A current-valued solution of the Euler equation and its application

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1 Euler Flows and Point Vortex Dynamics

In addition to the Euler flow, there is another model that describes the motion of an incompressible and inviscid fluid in the plane, called the point vortex dynamics, which is derived formally from the Euler flow. The Euler flow is defined as a solution (v_t, p_t) of the 2D Euler equation:

$$\partial_t v_t + (v_t \cdot \nabla) v_t = -\operatorname{grad} p_t, \quad \operatorname{div} v_t = 0.$$

Here, the vorticity $\omega_t = \operatorname{curl} v_t$ satisfies the vorticity equation:

$$\partial_t \omega_t + (-\mathcal{J}\operatorname{grad}\langle G_H, \omega_t\rangle \cdot \nabla)\omega_t = 0,$$

where $\mathcal{J}$ is the symplectic matrix and G_H is the Green function for the Laplacian. Besides, the fluid velocity v_t and pressure p_t are written in terms of the vorticity:

$$v_t = -\mathcal{J}\operatorname{grad}\langle G_H, \omega_t\rangle, \quad p_t = \langle G_H, \operatorname{div}(v_t \cdot \nabla) v_t\rangle. \tag{1}$$

In particular, the explicit presentation of the fluid velocity by the vorticity in equation (1) is called the Biot-Savart law. Now let us consider the case where vorticity Ω_t is given by a linear combination of delta functions over real coefficients $\Omega_t = \sum_{n=1}^{N} \Gamma_n \delta_{q_n(t)}$. In formulating the evolution of the vorticity in time, Helmholtz proposed that $q_n(t)$ be determined by the following ordinary differential equation [8].

$$\begin{aligned} \dot{q}_n &= \lim_{q \to q_n} \left[-\mathcal{J}\operatorname{grad}\langle G_H, \Omega_t\rangle + \mathcal{J}\operatorname{grad}\langle G_H, \Gamma_n \delta_{q_n(t)}\rangle(q) \right] \qquad (2) \\ &= -\mathcal{J}\operatorname{grad} \sum_{\substack{m=1 \\ m \neq n}}^{N} \Gamma_m G_H(q_n, q_m) \equiv v_n(q_n). \end{aligned}$$

The equation (2) is called the point vortex equation, and its solution is called point vortex dynamics. By substituting ω_t for Ω_t in the sense of distributions in (1), the velocity $V_t \in \mathfrak{X}^\infty(\mathbb{R}^2 \setminus \{q_n(t)\})$ and the pressure $\Pi_t \in C^\infty(\mathbb{R}^2 \setminus \{q_n(t)\})$ for point vortex dynamics are formally derived.

Since point vortex dynamics is a finite-dimensional dynamical system, it is easier to deal with mathematically than the Euler flow. For this reason, point vortex dynamics is of importance in applications, since it is utilized as simple models for fluid phenomena with localized vortex structures. Meanwhile, since point vortex dynamics is derived formally from the Euler flow, it still remains open whether the insight obtained by using point vortex dynamics is also applicable to the Euler flow. Thus, to solve this problem, we need to establish that the point vortex dynamics is an Euler flow in a mathematically appropriate sense. In other words, the problem is to set up a solution space for the Euler equation that contains the pair (V_t, Π_t) of the fluid velocity and the pressure for the point vortex dynamics. For the pair (V_t, Π_t) to be an Euler flow, the time evolution of the vorticity Ω_t is to satisfy the vorticity equation. However, due to the singularity of Ω_t and the nonlinearity of the vorticity equation, the problem of mathematical justification is considered to be a difficult one in the analysis of the 2D Euler equation [4, 5, 6]. Although there have been many studies to date on the indirect characterization of point vortex dynamics by approximate sequences of weak solutions to the Euler equation, the problem remains open in terms of constructing a space of solutions that directly includes point vortex dynamics.

Since point vortex dynamics is mathematically simpler than Euler flows, various generalizations have been proposed for certain applications. Among them are the generalization to surfaces [2], and the point vortex dynamics in a background field, which is a dynamical system obtained by adding the velocity field of an Euler flow, called a *background field*, to the point vortex dynamics, is utilized to get an insight into two-dimensional turbulence and is highly consistent with a laboratory experiment [10]. For a given Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ $(r \geq 1)$, the point vortex dynamics in the background field is a solution to the point vortex equation in the background field:

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \dots N. \tag{3}$$

where $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ is a given parameter. As in the case with no background field, the fluid velocity $V_t \in \mathfrak{X}^r(\mathbb{R}^2 \setminus \{q_n(t)\})$ and the pressure $\Pi_t \in C^r(\mathbb{R}^2 \setminus \{q_n(t)\})$ are now defined:

$$V_t(q) = X_t(q) - \mathcal{J}\mathrm{grad} \sum_{n=1}^{N} \Gamma_n G_H(q, q_n(t)), \quad \Pi_t = \langle G_H, \mathrm{div}(V_t \cdot \nabla) V_t \rangle.$$

As for the ordinary point vortex dynamics with no background field, the problem is raised as to whether the point vortex dynamics in a background field is an Euler flow.

