On the Head-on Collision of Coaxial Vortex Rings

Masashi Aiki

Tokyo University of Science

Abstract

We consider the head-on collision of two coaxial vortex rings, which have circulations of opposite sign, described as the motion of two coaxial circular vortex filaments under the localized induction approximation. We prove the existence of solutions to a system of nonlinear partial differential equations modelling the interaction of two vortex filaments proposed by the author [M. Aiki, On the existence of leapfrogging pair of circular vortex filaments, *Stud. Appl. Math.*, **143** (2019), no.3, pp.213–243.] which exhibit head–on collision. We also give a necessary and sufficient condition for the initial configuration and parameters of the filaments for head-on collision to occur. Our results suggest that there exists a critical value $\gamma_* > 1$ for the ratio γ of the magnitude of the circulations satisfying the following. When $\gamma \in [1, \gamma_*]$, two approaching rings will collide, and when $\gamma \in (\gamma_*, \infty)$, the ring with the larger circulation passes through the other and then separate indefinitely. As far as the author knows, the existence of such threshold γ_* is only indirectly suggested via numerical investigations of the head-on collision of coaxial vortex rings. Hence, our paper is the first to obtain the threshold in a way that is possible to numerically calculate γ_*, as well as prove that the threshold exists in a framework of a mathematical model.

1 Introduction

In this paper, we are interested in the head-on collision of two vortex rings sharing the same axis of symmetry (coaxial vortex rings) in a incompressible and inviscid fluid. For the purposes of this paper, we make a distinction between the two terms "vortex ring" and "circular vortex filament" by the following. The term "vortex ring" will be used to describe a torus shaped structure in which the vorticity of the fluid is concentrated. The term "circular vortex filament" will be used to describe a circular curve in space for which the vorticity of the fluid is concentrated. For a vortex filament, the vorticity of the fluid at each point of the curve is directed at the direction of the tangent vector. Hence, under our terminology, a circular vortex filament can be considered as an approximation of a vortex ring in which the core size is taken to be zero.

The study of the interaction of coaxial vortex rings dates back to the pioneering paper by Helmholtz [1]. In [1], Helmholtz considered vortex motion in a incompressible and inviscid fluid based on the Euler equations. His study includes the motion of circular vortex filaments, and he observed that motion patterns such as head-on collision may occur. Since then, many researches have been done on head-on collision of coaxial vortex rings, and interaction of coaxial vortex rings in general.

Dyson [2, 3] further studied the interaction of coaxial vortex rings and proposed a system of ordinary differential equations describing the motion of the rings. From here, we will refer to this model as the Dyson model. Dyson numerically considered the head-on collision of two identical rings approaching each other and observed the dynamics of the rings as the distance between the two rings decreased. Gurzhii and Konstantinov [4] numerically investigated a model system obtained in Gurzhii, Konstantinov, and Meleshko [5], which can be seen as a generalization of the Dyson model, and observed the possible motion patterns for a pair of coaxial vortex rings with circulations of opposite sign. In Shariff, Leonard, and Ferziger [6] and Shariff, Leonard, Zabusky, and Ferziger [7], they extend the method of contour dynamics, introduced in Zabusky, Hughes, and Roberts [8], and numerically investigated the head-on collision of coaxial vortex rings. They also compare and contrast with dynamics described by the Dyson model. The method of contour dynamics was later employed by Tang and Ko [9] to investigate sound generation by the head-on collision of coaxial vortex rings. They consider rings with both equal and unequal circulation magnitudes and observe the effect it has on sound generation.

Direct numerical simulation of the Navier–Stokes equations was done by Stanaway, Shariff, and Hussain [10]. In particular, they considered the head-on collision of coaxial vortex rings and the effect of viscosity was observed. Inoue, Hattori, and Sasaki [11] also conducted numerical simulations of the Navier–Stokes equations to investigate the head-on collision of coaxial vortex rings when the translational velocity of the rings are varied. Nakashima [12] conducted numerical simulations of the Navier–Stokes equations to investigate the generation of sound when two rings collide head-on. He also considered the case when the two rings collide at different angles of incidence. Mansfield, Knio, and Meneveau [13] considered the head-on collision of coaxial vortex rings using a Lagrangian particle method which combines a vortex element method and dynamic eddy viscosity model. They investigate the collision of coaxial vortex rings at high Reynolds numbers and compare their results with numerical results obtained in previous works which utilize different numerical methods.

