $(\eta, \mathbf{u}) = \big(\mathcal{P}_\infty(\lambda), \mathcal{V}_\infty(\lambda)\big)(d, \mathbf{f}) \in H^{1,2}_p(\Omega)$ is a unique solution of (1.7) for any $\lambda \in V_{\varepsilon,\lambda_2}$ and any $(d, \mathbf{f}) \in H^{1,0}_p(\Omega)$. Moreover, we have

$$\|\eta\|_{H^1_p(\Omega)} + \|\mathbf{u}\|_{H^2_p(\Omega)} \le C\big(\|d\|_{H^1_p(\Omega)} + \|\mathbf{f}\|_{L_p(\Omega)}\big) \tag{4.1}$$

for some constant C depending solely on $\lambda_2, \varepsilon, p, \mu, \nu, \gamma_1, \gamma_2, N$.

The existence of the semigroup $\{T(t)\}_{t\geq 0}$ associated to (1.1) is immediate from Theorem 4.1. For $1 < p, q < \infty$, we define

$$\mathcal{D}_p(\mathcal{A}_\Omega) = \{(\eta, \mathbf{u}) \in H^{1,0}_p(\Omega) \mid \mathbf{u} \in H^2_p(\Omega)^N,\ \big(\mathbf{S}(\mathbf{u}) - \gamma_2 \eta \mathbf{I}\big)\mathbf{n}_\Gamma = 0\},$$
$$\mathcal{D}_{p,q}(\Omega) = \big(H^{1,0}_p(\Omega), \mathcal{D}_p(\mathcal{A}_\Omega)\big)_{1-1/q,q}.$$

Theorem 4.2. *The operator $\mathcal{A}_\Omega$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in $H^{1,0}_p(\Omega)$ for any $1 < p \leq r < \infty$, which is analytic as well. Denote the solution of (1.1) by $(\rho, \mathbf{v})(t) = T(t)(\rho_0, \mathbf{v}_0)$. Then there exists positive constants γ_0 and C such that the following assertions hold.*

1. *For $(\rho_0, \mathbf{v}_0) \in H^{1,0}_p(\Omega)$, we have*

$$\|(\rho, \mathbf{v})(t)\|_{H^{1,0}_p(\Omega)} + t\big(\|\partial_t(\rho, \mathbf{v})(t)\|_{H^{1,0}_p(\Omega)} + \|(\rho, \mathbf{v})(t)\|_{\mathcal{D}_p(\mathcal{A}_\Omega)}\big) \leq Ce^{\gamma_0 t}\|(\rho_0, \mathbf{v}_0)\|_{H^{1,0}_p(\Omega)}.$$

2. *For $(\rho_0, \mathbf{v}_0) \in \mathcal{D}_p(\mathcal{A}_\Omega)$, we have*

$$\|\partial_t(\rho, \mathbf{v})(t)\|_{H^{1,0}_p(\Omega)} + \|(\rho, \mathbf{v})(t)\|_{\mathcal{D}_p(\mathcal{A}_\Omega)} \leq Ce^{\gamma_0 t}\|(\rho_0, \mathbf{v}_0)\|_{\mathcal{D}_p(\mathcal{A}_\Omega)}.$$

3. *For $(\rho_0, \mathbf{v}_0) \in \mathcal{D}_{p,q}(\Omega)$, we have*

$$\|e^{-\gamma_0 t}(\partial_t \rho, \rho)\|_{L_q(\mathbb{R}_+;H^1_p(\Omega))} + \|e^{-\gamma_0 t}\partial_t \mathbf{v}\|_{L_q(\mathbb{R}_+;L_p(\Omega))} + \|e^{-\gamma_0 t}\mathbf{v}\|_{L_q(\mathbb{R}_+;H^2_p(\Omega))}$$
$$\leq C\big(\|\rho_0\|_{H^1_p(\Omega)} + \|\mathbf{v}_0\|_{B^{2(1-1/q)}_{p,q}(\Omega)}\big).$$

4.2 Resolvent problem for λ in some compact subset

Thanks to Theorem 4.1 and Theorem 3.1, it remains to study (1.7) whenever λ is uniformly bounded from above and also from below. To this end, let us take some suitable positive constants λ_1' and λ_2' such that

$$0 < \lambda_1 - \lambda_1' \ll 1, \quad 0 < \lambda_2' - \lambda_2 \ll 1,$$

with λ_1 and λ_2 given by Theorem 3.1 and Theorem 4.1 respectively. For fixed constants $\mu, \nu, \gamma_1, \gamma_2 > 0$, we set

$$\begin{aligned} K_\varepsilon &= \Big\{\lambda \in \mathbb{C}\backslash\{0\} : \big(\Re\lambda + \frac{\gamma_1\gamma_2}{\mu+\nu} + \varepsilon\big)^2 + \Im\lambda^2 \geq \big(\frac{\gamma_1\gamma_2}{\mu+\nu} + \varepsilon\big)^2\Big\}, \\ D_\varepsilon' &= \{\lambda \in \Sigma_\varepsilon \cap K_\varepsilon : \lambda_1' \leq |\lambda| \leq \lambda_2'\}. \end{aligned} \tag{4.2}$$

Now, we address the resolvent problem (1.7) whenever λ lies in D_ε' above.

Theorem 4.3. *Suppose that Ω is a C^2 exterior domain in $\mathbb{R}^N$ for $N \geq 3$. Let $0 < \varepsilon < \pi/2$, $N < r < \infty$, $1 < p \leq r$, and $\lambda \in D'_\varepsilon$. Then there exist two families of operators*

$$\big(\mathcal{P}_{mid}(\lambda), \mathcal{V}_{mid}(\lambda)\big) \in \mathrm{Hol}\left(D'_\varepsilon; \mathcal{L}\big(H^{1,0}_p(\Omega); H^{1,2}_p(\Omega)\big)\right),$$

such that $(\eta, \mathbf{u}) = \big(\mathcal{P}_{mid}(\lambda), \mathcal{V}_{mid}(\lambda)\big)(d, \mathbf{f}) \in H^{1,2}_p(\Omega)$ is a unique solution of (1.7) for any $\lambda \in D'_\varepsilon$ and for any $(d, \mathbf{f}) \in H^{1,0}_p(\Omega)$. Moreover, we have

$$\|\eta\|_{H^1_p(\Omega)} + \|\mathbf{u}\|_{H^2_p(\Omega)} \leq C\big(\|d\|_{H^1_p(\Omega)} + \|\mathbf{f}\|_{L_p(\Omega)}\big)$$

for some constant C depending solely on $\lambda'_1, \lambda'_2, \varepsilon, p, r, \mu, \nu, \gamma_1, \gamma_2, N$.

The proof of Theorem 4.3 relies on the compactness of the set D'_ε. In [13], we first study (1.7) for any fixed $\lambda \in D'_\varepsilon$, where the elliptic estimates depend on λ. Then, using the finite covering property of D'_ε, we can remove such dependence of λ and obtain the uniform estimates as in Theorem 4.3.

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