The purpose of this paper is to justify the point vortex dynamics in the background field as an Euler flow. The problem originated in the analysis of the 2D Euler equation, but it is solved by a geometric method in this paper. In particular, we note that the case with no background field is regarded as a special case. We will provide a weak formulation of the Euler equation in the space of de Rham currents, and show that the point vortex dynamics in the background field is a current-valued solution of the Euler equation. The de Rham current is a continuous linear functional defined on the space of differential forms, which is used mainly in geometric analysis and geometric measure theory. Roughly speaking, currents are locally differential forms over distribution coefficients, which are characterized as a weak extension of differential forms. We then rewrite the notions involved in the analysis of the Euler equation, such as the divergence operator and the curl operator, in terms of differential forms.

The weak formulation obtained in this procedure is applicable not only to the plane but also to curved surfaces in general. Taking into account that point vortex dynamics in a rotational field on a sphere has been adopted as a mathematical model of geophysical fluids to investigate the effect of Coriolis force on non-viscous fluid motion [7], It is of significance for applications to justify the point vortex dynamics in the background field as an Euler flow for curved surfaces. For the generalization to curved surfaces, however, we refer to the original paper [9]. In this paper, we focus on the planar case in order to provide the essence of what is discussed to the reader who is not familiar with the notions of differential geometry. Therefore, we note that the main results presented in this paper hold true for general curved surfaces as well.

This paper is organized in the following sections. In section 2, weak formulations by currents of the notions involved in the analysis of the Euler equation are presented. Since the weak formulations by currents are obtained as weak extensions of differential forms, these notions are rewritten in terms of differential forms in Section 2.1, and then their weak formulations by currents are carried out in Section 2.2. In Section 3, current-valued solutions of the Euler equation and its regular-singular decomposition are formulated. The main result is stated in Section 4, and its applications are discussed in Section 5.

2 Weak Formulation of Vector Calculus

2.1 Reformulations in Differential Forms

The curl operator curl : $X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2) \to \operatorname{curl} X \in C^{r-1}(\mathbb{R}^2)$ and the divergence operator div : $X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2) \to \operatorname{div} X \in C^{r-1}(\mathbb{R}^2)$ are defined as follows:

$$\operatorname{curl} X = \partial_1 X^2 - \partial_2 X^1, \quad \operatorname{div} X = \partial_1 X^1 + \partial_2 X^2.$$

The value $\omega = \operatorname{curl} X \in C^{r-1}(\mathbb{R}^2)$ is referred to as the vorticity, and a vector field with $\operatorname{div} X = 0$ is said to be incompressible. A vector field $X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2)$ is called a gradient vector field if there exists a function $\phi \in C^{r+1}(\mathbb{R}^2)$ such that $X = \operatorname{grad} \phi = (\partial_1 \phi, \partial_2 \phi)$. A vector field $X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2)$ is called a Hamiltonian vector field if there exists a function $\psi \in C^{r+1}(\mathbb{R}^2)$ such that $X = -\mathcal{J} \operatorname{grad} \psi = (\partial_2 \psi, -\partial_1 \psi)$. where $\mathcal{J}$ is the matrix defined by

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The function ψ is then called the Hamiltonian or the stream function. Since every incompressible vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is a Hamiltonian vector field $X = -\mathcal{J} \operatorname{grad} \psi$, the vorticity $\omega = \operatorname{curl} X$ and the stream function ψ satisfy

$$\omega = \operatorname{curl}(-\mathcal{J} \operatorname{grad} \psi) = -\triangle \psi.$$

Accordingly, we obtain the Biot-Savart law:

$$X = -\mathcal{J} \operatorname{grad} \langle G_H, \omega \rangle,$$

where $\langle G_H, \omega\rangle$ is the convolution of the Green's function $G_H \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta)$ for $-\triangle$ and the vorticity ω.

What is essential in rewriting those facts on vector calculus in terms of differential forms is an isomorphism between vector fields and differential 1 forms, which is called the musical isomorphism. A vector field $X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2)$ is denoted by $X = X^1\partial_1 + X^2\partial_2$ using a basis of the tangent space (∂_1, ∂_2). For this, by replacing the basis of the tangent space with the basis of the cotangent space $(\mathrm{d}\,x^1, \mathrm{d}\,x^2)$, we obtain a 1 form $X^\flat = X^1\,\mathrm{d}\,x^1 + X^2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2)$. The operator $\flat : X \in \mathfrak{X}^r(\mathbb{R}^2) \to X^\flat \in \Omega^1_{[r]}(\mathbb{R}^2)$ is called the flat operator, and the corresponding 1 form is called the velocity form. Conversely, for each 1-form $\alpha = \alpha_1\,\mathrm{d}\,x^1 + \alpha_2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2)$, a vector field is defined by $\alpha_\sharp = \alpha_1\partial_1 + \alpha_2\partial_2 \in \mathfrak{X}^r(\mathbb{R}^2)$. The operator $\sharp : \alpha \in \Omega^1_{[r]}(\mathbb{R}^2) \to \alpha_\sharp \in \mathfrak{X}^r(\mathbb{R}^2)$ is called the sharp operator, and the corresponding vector field is called the dual vector field. These two operators are isomorphisms between the space of vector fields $\mathfrak{X}^r(\mathbb{R}^2)$ and the space of 1-forms $\Omega^1_{[r]}(\mathbb{R}^2)$, which together are called musical isomorphisms.

