Indestructible Guessing Models And The Approximation Property

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Abstract

In this short note, we shall prove some observations regarding the connection between indestructible ω_1-guessing models and the ω_1-approximation property of forcing notions.

Keywords. Approximation Property, Guessing Model, Indestructible Guessing Model

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1 Introduction and Basics

Viale and Weiß [4] introduced and used the notion of an ω_1-guessing model to reformulate the principle ISP(ω_2) and to show, among other things, that ISP(ω_2) follows form PFA. Cox and Krueger [1] introduced and studied indestructible ω_1-guessing sets of size ω_1, i.e., the ω_1-guessing sets which remains valid in generic extensions by any ω_1-preserving forcing. They formulated an analogous principle, denoted by IGMP(ω_2), and showed that it follows from PFA. Among other things, they showed that IGMP(ω_2) implies the Suslin Hypothesis. More generally, they proved that under IGMP(ω_2), if $(T, <_T)$ is a nontrivial tree of height and size ω_1, then the forcing notion $(T, \geq_T)$ collapses ω_1. This theorem establishes a connection between indestructible ω_1-guessing sets and the ω_1-approximation property of forcing notions. In this short paper, we examine a close inspection of the connection between the indestructibility of ω_1-guessing models and the ω_1-approximation property of forcing notions. In particular, we shall show that under GMP(ω_2), if $\mathbb{P}$ is an ω_1-preserving forcing which is proper for ω_1-guessing models of size ω_1, then $\mathbb{P}$ has the ω_1-approximation property if and only if the guessing models are indestructible by $\mathbb{P}$.

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Guessing models

Throughout this paper, by the stationarity of a set $\mathscr{S} \subseteq \mathscr{P}_{\omega_2}(H_\theta)$, we shall mean that for every function $F : \mathscr{P}_\omega(H_\theta) \to \mathscr{P}_{\omega_2}(H_\theta)$, there is $M \prec H_\theta$ in S with $M \cap \omega_2 \in \omega_2$ such that M is closed under F. We say a set x is *bounded* in a set or class M if there exists $X \in M$ with $x \subseteq X$.

Definition 1.1 (Viale-Weiß [4]). *A set M is called **ω_1-guessing** if and only if the following are equivalent for every x which is bounded in M.*

1. *x is ω_1-approximated in M, i.e., for every countable $a \in M$, $a \cap x \in M$.*
2. *x is guessed in M, i.e., there exists $x^* \in M$ with $x^* \cap M = x \cap M$.*

Definition 1.2 (GMP(ω_2)). GMP(ω_2) *states that for every regular $\theta \geq \omega_2$, the set of ω_1-guessing elementary submodels of H_θ of size ω_1 is stationary in $\mathscr{P}_{\omega_2}(H_\theta)$.*

Definition 1.3 (Cox–Krueger [1]).

1. *An ω_1-guessing set is said to be **indestructibly ω_1-guessing** if it remain ω_1-guessing in any ω_1-preserving forcing extension.*
2. *Let* IGMP(ω_2) *state that for every regular cardinal $\theta \geq \omega_2$, there exist stationarily many $M \in \mathscr{P}_{\omega_2}(H_\theta)$ such that M is indestructibly ω_1-guessing.*

We shall use the following without mentioning.

Fact 1.4. *Let $\theta \geq \omega_2$ be a cardinal. Assume $M \prec H_\theta$ is ω_1-guessing. Then $\omega_1 \subseteq M$.*

Proof. See [1, Lemma 2.3] 1.4

Generalised Proper Forcing

Let $\mathbb{P}$ be a forcing. Assume that $M \prec H_\theta$ with $\mathbb{P}, \mathscr{P}(\mathbb{P}) \in M$. A condition $p \in \mathbb{P}$ is $(M,\mathbb{P})$-generic, if for every dense set $D \subseteq \mathbb{P}$ which belongs to M, $M \cap D$ is pre-dense below p. The proof of the following is standard.

