Triplet of Fibonacci Duals

— Identical Duality —

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Abstract

We consider three pairs of quadratic optimization problems from a view point of identical duality. An identity

$$\text{(CI)} \quad \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n \;=\; x_0\mu_1$$

is called complementary. We show that a complementary identity via *conditional complementarity* produces a pair of conditional minimization (primal) problem and conditional maximization (dual) problem, together with an equality condition. It is shown that both the problems have an identical optimal solution (point and value). Moreover, we show that a primal and its dual satisfy *Fibonacci Identical Duality.*

1 Introduction

Bellman and others [1–12, 26] have analyzed a wide class of quadratic optimization problems. Dynamic programming has solved its partial class [2, 17, 18, 29]. Further a dual approach has been discussed through convex-concavity [14, 16, 28].

Recently some dual approaches — (1) extended Lagrangean method, (2) plus-minus method, (3) ineqlualty method , (4) complementary method and others — have been proposed in [18, 20–25].

In this paper, we show an *identical duality* for three pairs of minimization (primal) problems and maximization (dual) problems — (P_1^*) vs (D_1^*), (P_2^*) vs (D_2^*) and (P_3^*) vs (D_3^*) —. These three are the respective identical versions of (P_1) vs (D_1), (P_2) vs (D_2) and (P_3) vs (D_3), which have analyzed the *complementary duality* [23].

It is shown that each pair is dual to each other. It turns out that the duality is based upon the complementary identity and an elementary inequality with equality

$$2xy \;\le\; x^2+y^2 \quad \text{on} \;\; R^2 \;;\;\; x=y. \tag{1}$$

2 Identical Duality

Let $x=\{x_k\}_0^n$, $\mu=\{\mu_k\}_1^n$ be any two sequences of real number with $x_0=c$.

2.1 (P_1^*) vs (D_1^*)

Then a complementary identity

$$(\mathrm{C}_1)\quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n$$

holds true.

Let us define two sequences $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{gathered} y_1 = c - x_1,\ y_2 = x_1,\ y_3 = x_1 - x_2,\ y_4 = x_2,\ y_5 = x_2 - x_3 \\ \ldots,\ y_{2n-2} = x_{n-1},\ y_{2n-1} = x_{n-1} - x_n,\ y_{2n} = x_n \\ \nu_1 = \mu_1,\ \nu_2 = \mu_1 - \mu_2,\ \nu_3 = \mu_2,\ \nu_4 = \mu_2 - \mu_3,\ \nu_5 = \mu_3 \\ \ldots,\ \nu_{2n-2} = \mu_{n-1} - \mu_n,\ \nu_{2n-1} = \mu_n,\ \nu_{2n} = \mu_n \end{gathered} \tag{2}$$

, respectively. Then an identity

$$(\mathrm{C}_1^*)\quad c\nu_1 \;=\; \sum_{k=1}^{2n} y_k\nu_k \tag{3}$$

holds *under a constraint* – a linear system of $4n$-variable (y,ν) on $2n$-equation – :

$$(\mathrm{C}^{y\nu})\qquad \begin{aligned} c &= y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\ y_2 &= y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\ &\ \vdots & &\ \vdots \\ y_{2n-4} &= y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &= y_{2n-1} + y_{2n} & \nu_{2n-1} &= \nu_{2n}. \end{aligned} \tag{4}$$

An equality (3) with constraint (4) is called a $2n$-variable *conditional complementarity.* This is simply written as (C_1^*) under $(\mathrm{C}^{y\nu})$.