There are two operators that act on differential forms: the (exterior) differential operator d and the Hodge-$*$ operator. For each p-form $\alpha \in \Omega^p_{[r]}(\mathbb{R}^2)(p = 0, 1, 2)$, $\mathrm{d}\,\alpha \in \Omega^{p+1}_{[r-1]}(\mathbb{R}^2)$ and $*\alpha \in \Omega^{2-p}_{[r]}(\mathbb{R}^2)$ satisfy respectively

$$\mathrm{d}\,\alpha = \begin{cases} \partial_1\alpha_0\,\mathrm{d}\,x^1 + \partial_2\alpha_0\,\mathrm{d}\,x^2, & \text{if } \alpha = \alpha_0 \in \Omega^0_{[r]}(\mathbb{R}^2), \\ (\partial_1\alpha_2 - \partial_2\alpha_1)\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2, & \text{if } \alpha = \alpha_1\,\mathrm{d}\,x^1 + \alpha_2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2), \\ 0, & \text{if } \alpha = \alpha_{12}\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 \in \Omega^2_{[r]}(\mathbb{R}^2), \end{cases}$$

$$*\alpha = \begin{cases} \alpha_0\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2, & \text{if } \alpha = \alpha_0 \in \Omega^0_{[r]}(\mathbb{R}^2), \\ -\alpha_2\,\mathrm{d}\,x^1 + \alpha_1\,\mathrm{d}\,x^2, & \text{if } \alpha = \alpha_1\,\mathrm{d}\,x^1 + \alpha_2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2), \\ \alpha_{12}, & \text{if } \alpha = \alpha_{12}\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 \in \Omega^2_{[r]}(\mathbb{R}^2). \end{cases}$$

Moreover, by using the differential operator and the Hodge-$*$ operator, the codifferential operator $\delta = *\,\mathrm{d}\,* : \Omega^p_{[r]}(\mathbb{R}^2) \to \Omega^{p-1}_{[r-1]}(\mathbb{R}^2)$ and the Hodge Laplacian $\triangle = \delta\,\mathrm{d} + \mathrm{d}\,\delta : \Omega^p_{[r]}(\mathbb{R}^2) \to \Omega^p_{[r-2]}(\mathbb{R}^2)$ are defined. The inner product (α, β) of 1 forms is defined by $(\alpha, \beta) = \alpha_1\beta_1 + \alpha_2\beta_2$.

Using the above notion of differential forms, the curl operator and the divergence operator are thus rewritten as

$$\operatorname{curl} X = *\,\mathrm{d}\,X^\flat, \quad \operatorname{div} X = \delta X^\flat$$

since

$$\begin{aligned} *\,\mathrm{d}\,X^\flat &= *\,\mathrm{d}(X^1\,\mathrm{d}\,x^1 + X^2\,\mathrm{d}\,x^2) = *(\partial_1X^2 - \partial_2X^1)\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 \\ &= \partial_1X^2 - \partial_2X^1, \\ \delta X^\flat &= *\,\mathrm{d}\,*(X^1\,\mathrm{d}\,x^1 + X^2\,\mathrm{d}\,x^2) = *\,\mathrm{d}(-X^2\,\mathrm{d}\,x^1 + X^1\,\mathrm{d}\,x^2) \\ &= *(\partial_1X^1 + \partial_2X^2)\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 = \partial_1X^1 + \partial_2X^2. \end{aligned}$$

Similarly, since

$$\mathrm{d}\,\phi = \partial_1\phi\,\mathrm{d}\,x^1 + \partial_2\phi\,\mathrm{d}\,x^2, \quad *\,\mathrm{d}\,\psi = -\partial_2\psi\,\mathrm{d}\,x^1 + \partial_1\psi\,\mathrm{d}\,x^2,$$

we obtain

$$\operatorname{grad}\phi = (\mathrm{d}\,\phi)_\sharp, \quad \mathcal{J}\operatorname{grad}\psi = (*\,\mathrm{d}\,\psi)_\sharp.$$

The Euler equation is now presented as follows:

$$\partial_t v^\flat + (*\,\mathrm{d}\,v^\flat) * v^\flat + \mathrm{d}\,|v^\flat|^2/2 = -\,\mathrm{d}\,p. \tag{4}$$

It is easy to see that the equation (4) is equivalent to the Euler equation since

$$\begin{aligned} \partial_t v^\flat &= \partial_t v^1 \,\mathrm{d}\,x^1 + \partial_t v^2 \,\mathrm{d}\,x^2, \\ (*\,\mathrm{d}\,v^\flat) * v^\flat &= (-v^2\partial_1 v^2 + v^2\partial_2 v^1)\,\mathrm{d}\,x^1 + (v^1\partial_1 v^2 - v^1\partial_2 v^1)\,\mathrm{d}\,x^2, \\ \mathrm{d}\,|v|^2/2 &= (v^1\partial_1 v^1 + v^2\partial_1 v^2)\,\mathrm{d}\,x^1 + (v^1\partial_2 v^1 + v^2\partial_2 v^2)\,\mathrm{d}\,x^2, \\ \mathrm{d}\,p &= \partial_1 p\,\mathrm{d}\,x^1 + \partial_2 p\,\mathrm{d}\,x^2. \end{aligned}$$

Although the equation (4) is introduced heuristically in this paper, the derivation of the equation (4) from the Euler equation is based on an idea in differential geometry, the symmetric-antisymmetric decomposition of the Levi-Civita connection [9]. It is still open whether a similar presentation can be obtained for the three-dimensional Euler equation.