Lemma 1.5. *Suppose that $\mathbb{P}$ is a forcing. Assume that $M \prec H_\theta$ with $\mathbb{P}, \mathscr{P}(\mathbb{P}) \in M$. Let $p \in \mathbb{P}$. Then p is $(M,\mathbb{P})$-generic if and only if $p \Vdash$ "$M[\dot{G}] \cap H_\theta^V = M$".*

1.5

Let θ be a sufficiently large regular cardinal. A forcing $\mathbb{P}$ is said to be **proper for** $\mathscr{S}$, where $\mathscr{S} \subseteq \mathscr{P}_{\omega_2}(H_\theta)$ consists of elementary submodels of $(H_\theta, \in, \mathbb{P})$, if for every $M \in \mathscr{S}$ and every $p \in M \cap \mathbb{P}$, there is an $(M,\mathbb{P})$-generic condition $q \leq p$. A forcing is said to be **proper for models of size ω_1**, if for every sufficiently large regular cardinal θ, $\mathbb{P}$ is proper for $\{M \prec (H_\theta, \in, \mathbb{P}) : \omega_1 \subseteq M \text{ and } |M| = \omega_1\}$. It is easy to see that every forcing which is proper for a stationary set $\mathscr{S} \subseteq \mathscr{P}_{\omega_2}(H_\theta)$ preserves ω_2.

Lemma 1.6. *Suppose that $\mathbb{P}$ is proper for a stationary set $\mathscr{S} \subseteq \mathscr{P}_{\omega_2}(H_\theta)$. Then $\mathbb{P}$ preserves the stationarity of $\mathscr{S}$.*

Proof. Assume that $p \in \mathbb{P}$ forces that "$\dot{F} : \mathscr{P}_\omega(H_\theta^V) \to \mathscr{P}_{\omega_2}(H_\theta^V)$ is a function". Pick a sufficiently large regular cardinal $\theta^* > \theta$ with $\dot{F} \in H_{\theta^*}$. Pick $M^* \prec H_{\theta^*}$ with $\omega_1 \cup \{H_\theta, \dot{F}, p\} \subseteq M^*$ and $M := M^* \cap H_\theta \in \mathscr{S}$. Such a model exists by our assumption on the stationarity of $\mathscr{S}$. Since $\mathbb{P}$ is proper for $\mathscr{S}$, we can extend p to an $(M, \mathbb{P})$-generic condition q. Assume that $G \subseteq \mathbb{P}$ is a V-generic filter with $q \in G$. Now in $V[G]$, $M[G]$ is closed under F, as $\omega_1 \subseteq M$. By Lemma 1.5, $M[G] \cap H_\theta^V = M$, and hence M is closed under F. Thus q forces that $\check{M}$ is closed under $\dot{F}$. Since p was arbitrary, the maximal condition forces that $\mathscr{S}$ is stationary.

1.6

Let us recall the definition of the ω_1-approximation property of a forcing notion.

Definition 1.7 (Hamkins [2])**.** *A forcing notion $\mathbb{P}$ has the ω_1-approximation property in V if for every V-generic filter $G \subseteq \mathbb{P}$, and for every $x \in V[G]$ which is bounded in V so that for every countable $a \in V$, $a \cap x \in V$, then $x \in V$.*

2 IGMP and the Approximation Property

Lemma 2.1. *Suppose that $\mathbb{P}$ has the ω_1-approximation property. Assume that $M \prec H_\theta$ is ω_1-guessing, for some $\theta \geq \omega_2$. Then $\mathbb{P}$ forces M to be ω_1-guessing.*

Proof. Let $G \subseteq \mathbb{P}$ be a V-generic filter. Fix $x \in V[G]$ and assume that $x \subseteq X \in M$ is ω_1-approximated in M. We claim that $x \cap M$ is ω_1-approximated in V, which in turn implies that $x \cap M \in V$. Then, since M is ω_1-guessing in V, x is guessed in M. To see that $x \cap M$ is ω_1-approximated in V, fix a countable set $a \in V$. By [3, Theorem 1.4], there is a countable set $b \in M$ with $a \cap M \cap X \subseteq b$. Thus $a \cap x \cap M = a \cap x \cap b \in V$, since $a \in V$ and $x \cap b \in M \subseteq V$.

2.1

Definition 2.2. *For an ω_1-preserving forcing notion $\mathbb{P}$, we let $\mathbb{P}-\mathrm{IGMP}(\omega_2)$ states that for every sufficiently large regular θ, the set of ω_1-guessing sets of size ω_1 which remain ω_1-guessing after forcing with $\mathbb{P}$, is stationary in $\mathscr{P}_{\omega_2}(H_\theta)$.*

It is clear that $\mathrm{IGMP}(\omega_2)$ implies that $\mathbb{P}-\mathrm{IGMP}(\omega_2)$ holds, for all ω_1-preserving forcing $\mathbb{P}$. Note that $\mathrm{IGMP}(\omega_2)$ is a diagonal version of the statement that, for every ω_1-preserving forcing $\mathbb{P}$, $\mathbb{P}-\mathrm{IGMP}(\omega_2)$ holds. It is also worth mentioning that the $\mathrm{IGMP}(\omega_2)$ obtained by Cox and Kruger has the property that every indestructible ω_1-guessing model remains ω_1-guessing in any outer transitive extension with the same ω_1.