Now let $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ satisfy $(\mathrm{C}_1^{y\nu})$. Then an elementary inequality (1) yields

$$2c\nu_1 \;\le\; \sum_{k=1}^{2n}(y_k^2 + \nu_k^2).$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n}\nu_k^2 \;\le\; \sum_{k=1}^{2n} y_k^2.$$

The sign of equality holds iff

$$(\mathrm{EC}_1^*)\quad y_k \;=\; \nu_k \quad 1 \le k \le 2n. \tag{5}$$

Hence we have a pair of conditional minimization problem:

$$(\mathrm{P}_1^*)\qquad \begin{aligned} &\text{minimize} \quad y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 + y_{2n}^2 \\ &\text{subject to} \quad (1) \ \ y_1 + y_2 = c \\ &\qquad\qquad\quad (2) \ \ y_3 + y_4 = y_2 \\ &\qquad\qquad\qquad \vdots \\ &\qquad\quad (n-1) \ \ y_{2n-3} + y_{2n-2} = y_{2n-4} \\ &\qquad\qquad\quad (n) \ \ y_{2n-1} + y_{2n} = y_{2n-2} \\ &\qquad\quad (n+1) \ \ y \in R^{2n} \end{aligned}$$

and conditional maximization problem:

$$(\mathrm{D}_1^*)\qquad \begin{aligned} &\text{Maximize} \quad 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 + \nu_{2n}^2) \\ &\text{subject to} \quad [1] \ \ \nu_2 + \nu_3 = \nu_1 \\ &\qquad\qquad\quad [2] \ \ \nu_4 + \nu_5 = \nu_3 \\ &\qquad\qquad\qquad \vdots \\ &\qquad\quad [n-1] \ \ \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ &\qquad\qquad\quad [n] \ \ \nu_{2n} = \nu_{2n-1} \\ &\qquad\quad [n+1] \ \ \nu \in R^{2n}. \end{aligned}$$

Let (AC_1) be an *augmentation* of the system $(\mathrm{C}_1^{y\nu})$ with the additinal equality condition (EC_1^*):

$$(\mathrm{AC}_1)\qquad \begin{aligned} c &= y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\ y_2 &= y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\ &\ \vdots & &\ \vdots \\ y_{2n-4} &= y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &= y_{2n-1} + y_{2n} & \nu_{2n-1} &= \nu_{2n} \end{aligned}$$
$$y_k = \nu_k \quad 1 \le k \le 2n.$$

The linear system (AC_1) is of $4n$-variable on $4n$-equation. Let (y, ν) satisfy (AC_1). Then both sides become a common value with five expressions.

$$(5\mathrm{V}_1)\qquad \begin{aligned} & y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\ &= cy_1 \\ &= 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2) \\ &= \nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2 \\ &= c\nu_1. \end{aligned}$$

The system (AC_1) has indeed a unique common solution:

$$\begin{aligned} y &= (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-1},\ y_{2n}) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1), \\ \nu &= (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-1},\ \nu_{2n}) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1) \end{aligned}$$

where $\{F_n\}$ is the *Fibonacci sequence* [13,15,27,30]. This is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1,\ x_0 = 0. \tag{6}$$

n	$\cdots$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
F_n	$\cdots$	-1	1	0	1	1	2	3	5	8	13	21	34	55	89

Table 1 Fibonacci sequence $\{F_n\}$

The primal (P_1^*) has a minimum value $m_1 = \dfrac{F_{2n}}{F_{2n+1}}c^2$ at a path

$$\begin{aligned} \hat{y} &= (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_k,\ \ldots,\ \hat{y}_{2n-1},\ \hat{y}_{2n}) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1). \end{aligned}$$

The dual (D_1^*) has a maximum value $M_1 = \dfrac{F_{2n}}{F_{2n+1}}c^2$ at a path

$$\begin{aligned} \nu^* &= (\nu_1^*,\ \nu_2^*,\ \ldots,\ \nu_k^*,\ \ldots,\ \nu_{2n-1}^*,\ \nu_{2n}^*) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1). \end{aligned}$$

Both optimal solutions (point and value) are identical:

$$\hat{x} = \mu^*, \quad m_1 = M_1.$$

Further both are Fibonacci:

$$\hat{x} = \mu^* = \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1),$$

$$m_1 = M_1 = \frac{F_{2n}}{F_{2n+1}}c^2.$$

Thus *Fibonacci Identical Duality* (FID) [18–20,22,24,25] holds between (P_1^*) and (D_1^*).