2.2 Weak Formulation by Currents

We begin with a brief review of the basics of currents. For details, we refer the reader to [1]. A p-current ($p = 0, 1, 2$) is defined as a continuous linear functional on the space $\mathcal{D}^{2-p}(\mathbb{R}^2)$ of $(2-p)$-forms with compact support. The space of all p-currents is denoted by $\mathcal{D}'_p(\mathbb{R}^2)$. Then, a $(2-p)$-form with compact support is called a test form. Let $T[\phi]$ denote the coupling of the p-current T and the test form ϕ.

For example, differential forms and distributions are types of currents. For each p-form $\alpha \in \Omega^p_{[r]}(\mathbb{R}^2)$, a p-current $I(\alpha) \in \mathcal{D}'_p(\mathbb{R}^2)$ is defined as

$$I(\alpha)[\phi] = \int_{\mathbb{R}^2} \alpha \wedge \phi.$$

for each test form $\phi \in \mathcal{D}^{2-p}(\mathbb{R}^2)$. Since the topology of the space of currents is weak-$*$ topology, the inclusion map $I : \Omega^p_{[r]}(\mathbb{R}^2) \to \mathcal{D}'_p(\mathbb{R}^2)$ is now an embedding map. Therefore, currents are characterized as a weak extension of differential forms. In addition, for each distribution $f : \varphi \in \mathcal{D}^0(\mathbb{R}^2) \to f(\varphi) \in \mathbb{R}$, it is identified with a current $T_f \in \mathcal{D}'_0(\mathbb{R}^2)$ that is defined by $T_f[\phi] = f(*\phi)$. In particular, the delta function centered at $p \in \mathbb{R}^2$ corresponds to a current $\delta_p \in \mathcal{D}'_0(\mathbb{R}^2)$ with $\delta_p[\phi] = *\phi(p)$, called the *delta current*.

In the same manner as distributions, calculations on currents are defined via calculations on test forms. The differential operator $\mathrm{d} : T \in \mathcal{D}'_p(\mathbb{R}^2) \to \mathrm{d}\,T \in \mathcal{D}'_{p+1}(\mathbb{R}^2)$ for currents is defined as $\mathrm{d}\,T[\phi] = (-1)^{p+1}T[\mathrm{d}\,\phi]$, and the Hodge-$*$ operator $* : T \in \mathcal{D}'_p(\mathbb{R}^2) \to *T \in \mathcal{D}'_p(\mathbb{R}^2)$ is defined as $*T[\phi] = (-1)^{p(2-p)}T[*\phi]$. As for the codifferential operator and the Hodge Laplacian for currents, it is immediately obtained from the composition of the differential operator and the Hodge-$*$ operator.

Let $\Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ denote the space of all p-forms on a nonempty open subset $U \subset \mathbb{R}^2$, which are called local p-forms. In the same manner as $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, the topology of $\Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ can be defined. For each local p-form $\alpha \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$, there exists a maximal open subset U such that $\alpha \in \Omega^p_{[r]}(U)$. Then, the closed subset $\mathsf{S}^r(\alpha) = \mathbb{R}^2 \setminus U$ is called the singular support of α. A p-current $T \in \mathcal{D}'_p(\mathbb{R}^2)$ is said to be of class C^r, if there exists a local p-form $\alpha_T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ such that for every

$\phi \in \mathcal{D}^{2-p}(M \setminus \mathsf{S}^r(\alpha_T))$, $T[\phi] = I(\alpha_T)[\phi]$. Let $\mathcal{D}'^r_p(\mathbb{R}^2)$ denote the space of all C^r p-currents. For each $T \in \mathcal{D}'^r_p(\mathbb{R}^2)$, the subset $\mathsf{S}^r(T) = \mathsf{S}^r(\alpha_T)$ and the local p-form α_T is called the singular support of T and the density of T. Owing to the fundamental lemma of calculus of variation, the density is uniquely determined. Thus, the map $K : T \in \mathcal{D}'^r_p(\mathbb{R}^2) \to K(T) = \alpha_T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ is well-defined and called the derivative. For example, for each $p \in M$, the delta current δ_p is of class C^∞ since $\delta_p = I(0)$ in $M \setminus \{p\}$. We thus obtain $\mathsf{S}^\infty(\delta_p) = \{p\}$ and $K(\delta_0) = 0$. In this paper, all currents are of class C^r and the singular support consists of a finite set of points.

We define the principal value p.v. : $T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2) \to \mathrm{p.v.}\, T \in \mathcal{D}'_p(\mathbb{R}^2)$ by

$$\mathrm{p.v.}\, T[\phi] = \lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(\mathsf{S}(T))} T \wedge \phi$$

for each $\phi \in \mathcal{D}^{2-p}(\mathbb{R}^2)$ if the limit exists. The domain of p.v., say Dom(p.v.), is defined by the space of p-currents in which the limit exists for every $\phi \in \mathcal{D}^{2-p}(\mathbb{R}^2)$.