Proposition 2.3. *Assume that $\mathbb{P}$ is an ω_1-preserving forcing. Suppose that for every sufficiently large regular cardinal θ, $\mathbb{P}$ is proper for a stationary set $\mathfrak{G}_\theta \subseteq \mathscr{P}_{\omega_2}(H_\theta)$ of ω_1-guessing elementary submodels of H_θ. Then the following are equivalent.*

1. *$\mathbb{P}$ has the ω_1-approximation property.*
2. *Every ω_1-guessing model is indestructible by $\mathbb{P}$.*

Proof. Observe that the implication 1. $\Rightarrow$ 2. follows from Lemma 2.1. To see that the implication 2. $\Rightarrow$ 1. holds true, fix an ω_1-preserving forcing $\mathbb{P}$ and assume that the maximal condition of $\mathbb{P}$ forces $\dot{A}$ is a countably approximated subset of an ordinal γ. Pick a regular θ, with $\gamma, \dot{A}, \mathscr{P}(\mathbb{P}) \in H_\theta$. Assume that $\mathfrak{G} := \mathfrak{G}_\theta \subseteq \mathscr{P}_{\omega_2}(H_\theta)$ is a stationary set of ω_1-guessing elementary submodels of H_θ for which $\mathbb{P}$ is proper. We shall show that $\mathbb{P} \Vdash$ "$\dot{A} \in V$". Let $G \subseteq \mathbb{P}$ be a V-generic filter, and set

$$\mathscr{S} := \{M \in \mathfrak{G} : p, \gamma, \dot{A}, \mathbb{P} \in M \text{ and } M[G] \cap H^V_\theta = M\}.$$

In $V[G]$, $\mathscr{S}$ is stationary in $\mathscr{P}_{\omega_2}(H^V_\theta)$. To see this, let $F : \mathscr{P}_\omega(H^V_\theta) \to \mathscr{P}_{\omega_2}(H^V_\theta)$ be defined by $F(x) = \{\dot{y}^G\}$ if $x = \{\dot{y}\}$ for some $\mathbb{P}$-name $\dot{y}$ with $\dot{y}^G \in H^V_\theta$, and otherwise let $F(x) = \{p, \gamma, \dot{A}, \mathbb{P}\}$. By Lemma 1.6, the set of models in $\mathfrak{G}$ which are closed under F is stationary. Observe that a model $M \in \mathfrak{G}$ is closed under F if and only if $M \in \mathscr{S}$.

Let $A = \dot{A}^G$ and fix $M \in \mathscr{S}$. We claim that A is countably approximated in M. Let $a \in M$ be a countable subset of γ. Let D_a be the set of conditions deciding $\dot{A} \cap a$. Then D_a belongs to M and is dense in $\mathbb{P}$, as the maximal condition forces that $\dot{A}$ is countably approximated in V. By the elementarity of $M[G]$ in $H_\theta[G]$, there is $p \in G \cap D_a \cap M[G]$. But then $p \in M$, as $D_a \in H^V_\theta$. Working in V, the elementarity of M in H_θ implies that there is some $b \in M$ such that, $p \Vdash$ "$\check{b} = \dot{A} \cap a$". Since $p \in G$, we have $A \cap a = b \in M$. Thus A is countably approximated in M. By our assumption, M is an ω_1-guessing set in $V[G]$. Thus there is A^* in M, and hence in V, such that $A^* \cap M = A \cap M$.

