DYNAMICAL PROPERTIES OF DOUBLY 0-DIMENSIONAL MAPS

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1. Introduction

In this article, we will consider the following problems of one-sided dynamical systems:

Problem 1.1. *Is it possible to study arbitrary one-sided dynamical systems by use of appropriate 0-dimensional dynamical systems?*

Problem 1.2. *Is it possible to reconstruct arbitrary one-sided dynamical systems by use of appropriate time series analysis?, i.e., is it possible to extend Takens' reconstruction theorem to one-sided dynamical systems?*

In this article, we show that the above problems 1.1 and 1.2 have near-positive answers by using doubly 0-dimensional maps.

For a space X, $\dim X$ means the topological (covering) dimension of X (e.g. see [Eng95], [HW41] and [Nag65]). Let X be compact metric space and Y a space with a complete metric d_Y. Let $C(X,Y)$ denote the space consisting of all maps $f : X \to Y$. We equip $C(X,Y)$ with the metric d defined by

$$d(f,g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Recall that $C(X,Y)$ is a complete metric space and hence Baire's category theorem holds in $C(X,Y)$. A map $g : X \to Y$ of separable metric spaces is *n-dimensional* ($n = 0, 1, 2, ...$) if $\dim g^{-1}(y) \leq n$ for each $y \in Y$. Note that a closed map $g : X \to Y$ is 0-dimensional if and only if for any 0-dimensional subset D of Y, $\dim g^{-1}(D) \leq 0$ (see [Eng95, Hurewic's theorem (1.12.4)]). A map $T : X \to X$ is *doubly 0-dimensional* if for each closed set $A \subset X$ of dimension 0, one has $\dim T^{-1}(A) \leq 0$ and $\dim T(A) = 0$.

We have the following theorem ([Kat21] and [KOU16]) which is the key fact in this article.

Theorem 1.3. *Suppose that X is one of the following spaces: compact PL-manifolds, compact PL-manifolds with branched structures, Menger manifolds, Sierpinski carpet, Sierpinski gasket, dendrites. Then the followings hold.*
(1) *The set of all doubly 0-dimensional maps on X is dense in the space*

$C(X,X)$ (*see* [Kat21]).
(2) *The set of maps* T *with* $\dim P(T) \leq 0$ *contains a* G_δ*-dense subset of the set* $C(X,X)$ (*see* [KOU16]), *where* $P(T)$ *denotes the set of periodic points of* T.

So if we could study dynamical properties of doubly 0-dimensional maps, we can obtain approximate properties of any dynamical systems (X,T) and hence it is important to study the dynamical properties of "doubly 0-dimensional maps". In this article, we show that the above problems 1.1 and 1.2 have near-positive answers through Theorem 1.3.

2. Finite-to-one 0-dimensional covers of doubly 0-dimensional maps

In this section, we consider the problem 1.1. Throughout this article, all spaces are separable metric spaces and maps are continuous functions. Let $\mathbb{N}$ be the set of all nonnegative integers, i.e., $\mathbb{N} = \{0,1,2,...\}$ and let $\mathbb{Z}$ be the set of all integers and $\mathbb{R}$ the real line. A map $h : X \to Y$ is an *embedding* if $h : X \to h(X)$ is a homeomorphism. A pair (X,T) is called a *one-sided dynamical system* (abbreviated as *dynamical system*) if X is a separable metric space and $T : X \to X$ is any map. Moreover, if $T : X \to X$ is a homeomorphism, i.e., invertible, then (X,T) is called a *two-sided dynamical system*. Also if $T : X \to X$ is not a homeomorphism, (X,T) called a *non-invertible dynamical system*.

A dynamical system $(Z,\tilde{T})$ *covers* (X,T) *via* a map $p : Z \to X$ provided that p is an onto map and $p\tilde{T} = Tp$. We call the map $p : Z \to X$ a *factor mapping*. If Z is 0-dimensional, then we say that the dynamical system $(Z,\tilde{T})$ is a *0-dimensional cover* of (X,T). Moreover, if the factor mapping is a finite-to-one map, then we say that the dynamical system $(Z,\tilde{T})$ is a *finite-to-one 0-dimensional cover* of (X,T).

