Parallelogram laws and balanced mappings in smooth Banach spaces
滑らかなバナッハ空間における中線定理と均衡写像

東邦大学・理学部 木村泰紀
Yasunori Kimura
Department of Information Science
Toho University
東邦大学・理学研究科 須藤秀太
Shuta Sudo
Department of Information Science
Toho University

Abstract

A part of the structure of Hilbert spaces is characterized by the parallelogram law. In order for us to hold the parallelogram law by the norm, fairly good conditions are required for Banach space. In this paper, we propose new type parallelogram laws with a bifunction on Banach spaces, and consider alternative expression of affine combinations. Moreover, we introduce two different balanced mappings by using them.

1 Introduction

In real Hilbert spaces, it holds that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$$

for every two points x, y and every scalar $\alpha \in \mathbb{R}$. This equation is called the parallelogram law. If a real Banach space satisfies this equation, then it space has inner product, that is, such a space is a real Hilbert space.

In Banach spaces, Xu [7] proved inequalities like the parallelogram law by using uniform convexity and uniform smoothness.

Theorem 1.1 (Xu [7]). *Let E be a uniformly convex real Banach space. Then, for any $K > 0$, there exists a convex gauge function $\underline{g}_K$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_K(\|x-y\|)$$

for any $x, y \in E$ and $\alpha \in [0,1]$, where $\|x\| \leq K$ and $\|y\| \leq K$.

Theorem 1.2 (Xu [7]). *Let E be a uniformly smooth real Banach space. Then, for any $K > 0$, there exists a convex gauge function $\overline{g}_K$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \geq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_K(\|x-y\|)$$

for any $x, y \in E$ and $\alpha \in [0,1]$, where $\|x\| \leq K$ and $\|y\| \leq K$.

In this paper, we propose new type parallelogram laws in real Banach spaces for a bifunction defined with a bounded linear functional and norms.

2 Preliminaries

Let X be a metric space and T a mapping from X into itself. We denote the set of *fixed points* of T by $\operatorname{Fix} T$, that is, $\operatorname{Fix} T = \{x \in X \mid x = Tx\}$.

Let S be a nonempty set. Let $f\colon S \to]-\infty, \infty]$ be a function. We denote *the set of minimizers of f* by $\operatorname{argmin}_{x\in S} f(x)$, that is,

$$\operatorname*{argmin}_{x\in S} f(x) = \left\{ y \in S \;\middle|\; f(y) = \inf_{x\in S} f(x) \right\}.$$

In what follows, we always consider real linear spaces. Let E be a Banach space and let $S_E = \{x \in E \mid \|x\| = 1\}$ its unit sphere. E is said to be *strictly convex* if $\|x + y\| < 2$ holds for each $x, y \in S_E$ with $x \neq y$. E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S_E$.

Let E^* be a dual space of Banach space E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. A bifunction $\phi\colon E \times E^* \to \mathbb{R}$ is defined by

$$\phi(x, y^*) = \|x\|^2 - 2\langle x, y^* \rangle + \|y^*\|^2$$

for each $x \in E$ and $y^* \in E^*$. Also, a bifunction $\phi^*\colon E^* \times E^{**} \to \mathbb{R}$ is defined by

$$\phi^*(x^*, y^{**}) = \|x^*\|^2 - 2\langle x^*, y^{**} \rangle + \|y^{**}\|^2$$

for each $x^* \in E^*$ and $y^{**} \in E^{**}$. Since we can regard as $E \subset E^{**}$, $\phi(x, y^*) = \phi^*(y^*, x)$ holds for every $x \in E$ and $y^* \in E^*$. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \left\{x^* \in E^* \;\middle|\; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}$$

for each $x \in E$. We know $Jx \subset E^*$ is a nonempty bounded closed convex set for any $x \in E$ and $J0_E = \{0_{E^*}\}$.

Let E be a Banach space with its dual E^* and let $J\colon E \to 2^{E^*}$ is the normalized duality mapping. Then, we obtain the following results:

- E is reflexive if and only if E^* is reflexive;
- if E is reflexive, then E is strictly convex if and only if E^* is smooth;
- E is smooth if and only if $J\colon E \to 2^{E^*}$ is single-valued;
- if E is smooth, then E is reflexive if and only if $J\colon E \to E^*$ is surjective;
- if E is smooth, then E is strictly convex if and only if $J\colon E \to E^*$ is injective;
- if E is smooth, strictly convex and reflexive, then the normalized duality mapping $J^*\colon E^* \to E^{**} = E$ coinsides with $J^{-1}\colon E^* \to E$;
- if E is smooth and strictly convex, then $\phi(x, Jy) = 0$ if and only if $x = y$ for $x, y \in E$.

