Some kinds of continuity properties on composite functions of set-valued maps and scalarizing functions*

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Dedicated to the memory of Professor Wataru Takahashi

1 Introduction

A totally ordered space like the real field is very useful for preference, evaluation, computation, or comparison on the values of real-valued functions. However, multiobjective programming and vector optimization are a little complicated with multicriteria structure like some partial ordering, and minimal and maximal notions like Pareto optimal solution or efficient solution are defined with respect to a certain ordering cone (i.e., a dominance cone); see [19].

In vector optimization and set optimization, we have a typical approach by which optimization problems with vector-valued or set-valued maps can be easily handled by converting vectors or sets into real numbers; see [2] and [3, 5]. From the viewpoint of scalarization, the notion of weighted sum is a good tool for the scalarization of vectors in multicriteria problems, and it is regarded as the projection (i.e., inner product with the weight vector d) in $\mathbb{R}^n$. The average of elements is also a special case of weighted sum with the weight $d = (1/n, \ldots, 1/n)^{\mathrm{T}}$. They all are linear scalarization methods, and they can be regarded as a special case of a certain sublinear scalarization (introduced by Tammer [1, 3]):

$$h_C(v; d) := \inf \{t \in \mathbb{R} \mid v \in td - C\}$$

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where C is a convex cone in a real topological vector space and $d \in C$. This scalarizing functional $h_C(\cdot; d)$ is sublinear (i.e., $h_C(v_1 + v_2; d) \leq h_C(v_1) + h_C(v_2)$ and $h_C(tv; d) = th_C(v; d)$ for $t > 0$) and hence this conversion is called "sublinear scalarization." Therefore this special functional is a certain generalization of linear scalarization including the notions of weighted sum and inner product. Accordingly, this idea has inspired some researchers to develop particular scalarization methods for sets, leading to several applicable results shown in [4, 15].

On the other hand, we know that composite operation frequently preserves several mathematical properties of each nested function. For instance, a composition of continuous maps is continuous on topological spaces. Based on this property, we can characterize solutions for multicriteria problems through scalarization under certain assumptions. This leads to consider the mechanism by which composite functions of a set-valued map and a scalarization function transmit semicontinuity of parent set-valued maps through several scalarization for sets.

Recently, Ike, Liu, Ogata and Tanaka [6] show certain results on the inheritance property of some kinds of continuity of set-valued maps via scalarization functions for sets: if a set-valued map has a kind of continuity (lower continuity or upper continuity; see [3]) then the composition of its set-valued map and a certain scalarization function assures a similar semicontinuity to that of its scalarization function defined on the family of nonempty subsets of a real topological vector space. Their results are generalizations of results in earlier study by Kuwano, Tanaka and Yamada [15].

The aim of this paper is to review these background and to propose some idea how to generalize the inheritance property which is introduced by [6].

2 Set Relations and Scalarizing Functions for Sets

Throughout the paper, let X be a topological space and Y a real topological vector space. Let θ_Y be the zero vector in Y and $\mathcal{P}(Y)$ denote the set of all nonempty subsets of Y. The topological interior, topological closure, convex hull, and complement of a set $A \in \mathcal{P}(Y)$ are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{co} A$, and A^{c}, respectively. For given $A, B \in \mathcal{P}(Y)$ and $t \in \mathbb{R}$, the algebraic sum $A + B$ and the scalar multiplication tA are defined as follows:

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \quad tA := \{ta \mid a \in A\}.$$

In particular, we denote $A + \{y\}$ by $A + y$ and $(-1)A$ by $-A$ for $A \in \mathcal{P}(Y)$ and $y \in Y$.

Let X be a nonempty set and $\preccurlyeq$ a binary relation on X. The relation $\preccurlyeq$ is said to be

(i) reflexive if $x \preccurlyeq x$ for all $x \in X$;

(ii) irreflexive if $x \not\preccurlyeq x$ for all $x \in X$;

(iii) transitive if $x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$ for all $x, y, z \in X$;

(iv) antisymmetric if $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$ for all $x, y \in X$;

(v) complete if $x \preccurlyeq y$ or $y \preccurlyeq x$ for all $x, y \in X$.