The following theorem implies that the problem 1.1 has a near-positive answer (see [KM20]).

Theorem 2.1. *Suppose that* $T : X \to X$ *is a doubly 0-dimensional map of a compactum* X *with* $\dim X = n < \infty$. *If* $\dim P(T) \leq 0$, *then there exist a dense* G_δ*-set* H *of* X *and a zero-dimensional cover* $(Z,\tilde{T})$ *of* (X,T) *via an at most* 2^n*-to-one onto map* $p : Z \to X$ *such that* $P(T) \subset H$ *and* $|p^{-1}(x)| = 1$ *for* $x \in H$. *Moreover, if* X *is perfect, then* Z *can be chosen as a Cantor set. In particular,* $h(T) = h(\tilde{T})$, *where* $h(T)$ *denotes the topological entropy of* T.

For the special case of positively expansive maps, we have

Theorem 2.2. *Let* $T : X \to X$ *be a positively expansive map of a compactum* X *with* $\dim X = n < \infty$. *Then there exist* $k \geq 1$ *and a closed* σ*-invariant set* Σ *of the shift map* $\sigma : \{1,2,...,k\}^{\mathbb{N}} \to \{1,2,...,k\}^{\mathbb{N}}$ *such that* (Σ,σ) *is a zero-dimensional cover (= symbolic extension) of* (X,T) *via an at most* 2^n*-to-one map* $p : \Sigma \to X$.

An indexed family $(C_s)_{s\in S}$ of subsets of a set X will by abuse of notation also be denoted by $\{C_s\}_{s\in S}$ or $\{C_s : s \in S\}$. Hence if $\mathcal{C} = \{C_s\}_{s\in S}$ is such a family then its members C_s and C_t will be considered as different whenever $s \neq t$. We then put

$$\mathrm{ord}(\mathcal{C}) = \sup\{\mathrm{ord}_x(\mathcal{C}) : x \in X\}, \text{ where } \mathrm{ord}_x(\mathcal{C}) = |\{s \in S|\ x \in C_s\}|.$$

Note that $\mathrm{ord}(\mathcal{C})$ so defined is by 1 larger than it would be according to the usual definition, as e.g. in [Eng95, (1.6.6) Definition].

To prove the above theorems, we need the following lemma which is the key result to study the dynamical properties of doubly 0-dimensional maps (see [KM20]).

Lemma 2.3. *Suppose that $T : X \to X$ is a doubly 0-dimensional map of a compactum X such that $\dim X = n < \infty$ and $\dim P(T) \leq 0$. Let F be an F_σ-set of X with $\dim F \leq 0$. Then, for each $j \in \mathbb{N}$, there is a finite open cover $\mathcal{C}(j) = \{C(j)_i \mid 1 \leq i \leq m_j\}$ of X such that*
(1) $\mathrm{mesh}(\mathcal{C}(j)) < 1/j$ $(j \geq 1)$,
(2) $\mathrm{ord}(\mathcal{G}) \leq n$, *where* $\mathcal{G} = \{T^{-p}(\mathrm{bd}(C(j)_i)) \mid 1 \leq i \leq m_j,\ j \in \mathbb{N}$ *and* $p \in \mathbb{N}\}$, *and*
(3) $F \cap L = \emptyset$, *where* $L = \bigcup\{\mathrm{bd}(C(j)_i)|\ 1 \leq i \leq m_j, j \in \mathbb{N}\}$.

3. Dynamical decomposition theorem of doubly 0-dimensional maps

In dimension theory, the following decomposition theorem is well-known.