Let us introduce the operator $\mathfrak{L}$ by $\mathfrak{L} = \mathrm{d}\,\mathrm{p.v.} : T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2) \to \mathfrak{L}T \in \mathcal{D}'_{p+1}(\mathbb{R}^2)$, which is called the *localizing operator*. For each $\phi \in \mathcal{D}^{p+1}(\mathbb{R}^2)$, we have

$$\mathfrak{L}T[\phi] = \mathrm{d}\,\mathrm{p.v.}\, T[\phi] = (-1)^{p+1}\, \mathrm{p.v.}\, T[\mathrm{d}\,\phi].$$

The domain of $\mathfrak{L}$, Dom($\mathfrak{L}$), is the space of p-currents T in which $\mathrm{p.v.}\, T[\mathrm{d}\,\phi]$ is well-defined for every $\phi \in \mathcal{D}^{1-p}(\mathbb{R}^2)$. If $T \in \mathrm{Dom}(\mathfrak{L})$ satisfies $\mathrm{d}\, T = 0$ in $M \setminus \mathsf{S}(T)$, then we obtain

$$\begin{aligned}\mathfrak{L}T[\phi] &= (-1)^{p+1}\, \mathrm{p.v.}\, T[\mathrm{d}\,\phi] = (-1)^{p+1} \lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(\mathsf{S}(T))} T \wedge \mathrm{d}\,\phi \\ &= -\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(\mathsf{S}(T))} T \wedge \phi,\end{aligned}$$

since $(-1)^{p+1} T \wedge \mathrm{d}\,\phi = -\mathrm{d}(T \wedge \phi) + \mathrm{d}\, T \wedge \phi$. Hence, $\mathfrak{L}T$ is determined by the asymptotic behavior of T near the singular support $\mathsf{S}(T)$. The name of *localizing* operator is named after this property.

In light of the above basics for currents, let us apply it to a weak formulation of vector analysis in terms of currents. What remains to be done is basically to replace the part of the velocity form with a 1-current. The divergence of a 1 current $\alpha \in \mathcal{D}'_1(\mathbb{R}^2)$ is defined by $\delta\alpha \in \mathcal{D}'_0(\mathbb{R}^2)$, based on the presentation of the divergence operator by the velocity form: $\operatorname{div} X = \delta X^\flat$. But for the curl operator $\operatorname{curl} X = *\,\mathrm{d}\, X^\flat$, the principal value is used so that the vorticity can be calculated from the density of the current. Namely, the vorticity of a local 1 form $\alpha \in \mathrm{Dom}(\mathfrak{L})$ is defined by $*\,\mathfrak{L}\,\alpha \in \mathcal{D}'_0(\mathbb{R}^2)$.

Let us remember that $\Omega^1_{[r]}(\mathbb{R}^2)$ is identified with $\mathfrak{X}^r(\mathbb{R}^2)$ by the musical isomorphism $\sharp : \alpha \in \Omega^1_{[r]}(\mathbb{R}^2) \to \alpha_\sharp \in \mathfrak{X}^r(\mathbb{R}^2)$, satisfying $\alpha[X] = g(\alpha_\sharp, X)$ for all $X \in \mathfrak{X}^r(\mathbb{R}^2)$. Every C^r 1-current T generates a vector field allowing singularities in $\mathsf{S}(T)$. For each $T \in \mathcal{D}'^r_1(\mathbb{R}^2)$, we define $T_\sharp \in \mathfrak{X}^r(M \setminus \mathsf{S}^r(T))$ by $T_\sharp = K(T)_\sharp$. As an example, for each $\psi \in \mathcal{D}'^r_0(\mathbb{R}^2)$, we define $\mathcal{J}\operatorname{grad}\psi \in \mathfrak{X}^{r-1}(M \setminus \mathsf{S}^{r-1}(\mathrm{d}\,\psi))$ by $\mathcal{J}\operatorname{grad}\psi = K(*\,\mathrm{d}\,\psi)_\sharp$. The vector field $\mathcal{J}\operatorname{grad}\psi$ stands for the Hamiltonian vector field induced from the Hamiltonian ψ with singularities in $\mathsf{S}(\psi)$. We will use this notion when we take a vector field generated by point vortices.

Convention. *In what follows, for a given $T \in \mathcal{D}'^r_p(\mathbb{R}^2)$, we abbreviate $\mathsf{S}^r(T)$ to $\mathsf{S}(T)$. Moreover, we denote $K(T)$ briefly by T as long as no confusion arises.*

In particular, for a given $T \in \mathcal{D}'^r_1(\mathbb{R}^2)$, when we write $|T|^2$, it stands for not the multiplication of currents but the multiplication of the local 1-form $|K(T)|^2$. Similarly, $(*\,\mathrm{d}\,T) * T$ means the multiplication of the local 0-form $*\,\mathrm{d}\,K(T)$ and the local 1-form $*K(T)$. This treatment is sensitive when we formulate the nonlinear term in the Euler equation (4) in the sense of currents.