We remark that the $2n$-*variable* pair is a transliteration from n-*variable* one

$$
(\mathrm{P}_1) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2+x_n^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c \end{array}
$$

$$
(\mathrm{D}_1) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \displaystyle\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2-\mu_n^2 \\ \text{subject to} & \text{(i)}\ \ \mu \in R^n. \end{array}
$$

2.2 (P_2^*) vs (D_2^*)

On the other hand, we assume that $\underline{\mu_n = 0}$. Then an identity

$$
(\mathrm{C}_2) \quad c\mu_1 = \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n
$$

holds true.

Let us define two sequences $y = \{y_k\}_1^{2n-1}$, $\nu = \{\nu_k\}_1^{2n-1}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$
\begin{aligned}
&y_1 = c - x_1,\ y_2 = x_1,\ y_3 = x_1 - x_2,\ y_4 = x_2,\ y_5 = x_2 - x_3,\ \ldots \\
&,\ y_{2n-4} = x_{n-2},\ y_{2n-3} = x_{n-2} - x_{n-1},\ y_{2n-2} = x_{n-1},\ y_{2n-1} = x_{n-1} - x_n \\
&\nu_1 = \mu_1,\ \nu_2 = \mu_1 - \mu_2,\ \nu_3 = \mu_2,\ \nu_4 = \mu_2 - \mu_3,\ \nu_5 = \mu_3,\ \ldots \\
&,\ \nu_{2n-4} = \mu_{n-2} - \mu_{n-1},\ \nu_{2n-3} = \mu_{n-1},\ \nu_{2n-2} = \mu_{n-1} - \mu_n,\ \nu_{2n-1} = \mu_n
\end{aligned} \tag{7}
$$

, respectively. Then an identity

$$
(\mathrm{C}_2^*) \quad c\nu_1 = \sum_{k=1}^{2n-1} y_k\nu_k
$$

holds under a constraint – a linear system of $(4n-2)$-variable on $(2n-1)$-equation – :

$$
(\mathrm{C}_2^{y\nu}) \quad \begin{array}{rclrcl}
c &=& y_1 + y_2 & \nu_1 &=& \nu_2 + \nu_3 \\
y_2 &=& y_3 + y_4 & \nu_3 &=& \nu_4 + \nu_5 \\
&\vdots& & &\vdots& \\
y_{2n-6} &=& y_{2n-5} + y_{2n-4} & \nu_{2n-5} &=& \nu_{2n-4} + \nu_{2n-3} \\
y_{2n-4} &=& y_{2n-3} + y_{2n-2} & \nu_{2n-3} &=& \nu_{2n-2} + \nu_{2n-1} \\
& & & \underline{\nu_{2n-1}} & \underline{=} & \underline{0}.
\end{array}
$$

Thus we have a $(2n-1)$-variable *conditional complementarity* (C_2^*) under $(\mathrm{C}_2^{y\nu})$.

Let $y = \{y_k\}_1^{2n-1}$, $\nu = \{\nu_k\}_1^{2n-1}$ satisfy ($C_2^{y\nu}$). Then the elementary inequality (1) yields

$$2c\nu_1 \le \sum_{k=1}^{2n-1}(y_k^2 + \nu_k^2).$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n-1}\nu_k^2 \le \sum_{k=1}^{2n-1} y_k^2.$$