Experiments were conducted by many researchers as well. Oshima [14] conducted an experiment in which she observed the formation of multiple small rings after the initial two coaxial rings collided. This occurs due to the reconnection of the two rings and her work was the first to observe this phenomenon. A more revealing experiment of the reconnection phenomenon and the related instability of the motion of vortex rings was given in Lim and Nickels [15]. Kambe and Minota [16] and Minota and Kambe [17] conducted experiments to observe the acoustic waves radiated by the head-on collision of coaxial vortex rings. Chu et.al. [18] conducted experiments and numerical calculations of the Navier-Stokes equations to investigate the head-on collision phenomenon and its relation to the change in enstrophy. Minota, Nishida, and Lee [19] conducted experiments and numerical calculations to investigate the head-on collision of coaxial vortex rings at high speeds. They observe that when the rings become close enough, a shock is emitted from the narrow region between the two rings. Mckeown et. al. [20] conducted experiments as well as numerical experiments of the Navier-Stokes equations to investigate the breakdown of the vortex core when two coaxial vortex rings collide at high Reynolds numbers.

Other phenomena which are related to head-on collision of coaxial vortex rings include collision of multiple rings at various angles of incidence, collision of vortex rings travelling along off-set and parallel axes, and collision of vortex rings which are initially linked. Collision of rings at various angles of incidence is investigated, for example, by Kida, Takaoka, and Hussain [21], Kambe, Minota, and Takaoka [22], Adachi, Ishii, and Kambe [23], Ishii, Adachi, and Kambe [24], and Hernández and Reyes [25]. Collision of vortex rings travelling along off-set and parallel axes is considered by Zawadzki and Aref [26], Smith and Wei [27], and Zhu et. al. [28]. The collision of initially linked vortex rings is considered by Aref and Zawadzki [29].

Shariff and Leonard [30] and Maleshko [31] give an in-depth review of the history of the research of vortex rings in which many aspects of motion, including head-on collision, are addressed.

Although the study of head-on collision of coaxial vortex rings have been done for a very long time, there is significantly fewer research of the head-on collision phenomenon in a mathematically rigorous framework. Giga and Miyakawa [32] and Feng and Šverák [33] proved the well-posedness of the initial value problem for the Navier–Stokes equations with initial data given as vortex rings. In principle, these results give a mathematically rigorous treatment of the interaction of vortex rings, but extracting the dynamics of specific motion patterns through this approach seems difficult. In Borisov, Kilin, and Mamaev [34], they analyze the Dyson model to determine the possible motion patterns of a pair of coaxial vortex rings for a wide range of configurations, but their work doesn't include head-on collision.

Another approach one may take is to consider circular vortex filaments instead of vortex rings. By simplifying the structure, it is possible that specific motion patterns can be treated in a mathematically rigorous framework, and this is the approach we adopt in this paper. As far as the author knows, the works by Banica and Miot [35] and Banica, Faou, and Miot [36, 37] are the only mathematically rigorous results considering the collision of vortex filaments. They considered the motion of nearly parallel vortex filaments described by the model system of partial differential equations proposed by Klein, Majda, and Damodaran [38]. Since the model system is derived by assuming that the vortex filaments are nearly straight and parallel, it is not suitable for describing motions of circular vortex filaments. In [39], the author proposed a system of nonlinear partial differential equations describing the interaction of vortex filaments. The paper [39] focused on deriving a new system describing the interaction of vortex filaments with general shape and proving the existence of solutions corresponding to leapfrogging in the case of coaxial circular vortex filaments. The aim of this paper is to prove the existence of solutions to the model system proposed in [39] which correspond to head-on collision of coaxial circular vortex filaments. We also give necessary and sufficient conditions on the filament configurations and parameters for head-on collision to occur. This further shows the capabilities of the model to describe vortex filament interaction. The results of this paper will also imply that the time-global solvability of initial value problems for the model system doesn't hold in general.