The relation $\preccurlyeq$ is called

(i) a preorder if it is reflexive and transitive;

(ii) a strict order if it is irreflexive and transitive;

(iii) a partial order if it is reflexive, transitive, and antisymmetric;

(iv) a total order if it is reflexive, transitive, antisymmetric, and complete.

Throughout the paper, we assume that C is a convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $\theta_Y \in C$. Then, $C + C = C$ holds, and $\operatorname{int} C$ and $\operatorname{cl} C$ are also convex cones. Accordingly, we can define a preoder $\leq_C$ on Y induced by C as follows:

$$\text{for } y_1, y_2 \in Y, y_1 \leq_C y_2 \overset{\text{def}}{\iff} y_2 - y_1 \in C.$$

This preorder is compatible with the linear structure of Y:

$$\text{for all } y_1, y_2, y_3 \in Y, \quad y_1 \leq_C y_2 \implies y_1 + y_3 \leq_C y_2 + y_3; \tag{1}$$

$$\text{for all } y_1, y_2 \in Y \text{ and } t > 0, \quad y_1 \leq_C y_2 \implies ty_1 \leq_C ty_2. \tag{2}$$

When C is pointed (i.e., $C \cap (-C) = \{\theta_Y\}$), $\leq_C$ is antisymmetric and then a partial order.

Proposition 1. *Let C, C' be convex cones in Y and $d \in Y$. Assume that $C + (0, +\infty)d \subset C'$. Then, for any $v_1, v_2 \in Y$ and $t, t' \in \mathbb{R}$ with $t > t'$,*

$$v_1 + td \leq_C v_2 \implies v_1 + t'd \leq_{C'} v_2.$$

As generalizations of partial orderings for vectors, we give a definition of certain binary relations between sets in Y, called set relations. This is a modified version of the original one proposed in [12].

Definition 2 (set relations, [12])**.** For $A, B \in \mathcal{P}(Y)$, we define the following eight types of binary relations on $\mathcal{P}(Y)$.

(i) $A \leq_C^{(1)} B \overset{\text{def}}{\iff} \forall a \in A,\ \forall b \in B,\ a \leq_C b \iff A \subset \bigcap_{b \in B}(b - C)$
$\iff B \subset \bigcap_{a \in A}(a + C)$;

(ii) $A \leq_C^{(2L)} B \overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B,\ a \leq_C b \iff A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;

(iii) $A \le_C^{(2U)} B \overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A,\ a \le_C b \iff \big(\bigcap_{a\in A}(a+C)\big) \cap B \neq \emptyset;$

(iv) $A \le_C^{(2)} B \overset{\text{def}}{\iff} A \le_C^{(2L)} B \text{ and } A \le_C^{(2U)} B \iff \begin{array}{l} A \cap \big(\bigcap_{b\in B}(b-C)\big) \neq \emptyset \\ \text{and } \big(\bigcap_{a\in A}(a+C)\big) \cap B \neq \emptyset; \end{array}$

(v) $A \le_C^{(3L)} B \overset{\text{def}}{\iff} \forall b \in B,\ \exists a \in A \text{ s.t. } a \le_C b \iff B \subset A + C;$

(vi) $A \le_C^{(3U)} B \overset{\text{def}}{\iff} \forall a \in A,\ \exists b \in B \text{ s.t. } a \le_C b \iff A \subset B - C;$

(vii) $A \le_C^{(3)} B \overset{\text{def}}{\iff} A \le_C^{(3L)} B \text{ and } A \le_C^{(3U)} B \iff B \subset A+C \text{ and } A \subset B-C;$

(viii) $A \le_C^{(4)} B \overset{\text{def}}{\iff} \exists a \in A,\ \exists b \in B \text{ s.t. } a \le_C b \iff \begin{array}{l} A \cap (B - C) \neq \emptyset \\ \iff (A + C) \cap B \neq \emptyset. \end{array}$

In the above definition, the letters L and U stand for "lower" and "upper," respectively. Each relation $\le_C^{(j)}$ is transitive for $j = 1, 2L, 2U, 3L, 3U$ and not transitive for $j = 4$. Since $\theta_Y \in C$, $\le_C^{(j)}$ is reflexive for $j = 3L, 3U, 4$ and hence a preorder for $j = 3L, 3U$. Besides, for each $j = 1, 2L, 2U, 3L, 3U, 4$, the relation $\le_C^{(j)}$ satisfies certain similar properties to conditions (1) and (2) for all $A, B \in \mathcal{P}(Y)$,