The sign of equality holds iff

$$(\mathrm{EC}_2^*) \quad y_k = \nu_k \quad 1 \le k \le 2n-1. \tag{8}$$

Hence we have a pair of conditional minimization problem:

$$(\mathrm{P}_2^*) \quad \begin{array}{lrl} \text{minimize} & \multicolumn{2}{l}{y_1^2 + y_2^2 + \cdots + y_{2n-2}^2 + y_{2n-1}^2} \\ \text{subject to} & (1) & y_1 + y_2 = c \\ & (2) & y_3 + y_4 = y_2 \\ & \vdots & \\ & (n-2) & y_{2n-5} + y_{2n-4} = y_{2n-6} \\ & (n-1) & y_{2n-3} + y_{2n-2} = y_{2n-4} \\ & (n) & y \in R^{2n-1} \end{array}$$

and conditional maximization problem:

$$(\mathrm{D}_2^*) \quad \begin{array}{lrl} \text{Maximize} & \multicolumn{2}{l}{2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-2}^2 + \nu_{2n-1}^2)} \\ \text{subject to} & [1] & \nu_2 + \nu_3 = \nu_1 \\ & [2] & \nu_4 + \nu_5 = \nu_3 \\ & \vdots & \\ & [n-1] & \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ & [n] & \underline{\nu_{2n-1} = 0} \\ & [n+1] & \nu \in R^{2n-1}. \end{array}$$

Let (AC_2) be an *augmentation* of the system ($C_2^{y\nu}$) with the additinal equality condition (EC_2^*):

$$(\mathrm{AC}_2)\quad \begin{aligned} c &= y_1 + y_2 \\ y_2 &= y_3 + y_4 \\ &\vdots \\ y_{2n-6} &= y_{2n-5} + y_{2n-4} \\ y_{2n-4} &= y_{2n-3} + y_{2n-2} \end{aligned} \qquad \begin{aligned} \nu_1 &= \nu_2 + \nu_3 \\ \nu_3 &= \nu_4 + \nu_5 \\ &\vdots \\ \nu_{2n-5} &= \nu_{2n-4} + \nu_{2n-3} \\ \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\ \underline{\nu_{2n-1}} &\ \underline{= 0} \end{aligned}$$

$$y_k = \nu_k \quad 1 \le k \le 2n-1.$$

The linear system (AC_2) is of $(4n-2)$-variable on $(4n-2)$-equation. Let (y, ν) satisfy (AC_2). Then both sides become a common value with five expressions.

$$(5\mathrm{V}_2)\quad \begin{aligned} & y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 \\ = {} & cy_1 \\ = {} & 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2) \\ = {} & \nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 \\ = {} & c\nu_1. \end{aligned}$$

The system (AC_2) has indeed a unique common solution:

$$\begin{aligned} & y = (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-2},\ y_{2n-1}) \\ = {} & \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-3},\ \ldots, F_{2n-k-1},\ \ldots,\ F_1,\ F_0), \\ & \nu = (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-2},\ \nu_{2n-1}) \\ = {} & \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-3},\ \ldots, F_{2n-k-1},\ \ldots,\ F_1,\ F_0). \end{aligned}$$

The primal (P_2^*) has a minimum value $m_2 = \dfrac{F_{2n-2}}{F_{2n-1}}c^2$ at a path

$$\begin{aligned} & \hat{y} = (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_k,\ \ldots,\ \hat{y}_{2n-2},\ \hat{y}_{2n-1}) \\ = {} & \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-3},\ \ldots, F_{2n-k-1},\ \ldots,\ F_1,\ F_0). \end{aligned}$$

The dual (D_2^*) has a maximum value $M_2 = \dfrac{F_{2n-2}}{F_{2n-1}}c^2$ at a path

$$\begin{aligned} & \nu^* = (\nu_1^*,\ \nu_2^*,\ \ldots,\ \nu_k^*,\ \ldots,\ \nu_{2n-2}^*,\ \nu_{2n-1}^*) \\ = {} & \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-3},\ \ldots, F_{2n-k-1},\ \ldots,\ F_1,\ F_0). \end{aligned}$$

Both optimal solutions (point and value) are identical:

$$\hat{x} = \mu^*, \quad m_2 = M_2.$$

Further both are Fibonacci:

$$\hat{x} = \mu^* = \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-3},\ \ldots, F_{2n-k-1},\ \ldots,\ F_1,\ F_0),$$

$$m_2 = M_2 = \frac{F_{2n-2}}{F_{2n-1}}c^2.$$

Thus FID holds between (P_2^*) and (D_2^*).