The rest of the paper is organized as follows. In Section 2, we formulate the problem and state our main theorem. Section 3 is devoted to the proof of the main theorem. Finally, in Section 4, we give some discussions and concluding remarks. In particular, we compare our theoretical results with numerical results obtained by Inoue, Hattori, and Sasaki [11] and Gurzhii and Konstantinov [4].

2 Problem Setting

In [39], the author proposed the following system of nonlinear partial differential equations.

$$
(2.1) \qquad \begin{cases} \boldsymbol{X}_t = \Gamma_1 \dfrac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} - \alpha\Gamma_2 \dfrac{\boldsymbol{Y}_\xi \times (\boldsymbol{X}-\boldsymbol{Y})}{|\boldsymbol{X}-\boldsymbol{Y}|^3}, \\[2ex] \boldsymbol{Y}_t = \Gamma_2 \dfrac{\boldsymbol{Y}_\xi \times \boldsymbol{Y}_{\xi\xi}}{|\boldsymbol{Y}_\xi|^3} - \alpha\Gamma_1 \dfrac{\boldsymbol{X}_\xi \times (\boldsymbol{Y}-\boldsymbol{X})}{|\boldsymbol{X}-\boldsymbol{Y}|^3}. \end{cases}
$$

Here, $\boldsymbol{X} = \boldsymbol{X}(\xi, t)$ and $\boldsymbol{Y} = \boldsymbol{Y}(\xi, t)$ are the position vectors of the filaments parametrized by ξ at time t, non-zero parameters Γ_1 and Γ_2 are the circulations of the filaments $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively, α is a positive parameter introduced in the derivation of the model, $\times$ is the exterior product in the three-dimensional Euclidean space, and subscripts denote differentiation with the respective variables. The model system (2.1) was derived from the Biot–Savart law by applying the localized induction approximation. The localized induction approximation was applied to the Biot–Savart law first by Da Rios [40] and later independently by Murakami et. al. [41] and Arms and Hama [42] to derive a model equation describing the motion of a single vortex filament. In [39], the concept of localized induction was applied to the case where two vortex filaments are present to derive system (2.1). We first rescale the time variable by a factor of Γ_2 and arrive at

$$
(2.2) \qquad \begin{cases} \boldsymbol{X}_t = \beta \dfrac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} - \alpha \dfrac{\boldsymbol{Y}_\xi \times (\boldsymbol{X}-\boldsymbol{Y})}{|\boldsymbol{X}-\boldsymbol{Y}|^3}, \\[2ex] \boldsymbol{Y}_t = \dfrac{\boldsymbol{Y}_\xi \times \boldsymbol{Y}_{\xi\xi}}{|\boldsymbol{Y}_\xi|^3} - \alpha\beta \dfrac{\boldsymbol{X}_\xi \times (\boldsymbol{Y}-\boldsymbol{X})}{|\boldsymbol{X}-\boldsymbol{Y}|^3}, \end{cases}
$$

where $\beta = \Gamma_1/\Gamma_2$.

Following [39], we formulate the problem for a pair of coaxial circular vortex filaments. Suppose that for some $R_{1,0}, R_{2,0} > 0$ and $z_{1,0}, z_{2,0} \in \mathbf{R}$, the initial filaments $\boldsymbol{X}_0$ and $\boldsymbol{Y}_0$

are parametrized by $\xi \in [0, 2\pi)$ as follows.

$$\boldsymbol{X}_0(\xi) = {}^t(R_{1,0}\cos(\xi), R_{1,0}\sin(\xi), z_{1,0}), \quad \boldsymbol{Y}_0(\xi) = {}^t(R_{2,0}\cos(\xi), R_{2,0}\sin(\xi), z_{2,0}),$$

where we assume that $(R_{1,0} - R_{2,0})^2 + (z_{1,0} - z_{2,0})^2 > 0$, which means that the two circles are not overlapping. Now, we make the ansatz

$$\boldsymbol{X}(\xi, t) = {}^t(R_1(t)\cos(\xi), R_1(t)\sin(\xi), z_1(t)), \quad \boldsymbol{Y}(\xi, t) = {}^t(R_2(t)\cos(\xi), R_2(t)\sin(\xi), z_2(t)),$$

for the solution and substitute it into (2.2). After some calculations, we arrive at the following initial value problem for a system of ordinary differential equations.