(i) $A \le_C^{(j)} B \implies A + y \le_C^{(j)} B + y \quad \text{for} \quad y \in Y;$

(ii) $A \le_C^{(j)} B \implies tA \le_C^{(j)} tB \quad \text{for} \quad t > 0.$

Also, we easily obtain the following implications:

$$\begin{cases} A \le_C^{(1)} B \implies A \le_C^{(2L)} B \implies A \le_C^{(3L)} B \implies A \le_C^{(4)} B; \\ A \le_C^{(1)} B \implies A \le_C^{(2U)} B \implies A \le_C^{(3U)} B \implies A \le_C^{(4)} B; \\ A \le_C^{(1)} B \implies A \le_C^{(2)} B \implies A \le_C^{(3)} B \implies A \le_C^{(4)} B \end{cases} \tag{3}$$

for $A, B \in \mathcal{P}(Y)$.

Proposition 3 ([6]). *Let C' and C be two nonempty convex cones in Y and $d \in Y$. Assume that $C' + (0, +\infty)d \subset C$. Then, for each $j = 1, 2L, 3L, 2U, 3U, 4$, any $A, B \in \mathcal{P}(Y)$, $s, s' \in \mathbb{R}$ with $s' < s$ and $t, t' \in \mathbb{R}$ with $t < t'$,*

$$\begin{aligned} A \le_{C'}^{(j)} B + s'd &\implies A \le_C^{(j)} B + sd, \\ \text{and} \quad A + t'd \le_{C'}^{(j)} B &\implies A + td \le_C^{(j)} B. \end{aligned}$$

Now, we recall the scalarization scheme [13] for sets in a real vector space related to the set relations, which are certain generalizations as unification of several nonlinear scalarizations proposed in [5].

Definition 4 ([7, 13]). For each $j = 1, 2L, 3L, 2U, 3U, 4$, we define

$$I_C^{(j)}(A; V, d) := \inf \left\{ t \in \mathbb{R} \,\middle|\, A \leq_C^{(j)} (V + td) \right\}, \tag{4}$$

$$S_C^{(j)}(A; V, d) := \sup \left\{ t \in \mathbb{R} \,\middle|\, (V + td) \leq_C^{(j)} A \right\}, \tag{5}$$

for any $A, V \in \mathcal{P}(Y)$ and $d \in Y$.

The idea of these scalarization functions is introduced in [13], which originates from the idea of Gerstewitz's (Tammer's) sublinear scalarizing functional in [1]; see [3, 7]. This type of scalarization measures how far a given reference set needs to be moved toward a specific direction to fulfill each set relation between a target set and its moved reference set. Note that V and d in (4) and (5) are index parameters for scalarization which play key roles as a reference set and a reference direction, respectively.

Proposition 5 ([7]). *Let C be a convex cone in V. The following inequalities hold between each scalarizing function for sets:*

$$\begin{gathered}
I_C^{(4)}(A; W, d) \leq I_C^{(3L)}(A; W, d) \leq I_C^{(2L)}(A; W, d) \leq I_C^{(1)}(A; W, d); \\
I_C^{(4)}(A; W, d) \leq I_C^{(3U)}(A; W, d) \leq I_C^{(2U)}(A; W, d) \leq I_C^{(1)}(A; W, d); \\
I_C^{(4)}(A; W, d) \leq I_C^{(3)}(A; W, d) \leq I_C^{(2)}(A; W, d) \leq I_C^{(1)}(A; W, d); \\
S_C^{(1)}(A; W, d) \leq S_C^{(2L)}(A; W, d) \leq S_C^{(3L)}(A; W, d) \leq S_C^{(4)}(A; W, d); \\
S_C^{(1)}(A; W, d) \leq S_C^{(2U)}(A; W, d) \leq S_C^{(3U)}(A; W, d) \leq S_C^{(4)}(A; W, d); \\
S_C^{(1)}(A; W, d) \leq S_C^{(2)}(A; W, d) \leq S_C^{(3)}(A; W, d) \leq S_C^{(4)}(A; W, d)
\end{gathered}$$

for $A, W \in \mathcal{P}(V) \setminus \{\emptyset\}$ and $d \in C$.