Note that the $(2n-1)$-*variable* pair is a transliteration from n-*variable* one

$$(P_2) \quad \begin{array}{ll} \text{minimize} & \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + x_k^2\right] + (x_{n-1} - x_n)^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c \end{array}$$

$$(D_2) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\ \text{subject to} & \text{(i)}\ \ \mu \in R^n, \quad \text{(ii)}\ \ \underline{\mu_n = 0}. \end{array}$$

2.3 (P_3^*) vs (D_3^*)

On the other hand, we assume that $\underline{x_n = 0}$. Then an identity

$$(C_3) \quad c\mu_1 = \sum_{k=1}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n$$

holds true.

Let us define two sequences $y = \{y_k\}_1^{2n-1}$, $\nu = \{\nu_k\}_1^{2n-1}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{array}{l} y_1 = c - x_1,\ y_2 = x_1,\ y_3 = x_1 - x_2,\ y_4 = x_2,\ y_5 = x_2 - x_3,\ \ldots \\ ,\ y_{2n-4} = x_{n-2},\ y_{2n-3} = x_{n-2} - x_{n-1},\ y_{2n-2} = x_{n-1},\ y_{2n-1} = x_{n-1} - x_n \\ \nu_1 = \mu_1,\ \nu_2 = \mu_1 - \mu_2,\ \nu_3 = \mu_2,\ \nu_4 = \mu_2 - \mu_3,\ \nu_5 = \mu_3,\ \ldots \\ ,\ \nu_{2n-4} = \mu_{n-2} - \mu_{n-1},\ \nu_{2n-3} = \mu_{n-1},\ \nu_{2n-2} = \mu_{n-1} - \mu_n,\ \nu_{2n-1} = \mu_n \end{array} \tag{9}$$

, respectively. Then an identity

$$(C_3^*) \quad c\nu_1 = \sum_{k=1}^{2n-1} y_k\nu_k \tag{10}$$

holds under a constraint – a linear system of $(4n-2)$-variable on $(2n-1)$-equation – :

$$(\mathrm{C}_3^{y\nu}) \qquad \begin{aligned} c &= y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\ y_2 &= y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\ &\vdots & &\vdots \\ y_{2n-4} &= y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\ \underline{y_{2n-2}} &= \underline{y_{2n-1}}. \end{aligned} \qquad (11)$$

Thus we have a $(2n-1)$-variable *conditional complementarity* (C_3^*) under $(\mathrm{C}_3^{y\nu})$.

Let $y = \{y_k\}_1^{2n-1}$, $\nu = \{\nu_k\}_1^{2n-1}$ satisfy (C_ν^y). Then the elementary inequality (1) yields

$$2c\nu_1 \le \sum_{k=1}^{2n-1} (y_k^2 + \nu_k^2).$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n-1} \nu_k^2 \le \sum_{k=1}^{2n-1} y_k^2.$$

The sign of equality holds iff

$$(\mathrm{EC}_3^*) \qquad y_k = \nu_k \quad 1 \le k \le 2n-1. \qquad (12)$$

Hence we have a pair of conditional minimization problem:

$$(\mathrm{P}_3^*) \qquad \begin{aligned} &\text{minimize} \quad y_1^2 + y_2^2 + \cdots + y_{2n-2}^2 + y_{2n-1}^2 \\ &\text{subject to} \quad (1) \ \ y_1 + y_2 = c \\ &\qquad\qquad\quad (2) \ \ y_3 + y_4 = y_2 \\ &\qquad\qquad\qquad \vdots \\ &\qquad\quad (n-1) \ \ y_{2n-3} + y_{2n-2} = y_{2n-4} \\ &\qquad\qquad\quad (n) \ \ \underline{y_{2n-1} = y_{2n-2}} \\ &\qquad\quad (n+1) \ \ y \in R^{2n} \end{aligned}$$