For more details about the properties of J and ϕ on Banach spaces, see [1, 3, 5].

Let E be a smooth Banach space and $J\colon E \to E^*$ the normalized duality mapping. Let C be a nonempty closed convex subset of E and T a mapping from C into E. If $\operatorname{Fix} T \neq \varnothing$ and $\phi(u, JTx) \leq \phi(u, Jx)$ for $x \in C$ and $u \in \operatorname{Fix} T$, then we say that T is *relatively quasinonexpansive*. If $\operatorname{Fix} T \neq \varnothing$ and $\phi(Tx, Ju) \leq \phi(x, Ju)$ for $x \in C$ and $u \in \operatorname{Fix} T$, then we say that T is *generalized quasinonexpansive*. For more details, see, for instance, [3, 5].

3 Parallelogram laws in Banach spaces

In this section, we prove the parallelogram laws with the normalized duality mapping:

Theorem 3.1. *Let E be a smooth Banach space with its dual E^*. Let $N \in \mathbb{N}$. Then, it holds that*

$$\phi\left(\sum_{k=1}^{N} \alpha_k x_k, z^*\right) = \sum_{k=1}^{N} \alpha_k \phi(x_k, z^*) - \sum_{k=1}^{N} \alpha_k \phi\left(x_k, J\left(\sum_{k=1}^{N} \alpha_k x_k\right)\right)$$

for every $x_1, x_2, \dots, x_N \in E$, $z^* \in E^*$ and $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}$ such that $\sum_{k=1}^{N} \alpha_k = 1$, where $J\colon E \to E^*$ is the normalized duality mapping.

Proof. From the assumptions, it follows that

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha_k \phi(x_k, z^*) - \sum_{k=1}^{N} \alpha_k \phi\left(x_k, J\left(\sum_{k=1}^{N} \alpha_k x_k\right)\right) \\
&= \sum_{k=1}^{N} \alpha_k \|x_k\|^2 - 2\left\langle \sum_{k=1}^{N} \alpha_k x_k, z^* \right\rangle + \|z^*\|^2 \\
&\quad - \sum_{k=1}^{N} \alpha_k \|x_k\|^2 + 2\left\| \sum_{k=1}^{N} \alpha_k x_k \right\|^2 - \left\| J\left(\sum_{k=1}^{N} \alpha_k x_k\right) \right\|^2 \\
&= \left\| \sum_{k=1}^{N} \alpha_k x_k \right\|^2 - 2\left\langle \sum_{k=1}^{N} \alpha_k x_k, z^* \right\rangle + \|z^*\|^2 = \phi\left(\sum_{k=1}^{N} \alpha_k x_k, z^*\right).
\end{aligned}$$

This is the desired result. □

Theorem 3.2. *Let E be a reflexive Banach space with its dual E^*. Suppose that E^* is smooth. Let $N \in \mathbb{N}$. Then, it holds that*

$$\phi\left(z, \sum_{k=1}^N \alpha_k x_k^*\right) = \sum_{k=1}^N \alpha_k \phi(z, x_k^*) - \sum_{k=1}^N \alpha_k \phi\left(J^*\left(\sum_{k=1}^N \alpha_k x_k^*\right), x_k^*\right)$$

for every $z \in E$, $x_1^*, x_2^*, \dots, x_N^* \in E^*$ and $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}$ such that $\sum_{k=1}^N \alpha_k = 1$, where $J^* \colon E^* \to E$ is the normalized duality mapping.