Theorem 3.1. *A separable metric space X is $\dim X \leq n$ $(n \in \mathbb{N})$ if and only if X can be represented as the union of $n+1$ subspaces $Z_0, Z_1, ..., Z_n$ of X such that $\dim Z_i \leq 0$ for each $i = 0, 1, ..., n$.*

In this section, we study "dynamical decomposition theorem" of doubly 0-dimensional maps. Let $T : X \to X$ be a map. A subset Z of X is a *bright space* of T except n times $(n \in \mathbb{N})$ if for any $x \in X$,

$$|\{p \in \mathbb{N}|\ T^p(x) \notin Z\}| \leq n.$$

Note that for any $x \in X$, the positive orbit $O(x)$ appears in Z except n times. Also we say that $L = X - Z$ is a *dark space* of T except n times. Note that for any $x \in X$, the positive orbit $O(x)$ disappears from L except n times. Bright spaces Z would be expected to be very large spaces. But we can choose very "small" bright spaces.

Theorem 3.2. [KM20] *Suppose that $T : X \to X$ is a doubly 0-dimensional map of a compactum X with $\dim X = n < \infty$. Then $\dim P(T) \leq 0$ if and only if there is a zero-dimensional bright space Z of T except n times such that Z is a dense G_δ-set of X and the dark space $L = X - Z$ of T is an $(n-1)$-dimensional F_σ-set of X.*

This theorem implies that the bright space Z is very small like "small dots"and the dark space is very large like "dark matters" in physics. Such Z as in Theorem 3.2 satisfies the dynamical decomposition theorem of doubly 0-dimensional maps (see [KM20]).

Theorem 3.3. *Suppose that X is a compactum with $\dim X = n$ $(< \infty)$ and $T : X \to X$ is a doubly 0-dimensional onto map. Then $\dim P(T) \leq 0$ if and only if there exists a zero-dimensional G_δ-dense set Z of X such that for any $n+1$ integers $k_0 < k_1 < \cdots < k_n$ $(k_i \in \mathbb{Z})$,*

$$X = T^{k_0}(Z) \cup T^{k_1}(Z) \cup \cdots \cup T^{k_n}(Z).$$

4. Takens-type reconstruction theorem of one-sided dynamical systems

In this section, we consider the problem 1.2. Reconstruction of dynamical systems from a scalar time series is a topic that has been extensively studied. The theoretical basis for methods of recovering dynamical systems on compact manifolds from one-dimensional data was studied by Takens [Tak81, Tak02]. In 1981, Takens [Tak81], by use of Whitney's embedding theorem, proved that under some conditions of (two-sided) diffeomorphisms on a manifold, the dynamical system can be reconstructed from the observations made with generic functions.

Theorem 4.1. (Takens' reconstruction theorem for diffeomorphisms [Tak81] and [Noa91]) *Suppose that M is a compact smooth manifold of dimension d. Let $D^r(M)$ be the space of all C^r-diffeomorphisms on M and $C^r(M,\mathbb{R})$ the set of all C^r-functions $(r \geq 1)$ to $\mathbb{R}$. If E is the set of all pairs $(T,f) \in D^r(M) \times C^r(M,\mathbb{R})$ such that the delay observation map $I_{T,f}^{(0,1,2,..,2d)} : M \to \mathbb{R}^{2d+1}$ defined by*

$$x \mapsto (fT^j(x))_{j=0}^{2d}$$

is an embedding, then E is open and dense in $D^r(M) \times C^r(M,\mathbb{R})$.

Moreover, in 2002 Takens [Tak02], extended his theorem for endomorphisms on compact smooth manifolds as follows.