and conditional maximization problem:

$$(\mathrm{D}_3^*) \qquad \begin{aligned} &\text{Maximize} \quad 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-2}^2 + \nu_{2n-1}^2) \\ &\text{subject to} \quad [1] \ \ \nu_2 + \nu_3 = \nu_1 \\ &\qquad\qquad\quad [2] \ \ \nu_4 + \nu_5 = \nu_3 \\ &\qquad\qquad\qquad \vdots \\ &\qquad\quad [n-1] \ \ \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ &\qquad\qquad\quad [n] \ \ \nu \in R^{2n-1}. \end{aligned}$$

Let (AC_3) be an *augmentation* of the system ($C_3^{y\nu}$) with the additinal equality condition (EC_3^*):

$$
(\mathrm{AC}_3)\qquad
\begin{aligned}
c &= y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\
y_2 &= y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\
&\vdots & &\vdots \\
y_{2n-4} &= y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\
\underline{y_{2n-2}} &= \underline{y_{2n-1}} & &
\end{aligned}
$$

$$y_k = \nu_k \quad 1 \le k \le 2n-1.$$

The linear system (AC_3) is of $(4n-2)$-variable on $(4n-2)$-equation. Let (y, ν) satisfy (AC_3). Then both sides become a common value with five expressions.

$$
(5\mathrm{V}_3)\qquad
\begin{aligned}
& y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 \\
= {} & c y_1 \\
= {} & 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2) \\
= {} & \nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 \\
= {} & c\nu_1.
\end{aligned}
$$

The system (AC_3) has indeed a unique common solution:

$$
\begin{aligned}
& y = (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-2},\ y_{2n-1}) \\
= {} & \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-2},\ \ldots, F_{2n-k},\ \ldots,\ F_2,\ F_1), \\
& \nu = (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-2},\ \nu_{2n-1}) \\
= {} & \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-2},\ \ldots, F_{2n-k},\ \ldots,\ F_2,\ F_1).
\end{aligned}
$$

The primal (P_3^*) has a minimum value $m_3 = \dfrac{F_{2n-1}}{F_{2n}}c^2$ at a path

$$
\begin{aligned}
& \hat{y} = (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_k,\ \ldots,\ \hat{y}_{2n-2},\ \hat{y}_{2n-1}) \\
= {} & \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-2},\ \ldots, F_{2n-k},\ \ldots,\ F_2,\ F_1).
\end{aligned}
$$

The dual (D_3^*) has a maximum value $M_3 = \dfrac{F_{2n-1}}{F_{2n}}c^2$ at a path

$$
\begin{aligned}
& \nu^* = (\nu_1^*,\ \nu_2^*,\ \ldots,\ \nu_k^*,\ \ldots,\ \nu_{2n-2}^*,\ \nu_{2n-1}^*) \\
= {} & \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-2},\ \ldots, F_{2n-k},\ \ldots,\ F_2,\ F_1).
\end{aligned}
$$

Both optimal solutions (point and value) are identical:

$$\hat{x} = \mu^*, \quad m_3 = M_3.$$

Further both are Fibonacci:

$$\hat{x} = \mu^* = \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-2},\ \ldots, F_{2n-k},\ \ldots,\ F_2,\ F_1),$$

$$m_3 = M_3 = \frac{F_{2n-1}}{F_{2n}}c^2.$$

Thus FID holds between (P_3^*) and (D_3^*).

Note that the $(2n-1)$-*variable* pair is a transliteration from n-*variable* one

$$(\mathrm{P}_3)\quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c,\ \underline{x_n = 0} \end{array}$$

$$(\mathrm{D}_3)\quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \displaystyle\sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\ \text{subject to} & \text{(i)}\ \ \mu \in R^n. \end{array}$$

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