$$\begin{cases} \dot{R}_1 = -\dfrac{\alpha R_2(z_1 - z_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ \dot{z}_1 = \dfrac{\beta}{R_1} + \dfrac{\alpha R_2(R_1 - R_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ \dot{R}_2 = \dfrac{\alpha\beta R_1(z_1 - z_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ \dot{z}_2 = \dfrac{1}{R_2} - \dfrac{\alpha\beta R_1(R_1 - R_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ (R_1(0), z_1(0), R_2(0), z_2(0)) = (R_{1,0}, z_{1,0}, R_{2,0}, z_{2,0}). \end{cases} \tag{2.3}$$

Here, a dot over a variable denotes the derivative with respect to t. The analysis in [39] shows that head-on collision can only occur when the circulation of the filaments have opposite signs, i.e. when $\beta < 0$. To simplify the notation, we take $\gamma = -\beta$ to rewrite (2.3) to obtain

$$\begin{cases} \dot{R}_1 = -\dfrac{\alpha R_2(z_1 - z_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ \dot{z}_1 = -\dfrac{\gamma}{R_1} + \dfrac{\alpha R_2(R_1 - R_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ \dot{R}_2 = -\dfrac{\alpha\gamma R_1(z_1 - z_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ \dot{z}_2 = \dfrac{1}{R_2} + \dfrac{\alpha\gamma R_1(R_1 - R_2)}{\left((R_1 - R_2)^2 + (z_1 - z_2)^2\right)^{3/2}}, \\ (R_1(0), z_1(0), R_2(0), z_2(0)) = (R_{1,0}, z_{1,0}, R_{2,0}, z_{2,0}). \end{cases} \tag{2.4}$$

We can further assume without loss of generality that $\gamma \geq 1$ since the case $\gamma < 1$ is reduced to the case $\gamma \geq 1$ by renaming the filaments. By direct calculation, we see that

$\gamma R_1^2 - R_2^2$ is conserved throughout the motion. This means that (R_1, R_2) lies on the set defined by $\gamma R_1^2 - R_2^2 = d$, where $d = \gamma R_{1,0}^2 - R_{2,0}^2$. Further analysis shows that head-on collision can only occur when $d = 0$.

When $d = 0$, we see that $R_2 = \gamma^{1/2} R_1$, and setting

$$\theta(t) := \log(R_1(t)) \quad \text{and} \quad W(t) := z_1(t) - z_2(t),$$

problem (2.4) reads

$$\begin{cases} \dot{\theta} = -\dfrac{\alpha\gamma^{1/2} W}{\big((\gamma^{1/2}-1)^2 e^{2\theta} + W^2\big)^{3/2}} =: F_1(\theta, W), \\ \dot{W} = -(\gamma + \dfrac{1}{\gamma^{1/2}})e^{-\theta} + \dfrac{\alpha\gamma^{1/2}(\gamma^{1/2}-1)^2 e^{2\theta}}{\big((\gamma^{1/2}-1)^2 e^{2\theta} + W^2\big)^{3/2}} =: F_2(\theta, W), \end{cases} \tag{2.5}$$

with initial data given by $(\theta(0), W(0)) = (\theta_0, W_0)$ with $\theta_0 = \log(R_{1,0})$ and $W_0 = z_{1,0} - z_{2,0}$. In order to describe the behavior of the two filaments, it is sufficient to consider the behavior of the solutions to system (2.5). From here, we analyze system (2.5) as a two-dimensional dynamical system. System (2.5) is of Hamiltonian form and the Hamiltonian $\mathcal{H}(\theta, W)$ is given by

$$\mathcal{H}(\theta, W) = -(\gamma + \frac{1}{\gamma^{1/2}})e^{-\theta} + \frac{\alpha\gamma^{1/2}}{\big((\gamma^{1/2}-1)^2 e^{2\theta} + W^2\big)^{1/2}}. \tag{2.6}$$

The phase space and the possible motion patterns vary depending on the value of γ. Setting $H_0 := \mathcal{H}(\theta_0, W_0)$, the conservation of the Hamiltonian yields

$$H_0 = -(\gamma + \frac{1}{\gamma^{1/2}})e^{-\theta} + \frac{\alpha\gamma^{1/2}}{\big((\gamma^{1/2}-1)^2 e^{2\theta} + W^2\big)^{1/2}},$$

which can be used to rewrite system (2.5) as follows.