Theorem 4.2. (Takens' reconstruction theorem for endomorphisms [Tak02]) *Suppose that M is a compact smooth manifold of dimension d. Then there is an open dense subset $\mathcal{U} \subset \mathrm{End}^1(M) \times C^1(M,\mathbb{R})$, where $\mathrm{End}^1(M)$ denotes the space of all C^1-endomorphisms on M, such that, whenever $(T,f) \in \mathcal{U}$, there is a map $\pi : I_{T,f}^{(0,1,..,2d)}(M) \to M$ with $\pi \cdot I_{T,f}^{(0,1,..,2d)} = T^{2d}$.*

For a space K, we consider the (one-sided) shift $\sigma : K^{\mathbb{N}} \to K^{\mathbb{N}}$ which is defined by

$$\sigma(x_0, x_1, x_2, x_3....) = (x_1, x_2, x_3....), \; x_i \in K.$$

Let (X, T) and (X', T') be dynamical systems. If a map $h : X \to X'$ satisfies the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \downarrow T & & \downarrow T' \\ X & \xrightarrow{h} & X' \end{array}$$

then we say that $h : (X, T) \to (X', T')$ is a *morphism* of dynamical systems.

In this article, we need the following definition from [Kat20].

Definition 4.3. *Let $T : X \to X$ be a map of a compact metric space X.*
(a) Given a set $S \subset \mathbb{N}$ and a map $f : X \to \mathbb{R}$, the map $(fT^j)_{j\in S} : X \to \mathbb{R}^S$ will be denoted by $I^S_{T,f}$. We call this map the delay observation map at times $j \in S$. Note that $I_{T,f} := I^{\mathbb{N}}_{T,f} : (X, T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$ is a morphism of dynamical systems. We call $I_{T,f}$ the infinite delay observation map for (T, f).
(b) We say that I^S_f is a trajectory-embedding if $I^S_f(x) \neq I^S_f(y)$ whenever $T^j(x) \neq T^j(y)$ for all $j \in S$.

Let (X, T) be a dynamical system of a compact metric space X. For $n \geq 1$, let $P_n(T)$ be the set of all periodic points of T with period $\leq n$ and $P(T)$ the set of all periodic points of T, i.e.

$$P_n(T) = \{x \in X |\text{ there is an } i \text{ such that } 1 \leq i \leq n \text{ and } T^i(x) = x\}$$

$$\text{and } P(T) = \bigcup_{n\geq 1} P_n(T).$$

Two points x and y of X are *trajectory-separated* for T if $T^j(x) \neq T^j(y)$ for $j \in \mathbb{N}$. A morphism $h : (X, T) \to (X', T')$ is a *trajectory-monomorphism* if $h(x), h(y)$ are trajectory-separated for T', whenever $x, y \in X$ are trajectory-separated for T.

For $x, y \in X$, let $o_T(x) = (T^i(x))_{i\in\mathbb{N}}$ and $o_T(y) = (T^i(y))_{i\in\mathbb{N}}$ be two orbits of T. We say that the orbit $o_T(x)$ is *eventually equivalent* to the orbit $o_T(y)$ if the orbits will be equal in the future, i.e., there exists an $n \in \mathbb{N}$ such that $T^i(x) = T^i(y)$ for each $i \geq n$. In this case, we wright $o_T(x) \sim_e o_T(y)$. We see that this relation is an equivalence relation. So we have the equivalence class

$$[o_T(x)] = \{o_T(y) |\ o_T(x) \sim_e o_T(y)\}$$

containing $o_T(x)$ and we put

$$[O(T)] = \{[o_T(x)] |\ x \in X\}.$$

Note that if $T : X \to X$ is injective, the function $o : X \to [O(T)]$ defined by $x \mapsto [o_T(x)]$ is bijective, i.e., $o : X \cong [O(T)]$. Also, note that if $h : (X, T) \to (X', T')$ is a morphism of dynamical systems, then h induces the function $h : [O(T)] \to [O(T')]$ defined by $h([o_T(x)]) = [o_{T'}(h(x))]$ for $x \in X$. A morphism $h : (X, T) \to (X', T')$ of dynamical systems is a *trajectory-isomorphism* if h induces the bijection $h : [O(T)] \cong [O(T')]$.