3 Current-Valued Solutions of the Euler Equation

In Section 2.1, we saw that the equation (4) is an equivalent presentation to the Euler equation. Before we replace differential forms in (4) with currents, we need to deal with the nonlinear term carefully in order to avoid multiplication of currents. Based on the fact that multiplication of local p-forms is still valid and a local p-form is converted to a current by taking the principle value, the Euler equation is reformulated for $\alpha_t \in \mathcal{D}'^r_1(\mathbb{R}^2)$ and $p_t \in \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ as follows.

$$\partial_t\, \mathrm{p.v.}\, \alpha_t + \mathrm{p.v.}\{(*\,\mathrm{d}\,\alpha_t) * \alpha_t + \mathrm{d}\,|\alpha_t|^2/2\} = -\,\mathrm{p.v.}\,\mathrm{d}\,p_t \quad \text{in } \mathcal{D}'_1(\mathbb{R}^2), \tag{5}$$

if each of terms is contained in Dom(p.v.), where we abbreviate $K(\alpha_t)$ to α_t. To obtain a weak formulation of the vorticity equation, let us apply the differential operator to the equation (5), as the vorticity equation is obtained by applying the curl operator to the Euler equation.

Definition 3.1. *A pair of time-dependent currents $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ is called a weak Euler flow, if the following conditions are satisfied:*

1. $\alpha_t \in \mathrm{Dom}(\mathfrak{L})$ *and* $(*\,\mathrm{d}\,\alpha_t) * \alpha_t + \mathrm{d}\,|\alpha_t|^2/2 \in \mathrm{Dom}(\mathfrak{L})$;
2. $\mathrm{d}\,p_t \in \mathrm{Dom}(\mathfrak{L})$;
3. $\partial_t\, \mathfrak{L}\, \alpha_t + \mathfrak{L}\,\{(*\,\mathrm{d}\,\alpha_t) * \alpha_t + \mathrm{d}\,|\alpha_t|^2/2\} = -\,\mathfrak{L}\,\mathrm{d}\,p_t$ *in* $\mathcal{D}'_2(\mathbb{R}^2)$;
4. $\delta\alpha_t = 0$ *in* $\mathcal{D}'_1(\mathbb{R}^2)$;

In particular, we call the third condition the weak Euler equation and α_t the velocity current.

Let us divide each term of the weak Euler equation into a regular part and a singular part, assuming that the regular part is given by a classical solution of the Euler equation. We decompose the velocity current $\alpha_t \in \mathcal{D}'^r_1(\mathbb{R}^2)$ into a regular part X_t and a singular part u_t. Namely, we assume that there exists an Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ and a time-dependent 1-current $u_t \in \mathcal{D}'^r_1(\mathbb{R}^2)$ such that $\alpha_t = X_t + u_t$ is satisfied at each time t. Unless otherwise stated, the subscript t is omitted hereafter. Denoting the vorticity of X by $\omega_X = *\,\mathrm{d}\,X^\flat$, we define the relative vorticity ω of u by $\omega = *\,\mathrm{d}(\alpha - X^\flat) \in \mathcal{D}'_0(\mathbb{R}^2)$. For the corresponding densities, it follows from

$$\begin{aligned}
\partial_t\, \mathfrak{L}\, \alpha - \partial_t\, \mathfrak{L}\, X^\flat &= \partial_t\, \mathfrak{L}\, u \\
\mathfrak{L}(*\,\mathrm{d}\,\alpha) * \alpha - \mathfrak{L}\,\omega_X * X^\flat &= \mathfrak{L}(\omega + \omega_X) * (u + X^\flat) - \mathfrak{L}\,\omega_X * X^\flat \\
&= \mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat\}, \\
\mathfrak{L}\,\mathrm{d}\,|\alpha|^2 - \mathfrak{L}\,\mathrm{d}\,|X|^2 &= \mathfrak{L}\,\mathrm{d}\,g(X^\flat + u, X^\flat + u) - \mathfrak{L}\,\mathrm{d}\,g(X^\flat, X^\flat) \\
&= \mathfrak{L}\,\mathrm{d}\,g(2X^\flat + u, u)
\end{aligned}$$

that, since $(X,P) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ is a classical Euler flow,

$$\partial_t X^\flat + \omega_X * X^\flat + \mathrm{d}\,|X|^2 = -\,\mathrm{d}\,P,$$

the weak Euler equation (Definition 3.1-3) is reduced to

$$\partial_t\,\mathfrak{L}\,u + \mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat + \mathrm{d}(2X^\flat + u, u)/2\} = -\,\mathfrak{L}\,\mathrm{d}(p - P). \tag{6}$$

We examine closely the role of each term in the equation (6). When the relative vorticity ω is given by a delta current $\omega = \delta_0$, for instance, a straightforward calculation of currents shows that $\mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat\} = 0$ is actually valid [9]. Furthermore, if the pressure term vanishes under some assumptions, the equation (6) can be written only in terms of the velocity current. We now consider the situation where the remaining terms for the advection and pressure terms, $\mathfrak{L}\,\mathrm{d}(X^\flat, u)$, $\mathfrak{L}\,\mathrm{d}\,|u|^2/2$ and $\mathfrak{L}\,\mathrm{d}\,p$, cancel each other out. In other words, suppose that there exists a parameter $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ such that

$$p = P + (2\beta_X - 1)(X^\flat, u) + (2\beta_\omega - 1)|u|^2/2.$$

The weak Euler equation then is reduced to the following form:

$$\partial_t\,\mathfrak{L}\,u + \mathfrak{L}\{(\omega_X + \omega) * u + \omega * X^\flat + \mathrm{d}(2\beta_X X^\flat + \beta_\omega u, u)\} = 0.$$

Definition 3.2. *A weak Euler flow $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ is said to be C^r decomposable $(r \geq 1)$, if there exists a classical Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ and $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ such that the following conditions are satisfied for each time t.*

1. $\alpha_t = X_t^\flat + u_t$;
2. $p_t = P_t + (2\beta_X - 1)(X_t^\flat, u_t) + (2\beta_\omega - 1)|u_t|^2/2.$

Then we call X_t a background field of α_t, $\alpha_t - X_t$ a relative velocity current and (β_X, β_ω) a growth rate of p_t.