Working in $V[G]$ again, for every $M \in \mathscr{S}$, there is, by the previous paragraph, a set $A^*_M \in M$ such that $A^*_M \cap M = A \cap M$. This defines a regressive function $M \mapsto A^*_M$ on $\mathscr{S}$. As $\mathscr{S}$ is stationary in H^V_θ, there are a set $A^* \in H^V_\theta$ and a stationary set $\mathscr{S}^* \subseteq \mathscr{S}$ such that for every $M \in \mathscr{S}^*$, we have $A^* \cap M = A \cap M$. Since $A \subseteq \bigcup \mathscr{S}^*$, we have $A^* = A$, which in turn implies that $A \in V$. 2.3

Corollary 2.4. *Assume* GMP(ω_2)*. Suppose that $\mathbb{P}$ is an ω_1-preserving forcing which is also proper for models of size ω_1. Then the following are equivalent.*

1. *$\mathbb{P}-$IGMP(ω_2) holds.*
2. *$\mathbb{P}$ has the ω_1-approximation property.*

2.4

Note that if $(T, <_T)$ is a tree of height and size ω_1, then $(T, \geq_T)$ is proper for models of size ω_1. However, it does not have the ω_1-approximation property if it is nontrivial as a forcing notion. We have the following generalisation of [1, Theorem 3.7].

Theorem 2.5. *Assume* IGMP(ω_2). *Then every ω_1-preserving forcing which is proper for models of size ω_1 has the ω_1-approximation property. In particular, under* IGMP(ω_2) *every ω_1-preserving forcing of size ω_1 has the ω_1-approximation property.*

Proof. Let $\mathbb{P}$ be an ω_1-preserving function which is proper for models of size ω_1. As IGMP(ω_2) holds, Proposition 2.3 implies that $\mathbb{P}$ has the ω_1-approximation property. $\boxed{2.5}$

For a class $\mathfrak{K}$ of forcing notions, we let FA$(\mathfrak{K}, \omega_1)$ state that for every $\mathbb{P} \in \mathfrak{K}$, and every ω_1-sized family $\mathscr{D}$ of dense subsets of $\mathbb{P}$, there is a $\mathscr{D}$-generic filter $G \subseteq \mathbb{P}$.

Lemma 2.6. *Assume* FA$(\{\mathbb{P}\}, \omega_1)$, *for some forcing notion $\mathbb{P}$. Suppose that M is an ω_1-guessing set of size ω_1. Then $\mathbb{P}$ forces that M is ω_1-guessing.*

Proof. Assume towards a contraction that for some $p_0 \in \mathbb{P}$, some ordinal $\delta \in M$, and some $\mathbb{P}$-name $\dot{A}$, p_0 forces that $\dot{A} \subseteq \delta$ is countably approximated in M, but is not guessed in M. We may assume that p_0 is the maximal condition of $\mathbb{P}$.

- For every $\alpha \in M \cap \delta$, let $D_\alpha := \{p \in \mathbb{P} : p \text{ decides } \alpha \in \dot{A}\}$.
- For every $x \in M \cap \mathscr{P}_{\omega_1}(\delta)$, let $E_x := \{p \in \mathbb{P} :\ \exists y \in M\ p \Vdash \text{“}\dot{A} \cap x = \check{y}\text{”}\}$.
- For every $B \in M \cap \mathscr{P}(\delta)$, let $F_B := \{p \in \mathbb{P} : \exists \xi \in M, (p \Vdash \text{“}\xi \in \dot{A}\text{”}) \Leftrightarrow \xi \notin B\}$.

By our assumptions, it is easily seen that the above sets are dense in $\mathbb{P}$. Let

$$\mathscr{D} = \{D_\alpha, E_x, F_B : \alpha, x, B \text{ as above }\}.$$

We have $|\mathscr{D}| = \omega_1$. By FA$(\{\mathbb{P}\}, \omega_1)$, there is a $\mathscr{D}$-generic filter $G \subseteq \mathbb{P}$. Let $A^* \subseteq \delta$ be defined by

$$\alpha \in A^* \text{ if and only if } \exists p \in G \text{ with } p \Vdash \text{“}\alpha \in \dot{A}.\text{”}$$

By the $\mathscr{D}$-genericity of G, A^* is a well-defined subset of δ which is countably approximated in M but not guessed in M, a contradiction! $\boxed{2.6}$

The following theorem is immediate from Corollary 2.4 and Lemma 2.6.

Theorem 2.7. *Let $\mathfrak{K}$ be a class of forcings which are proper for models of size ω_1. Assume that* FA$(\mathfrak{K}, \omega_1)$ *and* GMP(ω_2) *hold. Then, for every forcing $\mathbb{P} \in \mathfrak{K}$, $\mathbb{P}-$IGMP(ω_2) holds, and $\mathbb{P}$ has the ω_1-approximation property.*

$\boxed{2.7}$

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