Proposition 6 ([7]). *Let C be a convex cone in V. There are certain relations among the scalarizations of types* (2L), (2U), (2) *as well as* (3L), (3U), (3)*:*

(i) $I_C^{(2)}(A; W, d) = \max \left\{ I_C^{(2L)}(A; W, d), I_C^{(2U)}(A; W, d) \right\}$;

(ii) $I_C^{(3)}(A; W, d) = \max \left\{ I_C^{(3L)}(A; W, d), I_C^{(3U)}(A; W, d) \right\}$;

(iii) $S_C^{(2)}(A; W, d) = \min \left\{ S_C^{(2L)}(A; W, d), S_C^{(2U)}(A; W, d) \right\}$;

(iv) $S_C^{(3)}(A; W, d) = \min \left\{ S_C^{(3L)}(A; W, d), S_C^{(3U)}(A; W, d) \right\}$

for $A, W \in \mathcal{P}(V) \setminus \{\emptyset\}$ and $d \in C$.

Proposition 7 ([6])**.** *Let $A, V \in \mathcal{P}(Y)$ and $d \in Y$. Then the following statements hold*

$$\begin{aligned}
-I_C^{(1)}(-A;-V,d) &= S_C^{(1)}(A;V,d),\\
-I_C^{(2L)}(-A;-V,d) &= S_C^{(2U)}(A;V,d),\\
-I_C^{(3L)}(-A;-V,d) &= S_C^{(3U)}(A;V,d),\\
-I_C^{(2U)}(-A;-V,d) &= S_C^{(2L)}(A;V,d),\\
-I_C^{(3U)}(-A;-V,d) &= S_C^{(3L)}(A;V,d),\\
-I_C^{(4)}(-A;-V,d) &= S_C^{(4)}(A;V,d).
\end{aligned}$$

For each j without $j = 4$, scalarizing functions $I_C^{(j)}(\cdot\,;W,d)$ and $S_C^{(j)}(\cdot\,;W,d)$ with a nonempty reference set W and a direction d have the following monotonicity with respect to $\leq_C^{(j)}$, which is referred to as "j-monotonicity" in [10]:

$$\begin{cases} A \leq_C^{(j)} B \implies I_C^{(j)}(A;W,d) \leq I_C^{(j)}(B;W,d); \\ A \leq_C^{(j)} B \implies S_C^{(j)}(A;W,d) \leq S_C^{(j)}(B;W,d). \end{cases} \tag{6}$$

3 Transmission Mechanism on Semicontinuity of Set-Valued maps

Ike, Liu, Ogata and Tanaka [6] introduce a new concept of invariant property for set-valued map $F : X \to \mathcal{P}(Y)$ with respect to a binary relationship on a family of sets in Y, which is regarded as some kind of continuity from the viewpoint of order-monotonicity. Besides they show certain results on the inheritance property of some kinds of continuity of set-valued maps via scalarization functions for sets: if a set-valued map has a kind of continuity (lower continuity or upper continuity; see [3]) then the composition of its set-valued map and a certain scalarization function assures a similar semicontinuity to that of its scalarization function defined on the family of nonempty subsets of a real topological vector space. On the other hand, Sonda, Kuwano, and Tanaka [20] introduce two kinds of continuity with respect to cone, called "cone continuity," for set-valued maps by analogy with semicontinuity for real-valued functions, and they investigate the inheritance properties on cone continuity of parent set-valued maps via scalarization. Therefore, it is interesting to investigate the inheritance of cone continuity for set-valued maps via general scalarization functions for sets in the same manner as [6]. At first, we recall several definitions and results in [6].

Let $\mathcal{N}(x)$ and $\preccurlyeq$ be a neighborhood system of a point $x \in X$ and a binary relation on $\mathcal{P}(Y)$, respectively.

Definition 8 (Definition 3.2 in [6])**.** Let $F : X \to \mathcal{P}(Y)$ be a set-valued map, $x_0 \in X$ and $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$. We say F is $\preccurlyeq$-continuous at x_0 if $\forall W \subset Y$ with $W \preccurlyeq F(x_0)$, $\exists V \in \mathcal{N}(x_0)$ such that $W \preccurlyeq F(x), \forall x \in V$.