The C^r-decomposability was proposed based on a theoretical motivation to reduce the pressure term. On the other hand, C^r-decomposability is physically interpreted as a mathematical model in which the singular behavior of pressure is given by the interaction energy density $(X^\flat, u)$ and the kinetic energy density $|u|^2/2$. The parameter $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ is then characterized as the growth rate of the singular behavior of the pressure compared to these energy densities. Later, in Section 5, we will revisit some applications from this perspective.

Let us note that the C^r decomposability of the weak Euler flow guarantees the existence of the decomposition but there is no mention of the uniqueness of the decomposition. Hence, when we study a C^r decomposable weak Euler flow $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$, we need to fix a classical Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ and a parameter $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ such that the velocity field X_t is a background field of α_t and the parameter (β_X, β_ω) is a growth rate of p_t.

4 Main Results

Let us fix $N \in \mathbb{Z}_{\geq 1}$, $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$ and a C^r one-parameter family $\Phi : t \in [0,T] \to \Phi_t \in \mathrm{Diff}^r(Q_N)$ in what follows. Let us denote by $(q_n(t))_{n=1}^N = \Phi_t((q_n(0))_{n=1}^N)$ an orbit of Φ. In what follows, we focus on a time-dependent 0-current $\sum_{n=1}^N \Gamma_n \delta_{q_n(t)} \in \mathcal{D}_0'(\mathbb{R}^2)$ and call it a *singular vorticity of point vortices placed on* $\{q_n(t)\}_{n=1}^N$. First, we show that if the singular part of the vorticity of the C^r-decomposable weak Euler flow is given by the singular vorticity of point vortices, then $q_n(t)$ is a solution of the point vortex equation in the background field (3).

Theorem 4.1. *Let* $(\alpha_t, p_t) \in \mathcal{D}_1'^r(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ *be a* C^r*-decomposable weak Euler flow* $(r \geq 1)$*. Fix a background field* X_t *of* α_t*, a growth rate* (β_X, β_ω) *of* p_t*. Suppose the relative vorticity* ω_t *is a singular vorticity of point vortices placed on* $\{q_n(t)\}_{n=1}^N$*. Then,* $q_n(t)(n = 1, \ldots, N)$ *is a solution of the point vortex equation in the background field* (3)*.*

Conversely, if $q_n(t)$ is a solution of the point vortex equation in a background field (3), then there exists a C^r-decomposable weak Euler flow such that the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$.

Theorem 4.2. *Fix a classical Euler flow* $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ *and* $(\beta_X, \beta_\omega) \in \mathbb{R}^2$*. Let* $\omega_t \in \mathcal{D}_0'(\mathbb{R}^2)$ *be a singular vorticity of point vortices placed on* $\{q_n(t)\}_{n=1}^N$*. Define a time-dependent current* $u_t \in \mathcal{D}_1'^\infty(\mathbb{R}^2)$ *by* $u_t = - * \mathrm{d}\langle G_H, \omega_t\rangle$*. Suppose* $q_n(t)(n = 1, \ldots, N)$ *is a solution of the point vortex equation in the background field* X_t (3)*. Then, the following pair of time-dependent currents* α_t *and* p_t *defines a* C^r*-decomposable weak Euler flow.*

$$\alpha_t = X_t^\flat + u_t \in \mathcal{D}_1'^r(\mathbb{R}^2),$$
$$p_t = P_t + (2\beta_X - 1)(X_t^\flat, u_t) + (2\beta_\omega - 1)|u_t|^2/2 \in \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2).$$

We refer the reader to the paper [9] for the proof of the theorem.

5 Applications

First, Theorem 4.2 shows that the point vortex dynamics in the background field justified as a C^r-decomposable weak Euler flow although it is a mathematical model derived formally from the Euler flow. Next, from Theorem 4.1, we deduce that the parameter (β_X, β_ω) is not only a dynamical parameter in the point vortex equation in the background field, but also a physical parameter in terms of the growth rate of pressure. With these perspectives in mind, we revisit the following examples of point vortex dynamics in the background field.