Proposition 4.4. *Suppose that a morphism $h : (X,T) \to (X',T')$ is a trajectory-monomorphism and h is surjective, i.e., $h(X) = X'$. Then h is a trajectory-isomorphism:*

$$h : [O(T)] \cong [O(T')]$$

We need the definition of topological entropy and we give the definition by Bowen [Bow78]. Let $T : X \to X$ be any map of a compact metric space X. A subset E of X is (n,ϵ)-separated if for any $x, y \in E$ with $x \neq y$, there is an integer j such that $0 \leq j < n$ and $d(T^j(x), T^j(y)) \geq \epsilon$. If K is any nonempty closed subset of X, $s_n(\epsilon; K)$ denotes the largest cardinality of any set $E \subset K$ which is (n,ϵ)-separated. Also we define

$$s(\epsilon; K) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\epsilon; K),$$

$$h(T; K) = \lim_{\epsilon\to 0} s(\epsilon; K).$$

It is well known that the topological entropy $h(T)$ of T is equal to $h(T; X)$ (see [Bow78]).

Let (X,T) and (Y,S) be one-sided dynamical systems of compact metric spaces. The *inverse limit* of T is the space

$$\varprojlim(X,T) = \{(x_i)_{i=0}^{\infty} \mid T(x_{i+1}) = x_i \text{ for each } i \in \mathbb{N}\} \subset X^{\mathbb{N}}$$

which has the topology inherited as a subspace of the product space $X^{\mathbb{N}}$. If $h : (X,T) \to (Y,S)$ is a morphism of dynamical systems, then the map

$$\varprojlim h : \varprojlim(X,T) \to \varprojlim(Y,S)$$

is defied by $\varprojlim h((x_i)_i) = (h(x_i))_i$ for $(x_i)_i \in \varprojlim(X,T)$. Note that if T is a homeomorphism, then $X \cong \varprojlim(X,T)$.

By [Kat20, Theorem 3.1], we have the following result.

Theorem 4.5. *Let X be a compact metric space with $\dim X = d < \infty$ and let $T : X \to X$ be a doubly 0-dimensional map with $\dim P(T) \leq 0$. Then there is a dense G_δ-set D of $C(X,\mathbb{R})$ such that for all $f \in D$,*

$$I_{T,f} = T_{T,f}^{\mathbb{N}} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions:
(a) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$,
(b) $\varprojlim I_{T,f} : \varprojlim(X,T) \to \varprojlim(I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism,*
(c) $h(T) = h(\sigma_{T,f})$ *and*
(d) if $x, y \in X$ *are trajectory-separated for* T, *then*

$$|\{i \in \mathbb{N} | \ I_{T,f}(x)_i = I_{T,f}(y)_i\}| \leq 2d.$$

Now, we will introduce the notion of *reconstruction space* of dynamical systems (see [Kat21]).

Definition 4.6. *A compact metric space X is a reconstruction space of dynamical systems if there exists a G_δ-dense set E of $C(X,X) \times C(X,\mathbb{R})$ such that for $(T,f) \in E$, the infinite delay observation map*

$$I_{T,f} := I_{T,f}^{\mathbb{N}} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions (1) and (2):
(1) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$, *where* $\sigma_{T,f} = \sigma | I_{T,f}(X)$, *and*
(2) $\varprojlim I_{T,f} : \varprojlim (X,T) \to \varprojlim (I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism.*

$$\begin{array}{ccccc} X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \\ \downarrow T & & \downarrow \sigma_{T,f} & & \downarrow \sigma \\ X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \end{array}$$

Finally we obtain Theorem 4.7 by use of Theorem 1.3 (see [Kat21]). Theorem 4.7 implies that the problem 1.2 has a near-positive answer.

Theorem 4.7. *Let X be one of the following spaces: PL-manifold, PL-manifold with branch structures, Menger manifold, Sierpiński carpet, Sierpiński gasket and dendrite. Then X is a reconstruction space of dynamical systems.*

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