For $A, B \in \mathcal{P}(Y)$, we denote binary relations $\operatorname{int} A \cap B \neq \emptyset$ and $B \subset \operatorname{int} A$ by $A \preccurlyeq_1 B$ and $A \preccurlyeq_2 B$, respectively. Then $\preccurlyeq_1$-continuous and $\preccurlyeq_2$-continuous coincide with usual "lower (semi)continuity" and "upper (semi)continuity" for set-valued maps, respectively.

Definition 9 (Definition 3.3 in [6])**.** Let $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ be a scalarization function, $A_0 \in \mathcal{P}(Y)$, and $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$. Then

(i) we say φ is $\preccurlyeq$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y) \text{ such that } W \preccurlyeq A_0 \text{ and } r < \varphi(A), \forall A \in U(W, \preccurlyeq),$$

(ii) we say φ is $\preccurlyeq$-upper semicontinuous at A_0 if

$$\forall r > \varphi(A_0), \exists W \in \mathcal{P}(Y) \text{ such that } W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W, \preccurlyeq),$$

where $U(W, \preccurlyeq) := \{A \in \mathcal{P}(Y) \mid W \preccurlyeq A\}$.

Theorem 10 (Theorem 3.1 in [6])**.** *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, and $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$. If F is $\preccurlyeq$-continuous at x_0 and φ is $\preccurlyeq$-lower semicontinuous at $F(x_0)$, then $\varphi \circ F$ is lower semicontinuous at x_0 where $\varphi \circ F(x) := \varphi(F(x))$ for each $x \in X$.*

Theorem 11 (Theorem 3.2 in [6])**.** *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, and $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$. If F is $\preccurlyeq$-continuous at x_0 and φ is $\preccurlyeq$-upper semicontinuous at $F(x_0)$, then $\varphi \circ F$ is upper semicontinuous at x_0.*

In order to investigate the inheritance properties on cone continuity of parent set-valued maps via scalarization, we consider generalizations of semicontinuity for set-valued maps and real-valued functions.

Definition 12. Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open }, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

As special cases, $(\preccurlyeq_1, C)$-continuity and $(\preccurlyeq_2, C)$-continuity coincide with "C-lower continuity" and "C-upper continuity" for set-valued maps, respectively. Indeed, $F : X \to \mathcal{P}(Y)$ is $(\preccurlyeq_1, C)$-continuous at x_0 if and only if

$$\forall W \subset Y, W \text{ open }, W \cap F(x_0) \neq \emptyset, \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } (W + C) \cap F(x) \neq \emptyset, \forall x \in V,$$

that is, F is C-lower continuous at x_0. Similarly, F is $(\preccurlyeq_2, C)$-continuous at x_0 if and only if

$$\forall W \subset Y, W \text{ open }, F(x_0) \subset W, \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } F(x) \subset W + C, \forall x \in V,$$

that is, F is C-upper continuous at x_0; see Definition 2.5.16 of [3].

Remark 13. If $C = \{0\}$ then $(\preccurlyeq, C)$-continuity for set-valued maps becomes $\preccurlyeq$-continuity in Definition 8.

Definition 14. Let $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is

(i) $(\preccurlyeq, C)$-lower semicontinuous at A_0 if $\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r < \varphi(A), \forall A \in U(W + C, \preccurlyeq)$;

(ii) $(\preccurlyeq, C)$-upper semicontinuous at A_0 if $\forall r > \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r > \varphi(A), \forall A \in U(W + C, \preccurlyeq)$,

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}(Y) \,|V \preccurlyeq A\}$.

Remark 15. When $C = \{0\}$, $(\preccurlyeq, C)$-lower and $(\preccurlyeq, C)$-upper semicontinuities are coincident with $\preccurlyeq$-lower and $\preccurlyeq$-upper semicontinuities, respectively, which are introduced in Definition 3.3 of [6]. In Definition 14, we adopt that if $\varphi(A_0) = -\infty$ (resp. $+\infty$) then φ is $(\preccurlyeq, C)$-lower (resp. upper) semicontinuous at A_0.

Therefore, we can easily show the following results as generalizations of Theorems 10 and 11.

Theorem 16. *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $\varphi \circ F$ is lower semicontinuous at x_0.*

Theorem 17. *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-upper semicontinuous at $F(x_0)$, then $\varphi \circ F$ is upper semicontinuous at x_0.*

By above theorems, we can systematically unravel the inheritance mechanism related to lower and upper continuities for set-valued maps.

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