$$\begin{cases} \dot{\theta} = -\dfrac{1}{\alpha^2\gamma}\Big\{H_0 + (\gamma + \dfrac{1}{\gamma^{1/2}})e^{-\theta}\Big\}^2 \Big\{\alpha^2\gamma - (\gamma^{1/2}-1)^2 e^{2\theta}\big[H_0 + (\gamma + \dfrac{1}{\gamma^{1/2}})e^{-\theta}\big]^2\Big\}^{1/2}, \\ \dot{W} = -(\gamma + \dfrac{1}{\gamma^{1/2}})e^{-\theta} + \dfrac{(\gamma^{1/2}-1)^2}{\alpha^2\gamma}\Big\{H_0 + (\gamma + \dfrac{1}{\gamma^{1/2}})e^{-\theta}\Big\}^3 e^{2\theta}. \end{cases} \tag{2.7}$$

In particular, we utilized the relation

$$W = \left\{\frac{\alpha^2\gamma}{\big[H_0 + (\gamma + \frac{1}{\gamma^{1/2}})e^{-\theta}\big]^2} - (\gamma^{1/2}-1)^2 e^{2\theta}\right\}^{1/2}. \tag{2.8}$$

The above relation and the alternate form of system (2.5) will be used extensively throughout the paper. In order to state our main theorem, we give two definitions for the types of solution we consider.

Definition 2.1 (Head-on collision and asymmetric collision)
For a finite-time solution (θ, W) of system (2.5), we define the following two types of solutions. In what follows, $T_{max} \in (0, \infty)$ denotes the maximum existence time of the solution in consideration.

(i) *For $\gamma = 1$, we call a finite-time solution $(\theta, W) \in C^1([0, T_{max})) \times C^1([0, T_{max}))$ of system (2.5) with initial data (θ_0, W_0) satisfying $W_0 \neq 0$ a solution corresponding to head-on collision if $W(t) \to 0$ monotonically as $t \to T_{max}$.*

(ii) *For $\gamma > 1$, we call a finite-time solution $(\theta, W) \in C^1([0, T_{max})) \times C^1([0, T_{max}))$ of system (2.5) with initial data (θ_0, W_0) satisfying $W_0 \neq 0$ a solution corresponding to asymmetric collision if $W(t) \to 0$ monotonically as $t \to T_{max}$.*

We will use the terminology "colliding solution" to refer to either or both types of solutions, depending on the context.

The behavior of the solution in the above two definitions are the same. The only difference is the value of γ. We made this distinction because the term "head-on collision" seems to be mostly used when two identical rings collide. The term "asymmetric collision" was adopted from [11].

We state our main theorem.

Theorem 2.2 *For any $\alpha \in (0, 1)$, there exists a unique $\gamma_* \in (1, \infty)$ such that the following four statements hold.*

(i) *When $\gamma = 1$, the phase space is $\mathbf{R}^2 \setminus (\mathbf{R} \times \{0\})$ and the following two statements are equivalent.*

- **(a)** *The solution of (2.5) with initial data $(\theta_0, W_0) \in \mathbf{R}^2 \setminus (\mathbf{R} \times \{0\})$ is a solution corresponding to head-on collision.*
- **(b)** $W_0 > 0$.

(ii) *When $\gamma \in (1, \gamma_*)$, the phase space is $\mathbf{R}^2$ and there exists $\theta_* \in \mathbf{R}$ such that the following two statements are equivalent.*

- **(a)** *The solution of (2.5) with initial data $(\theta_0, W_0) \in \mathbf{R}^2$ corresponds to asymmetric collision.*
- **(b)** *$W_0 > 0$ and one of the following holds.*
  - **(c)** $\mathcal{H}(\theta_0, W_0) \leq 0$.
  - **(d)** *$\mathcal{H}(\theta_0, W_0) > 0$ and $\theta_0 \leq \theta_*$.*

(iii) *When $\gamma = \gamma_*$, the phase space is $\mathbf{R}^2$ and the following two statements are equivalent.*

- **(a)** *The solution of (2.5) with initial data $(\theta_0, W_0) \in \mathbf{R}^2$ corresponds to asymmetric collision.*

(b) $W_0 > 0$.