Twin Vortices in a Linear Shear The motion of twin vortices in a linear shear is governed by point vortex dynamics in a background field where the singular vorticity of point vortices is given by $\delta_{q_1(t)} + \delta_{q_2(t)}$, the background field is the shear $X = (cy, 0)$, and the parameters $(\beta_X, \beta_\omega) = (1, 1)$, respectively. The twin vortex in a linear shear is employed as a simplified model for qualitative understanding of vortex merger [10]. The vortex merger is characterized as a process in which vortices with similarly small scales are influenced by a shear flow generated by the surrounding vortices and combine to form a larger scale vortex. From the qualitative theory of dynamical systems, we see that when $c < 0$, the distance between point vortices becomes smaller than the initial value. Therefore, it is expected that the emergence of vortex merger is determined by the orientation of the background field. An experimental study by Trieling et al. shows that this observation is consistent with the laboratory experiment and also concludes that the orientation of the background field is a more critical determinant for the emergence of vortex merger than the initial vorticity [10]. On the other hand, without normalizing the parameter (β_X, β_ω) and performing the same consideration in terms of dynamical systems, it is found that when $c\beta_X/\beta_\omega < 0$, the distance between point vortices becomes smaller than the initial value. Therefore, from Theorem 4.1, we gain a physical insight into two-dimensional turbulence that the emergence of vortex merger is determined by the sign of the growth rate of the pressure in addition to the direction of the background field.

Point Vortices in an Irrotational Field The irrotational field $X \in \mathfrak{X}^\infty(\mathbb{R}^2)$ is defined as a vector field with zero vorticity and is a steady solution to the Euler equation. The corresponding pressure $P \in C^\infty(\mathbb{R}^2)$ satisfies the Bernoulli law $P = -|X|^2/2$. A vector field generated by a singular vorticity of point vortices can also be regarded as an irrotational field in the domain except for the singularities, since the vorticity is given by a linear combination of delta currents. Still, as for whether the pressure satisfies the Bernoulli law, it should be shown that it is the pressure of an Euler flow in the mathematically appropriate sense, which is confirmed in Theorem 2.

We investigate the Bernoulli law with point vortices in more detail using Theorem 4.1. Let $(\alpha_t, p_t) \in \mathcal{D}_1'^\infty(\mathbb{R}^2) \times \Omega^0_{[\infty]\mathrm{loc}}(\mathbb{R}^2)$ be a C^∞-decomposable weak Euler flow such that the background field is the irrotational field $X \in \mathfrak{X}^\infty(\mathbb{R}^2)$ and that the growth rate (β_X, β_ω) satisfies $\beta_X = 0$. Namely, the pressure p_t satisfies

$$\begin{aligned} p_t &= -|X|^2/2 - (X^\flat, u_t) + (\beta_\omega - 1/2)|u_t|^2 \\ &= -|\alpha_t|^2/2 + \beta_\omega |u_t|^2. \end{aligned} \tag{7}$$

Assuming the relative vorticity ω_t is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$, we deduce from Theorem 4.1 that $q_n(t)(n = 1, \ldots, N)$ is a solution of the point vortex equation in the background field:

$$\dot{q}_n = \beta_\omega v_n(q_n). \tag{8}$$

Based on the equation (7)-(8), we first examine the case where $\beta_\omega = 0, 1$. When $\beta_\omega = 0$, $\dot{q}_n(t) = 0$, which implies that each point vortex is stationary. Furthermore, the pressure is consistent with the Bernoulli law $p_t = -|\alpha_t|^2/2$. On the other hand, when $\beta_\omega = 1$, the motion of the point vortices is governed by the usual point vortex equation $\dot{q}_n(t) = v_n(q_n)$. In this case, the pressure is given by a modified Bernoulli law $p_t = -|\alpha_t|^2/2 + |u_t|^2$. Summarizing the above facts, we obtain the following as a corollary of Theorem 4.1.

Corollary 5.1 (Bernoulli law with point vortices). *If a C^∞-decomposable weak Euler flow (α_t, p_t) satisfies that the background field of α_t is an irrotational field $X \in \mathfrak{X}^\infty(M)$ and that the pressure is given by $p_t = -|\alpha_t|^2/2$ and that the relative vorticity is a singular vorticity of point vortices, then (α_t, p_t) is a steady solution of the weak Euler equations.*

Corollary 5.2 (Modified Bernoulli law with point vortices)**.** *If a C^∞-decomposable weak Euler flow (α_t, p_t) satisfies that the background field of α_t is an irrotational field and that $p_t = -|\alpha_t|^2/2 + |u_t|^2$ and that the relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$, then for every $n \in \{1, \ldots, N\}$, $q_n(t)$ is a solution of the point vortex equation:*

$$\dot{q}_n(t) = v_n(q_n(t)).$$

Finally, we study the role of the parameter β_ω in the point vortex dynamics and the pressure using the equation (7)-(8). Taking the solution $Q_n(t)$ of the usual point vortex equation $\dot{Q}_n = v_n(Q_n)$, the solution $q_n(t)$ of the equation (8) is presented as $q_n(t) = Q_n(\beta_\omega t)$. This tells us that as $\beta_\omega \to 0$, q_n moves quite slowly in the orbit of Q_n. In this sense, β_ω means the flexibility of the motion of point vortices. At the same time, the pressure converges to the Bernoulli law $p_t = -|\alpha_t|^2/2$ in $\Omega^0_{[\infty]\mathrm{loc}}$. As a result, we conclude that the parameter β_ω is the growth rate of the pressure as well as the flexibility of the motion of point vortices, and when it is sufficiently small, the pressure is close to the Bernoulli law and the point vortices move quite slowly.

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