(iv) *When $\gamma \in (\gamma_*, \infty)$, the phase space is $\mathbf{R}^2$ and none of the solutions correspond to asymmetric collision. More precisely, for any initial data $(\theta_0, W_0) \in \mathbf{R}^2$, there exists a unique time-global solution (θ, W) to problem* (2.5) *with the following properties.*

(a) $(\theta, W) \in C^1\big((0,\infty)\big) \times C^1\big((0,\infty)\big)$

(b) *W is monotonically decreasing and $W(t) \to -\infty$ as $t \to \infty$.*

Theorem 2.2 gives a necessary and sufficient condition for head-on collision and asymmetric collision to occur. Note that the solution described in (iv) doesn't correspond to asymmetric collision since it is a time-global solution, even though W could monotonically decrease to zero at some finite time.

Remark 2.3 (Note on the assumption for α in Theorem 2.2)
The parameter α is introduced when we derived the model system (2.1) *in* [39]. *α is explicitly given by*

$$\alpha = \frac{2\delta}{\log(\frac{L}{\varepsilon})},$$

where $L > 0$ is a cut-off parameter and $\varepsilon > 0$ and $\delta > 0$ are small parameters introduced in the localized induction approximation. Hence, although the choice of the upper bound on α in Theorem 2.2 *is technical, it is natural to assume that $\alpha > 0$ is small.*

Remark 2.4 (Note on the Threshold γ_*)
Theorem 2.2 *shows that γ_* can be interpreted as a threshold for γ. When $\gamma \in [1, \gamma_*]$, collision occurs, and when $\gamma \in (\gamma_*, \infty)$, collision doesn't occur. As Theorem* 2.2 *states, the threshold γ_* depends on α. In particular, $\gamma_* \to 1$ as $\alpha \to 0$, and since it is natural to assume that α is small, it is expected that γ_* is close to one. This may be one of the reasons that the existence of the threshold γ_* has not been considered in previous works, since it would be hard to confirm such threshold through experiments or numerical simulations.*

In principle, it is possible to obtain γ_* analytically. This will be apparent from the proof of Theorem 2.2 *where γ_* is obtained as the root of a fourth-order polynomial. We will not do this since it is not needed for the analysis carried out in this paper, and also because the explicit expression for γ_* is not practical for application.*

3 Proof of Theorem 2.2

We first note that since $F_1(\theta, W)$ and $F_2(\theta, W)$ are smooth with respect to θ and W, the time-local unique solvability of initial value problems relevant to Theorem 2.2 is known from general theory of ordinary differential equations. We denote the maximum existence time for a solution (θ, W) by T_{max}.

Next, we investigate the equilibria of system (2.5). This will introduce γ_* as stated in Theorem 2.2.

3.1 Equilibria of System (2.5)

First we determine the equilibria (or the lack there of) of system (2.5) with $\gamma > 1$. In this case, the phase space is $\mathbf{R}^2$ and from the form of $F_1(\theta, W)$, we see that any equilibrium must have the form $(\theta, 0)$. Hence, we set $f(\theta) := F_2(\theta, 0)$ and investigate the zeroes of f. From direct calculation, we have

$$f(\theta) = \frac{e^{-\theta}}{(\gamma^{1/2} - 1)} \Big\{ - (\gamma + \frac{1}{\gamma^{1/2}})(\gamma^{1/2} - 1) + \alpha\gamma^{1/2} \Big\}.$$

Now we set

$$g(\gamma) := -(\gamma + \frac{1}{\gamma^{1/2}})(\gamma^{1/2} - 1) + \alpha\gamma^{1/2},$$

and further setting $\eta := \gamma^{1/2} > 1$ we have

$$g(\eta^2) = \frac{1}{\eta}\big(- \eta^4 + \eta^3 + \alpha\eta^2 - \eta + 1 \big).$$

After some simple calculus, we see that for any $\alpha \in (0, 1)$, there exists a unique $\eta_* \in (1, \infty)$ such that $g(\eta_*^2) = 0$. Hence, when $\gamma = \gamma_* := \eta_*^2$, $f(\theta) = 0$ for all $\theta \in \mathbf{R}$ and $(\theta, 0)$ is an equilibrium for all $\theta \in \mathbf{R}$. Moreover, when $\gamma \in (1, \gamma_*)$, $f(\theta) > 0$ for all $\theta \in \mathbf{R}$ and when $\gamma \in (\gamma_*, \infty)$, $f(\theta) < 0$ for all $\theta \in \mathbf{R}$. In either cases, there are no equilibria.

When $\gamma = 1$, system (2.5) reduces to

$$\begin{cases} \dot{\theta} = -\dfrac{\alpha W}{|W|^3} \\ \dot{W} = -2e^{-\theta} \end{cases}$$

and the corresponding Hamiltonian reduces to

$$\mathcal{H}(\theta, W) = -2e^{-\theta} + \frac{\alpha}{|W|}.$$

This implies that the phase space is $\mathbf{R}^2 \setminus (\mathbf{R} \times \{0\})$, and there is no equilibrium.

We summarize the results in the following.

Lemma 3.1 *For any $\alpha \in (0, 1)$, there exists a unique $\gamma_* \in (1, \infty)$ such that the following holds.*

(i) *When $\gamma = 1$, the phase space is $\mathbf{R}^2 \setminus (\mathbf{R} \times \{0\})$ and there is no equilibrium.*

(ii) *When $\gamma > 1$, the phase space is $\mathbf{R}^2$ and the following hold.*

- **(a)** *If $\gamma = \gamma_*$, $(\theta, 0)$ for all $\theta \in \mathbf{R}$ are equilibria.*
- **(b)** *If $\gamma \neq \gamma_*$, there is no equilibrium.*

From here, the proof is divided according to the value of γ. We show the case $\gamma = 1$ to communicate the general idea of the proof.

3.2 The Case $\gamma = 1$

In this case, system (2.5) reduces to

$$\begin{cases} \dot{\theta} = -\dfrac{\alpha W}{|W|^3} \\ \dot{W} = -2e^{-\theta} \end{cases} \tag{3.1}$$

and the Hamiltonian is given by

$$\mathcal{H}(\theta, W) = -2e^{-\theta} + \frac{\alpha}{|W|}. \tag{3.2}$$

Particularly, we see that W is monotonically decreasing. Hence, for a solution to correspond to head-on collision, $W_0 > 0$ is necessary.

Conversely, for $W_0 > 0$, we consider the solution of (3.1) with initial data (θ_0, W_0). Since the Hamiltonian is divergent at $W = 0$, the conservation of the Hamiltonian implies that $W(t) > 0$ for all $t \in [0, T_{max})$. Hence, (3.1) and (3.2) is further simplified to

$$\begin{cases} \dot{\theta} = -\dfrac{\alpha}{W^2} \\ \dot{W} = -2e^{-\theta} \end{cases} \tag{3.3}$$

and

$$\mathcal{H}(\theta, W) = -2e^{-\theta} + \frac{\alpha}{W}.$$

The conservation of the Hamiltonian yields

$$-2e^{-\theta} = H_0 - \frac{\alpha}{W},$$

which can be utilized to decouple system (3.3). In particular, we have

$$\dot{W} = H_0 - \frac{\alpha}{W}. \tag{3.4}$$

When $H_0 = 0$, equation (3.4) can be explicitly solved to obtain

$$W(t) = \sqrt{W_0^2 - \frac{\alpha}{2} t}.$$

This shows that the solution is a finite-time solution with $T_{max} = \dfrac{2}{\alpha} W_0^2$, and $W(t) \to 0$ monotonically as $t \to T_{max}$. This proves that the solution corresponds to head-on collision.

When $H_0 \neq 0$, solving equation (3.4) gives the following implicit formula for W.

$$\frac{\alpha}{H_0^2} \log(\alpha - H_0 W) + \frac{W}{H_0} = t + \frac{\alpha}{H_0^2} \log(\alpha - H_0 W_0) + \frac{W_0}{H_0}.$$

Setting $G_1(W) = \frac{\alpha}{H_0^2}\log(\alpha - H_0 W) + \frac{W}{H_0}$, we have

$$G_1(W) = t + G_1(W_0).$$

We see that $G_1(W)$ is monotonically decreasing with respect to W and $G_1(W) \to \frac{\alpha}{H_0^2}\log\alpha < 0$ as $W \to 0$. This implies that $G_1(W_0) < \frac{\alpha}{H_0^2}\log\alpha < 0$. Hence, $T_{max} = \frac{\alpha}{H_0^2}\log\alpha - G_1(W_0)$ and $W(t) \to 0$ monotonically as $t \to T_{max}$, and corresponds to head-on collision.

In either cases, the solution corresponds to head-on collision and this proves (i) of Theorem 2.2.

The general idea of the proof of the other cases are the same. We compare the solution (θ, W) with a solution of an associated differential equation which exists only for finite time. $\square$.

4 Discussions and Concluding Remarks

In this paper, we considered the existence of solutions to system (2.1) which correspond to two coaxial circular vortex filaments colliding. This was done by reducing the problem to system (2.5) and analyzing the behavior of the solution in detail. We make some remarks.

4.1 On the Threshold γ_*

In most of the preceding works related to head-on collision of coaxial vortex rings, rings having circulations with equal absolute value were considered. In our formulation, this corresponds to $\gamma = 1$. The results of this paper suggests that if the absolute value of the circulation of the two rings are close enough, the two rings should collide. In our formulation, this corresponds to $\gamma \in [1, \gamma_*]$. This agrees with the numerical observation made by Inoue, Hattori, and Sasaki in [11].

In [11], they conducted a numerical simulation of the Navier–Stokes equations and investigated the head-on collision of coaxial vortex rings with equal size, but varying initial translational velocity. As is pointed out in [11], Saffman [43] showed that the translational velocity U of a vortex ring can be approximated by

$$U = \frac{\Gamma}{4\pi R_0}\left[\log\left(\frac{8R_0}{R_c}\right) - 0.558\right],$$

where R_0 is the radius of the ring, R_c is the radius of the core, and Γ is the circulation of the ring. Hence, varying the value of U for the two rings while keeping the size of the rings, i.e. R_0 and R_c, the same, is essentially equivalent to varying the value of the circulation of the two rings. In [11], they show numerical results for $M_1/M_2 = 1.0, 1.1, 1.33$, and 2.0 (in their notation, M_1/M_2 corresponds to our γ). When $M_1/M_2 = 1.0$ and 1.1, the two rings exhibit collision, and when $M_1/M_2 = 1.33$ and 2.0, the rings pass through one another.

On the other hand, when $\alpha = 0.2$, the threshold γ_* in our formulation is numerically obtained as $\gamma_* = 1.219$. Hence, Theorem 2.2 can then be interpreted as follows. When

$\gamma \in [1, \gamma_*]$, the two filaments collide, and when $\gamma \in (\gamma_*, \infty)$, the two filaments pass through one another. This agrees with the numerical findings of [11].

4.2 On the Possible Motion Patterns

Gurzhii and Konstantinov [4] numerically investigated a model system obtained in Gurzhii, Konstantinov, and Meleshko [5], which can be seen as a generalization of the Dyson model, and observed the following three types of motion patterns for a pair of coaxial vortex rings with circulations of opposite sign.

Head-on collision of two identical rings having circulations with equal magnitude. This corresponds to the case $\gamma = 1$ in our formulation and they observe head-on collision.

Pass-through of the rings when the radius and magnitude of the circulations are unequal. In this case, it is observed that the ring with the circulation of larger magnitude passes through the other ring. This corresponds to the case $\gamma > \gamma_*$ in our formulation and the motion patterns agree with our results.

Mutual capture. When the initial distance of the rings is small and the ratio of the radius and the magnitude of the circulation of the two rings are a specific value, the two rings become mutually captured and travel as a whole in one direction. This motion pattern corresponds to the equilibria in the case $\gamma = \gamma_*$ in our formulation. The observation by Gurzhii and Konstantinov [4] that this type of motion only occurs when the initial distance of the rings is small and the ratio of the radius and circulation must be a specific value agrees with the characteristics of the equilibria obtained in Lemma 3.1.

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Department of Mathematics
Faculty of Science and Technology, Tokyo University of Science
2641 Yamazaki, Noda, Chiba 278-8510, Japan
E-mail: aiki_masashi@ma.noda.tus.ac.jp