Proof. From the assumptions, we obtain

$$\begin{aligned}
&\sum_{k=1}^N \alpha_k \phi(z, x_k^*) - \sum_{k=1}^N \alpha_k \phi\left(J^*\left(\sum_{k=1}^N \alpha_k x_k^*\right), x_k^*\right)\\
&= \|z\|^2 - 2\left\langle z, \sum_{k=1}^N \alpha_k x_k^*\right\rangle + \sum_{k=1}^N \alpha_k \|x_k^*\|^2\\
&\quad - \left\|J^*\left(\sum_{k=1}^N \alpha_k x_k^*\right)\right\|^2 + 2\left\|\sum_{k=1}^N \alpha_k x_k^*\right\|^2 - \sum_{k=1}^N \alpha_k \|x_k^*\|^2\\
&= \|z\|^2 - 2\left\langle z, \sum_{k=1}^N \alpha_k x_k^*\right\rangle + \left\|\sum_{k=1}^N \alpha_k x_k^*\right\|^2 = \phi\left(z, \sum_{k=1}^N \alpha_k x_k^*\right).
\end{aligned}$$

This is the desired result. □

Consequently, we know the following results:

Theorem 3.3. *Let E be a smooth Banach space with its dual E^*. Then, it holds that*

$$\begin{aligned}
\phi(\alpha x + (1-\alpha)y, z^*) &= \alpha\phi(x, z^*) + (1-\alpha)\phi(y, z^*)\\
&\quad - \alpha\phi(x, J(\alpha x + (1-\alpha)y)) - (1-\alpha)\phi(y, J(\alpha x + (1-\alpha)y))
\end{aligned}$$

for every $x, y \in E$, $z^* \in E^*$ and $\alpha \in \mathbb{R}$, where $J \colon E \to E^*$ is the normalized duality mapping.

Theorem 3.4. *Let E be a reflexive Banach space with its dual E^*. Suppose that E^* is smooth. Then, it holds that*

$$\begin{aligned}
\phi(z, \alpha x^* + (1-\alpha)y^*) &= \alpha\phi(z, x^*) + (1-\alpha)\phi(z, y^*)\\
&\quad - \alpha\phi(J^*(\alpha x^* + (1-\alpha)y^*), x^*) - (1-\alpha)\phi(J^*(\alpha x^* + (1-\alpha)y^*), y^*)
\end{aligned}$$

for every $z \in E$, $x^*, y^* \in E^*$ and $\alpha \in \mathbb{R}$, where $J^* \colon E^* \to E$ is the normalized duality mapping.

Corollary 3.1. *Let E be a smooth Banach space. Then, it holds that*

$$\begin{aligned}\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 \\ - \alpha\phi(x, J(\alpha x + (1-\alpha)y)) - (1-\alpha)\phi(y, J(\alpha x + (1-\alpha)y))\end{aligned}$$

for every $x, y \in E$ and $\alpha \in \mathbb{R}$, where J is the normalized duality mapping.

If E is a Hilbert space, then, since $\phi(\cdot,\cdot)$ coincides with $\|\cdot - \cdot\|^2$, it follows that

$$\begin{aligned}\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 \\ - \alpha\|x - (\alpha x + (1-\alpha)y)\|^2 - (1-\alpha)\|y - (\alpha x + (1-\alpha)y)\|^2\end{aligned}$$

for every $x, y \in E$ and $\alpha \in \mathbb{R}$. Moreover, we can proceed with the following calculations:

$$\begin{aligned}&- \alpha\|x - (\alpha x + (1-\alpha)y)\|^2 - (1-\alpha)\|y - (\alpha x + (1-\alpha)y)\|^2 \\ &= -\left(\alpha\|(1-\alpha)(x-y)\|^2 + (1-\alpha)\|\alpha(x-y)\|^2\right) \\ &= -\left(\alpha(1-\alpha)^2\|x-y\|^2 + \alpha^2(1-\alpha)\|x-y\|^2\right) \\ &= -\alpha(1-\alpha)\left((1-\alpha)\|x-y\|^2 + \alpha\|x-y\|^2\right) \\ &= -\alpha(1-\alpha)\|x-y\|^2.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\|\alpha x + (1-\alpha)y\|^2 &= \alpha\|x\|^2 + (1-\alpha)\|y\|^2 \\ &\quad - \alpha\|x - (\alpha x + (1-\alpha)y)\|^2 - (1-\alpha)\|y - (\alpha x + (1-\alpha)y)\|^2 \\ &= \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2\end{aligned}$$

for every $x, y \in E$ and $\alpha \in \mathbb{R}$.

At the end of this section, we consider affine combinations in smooth Banach space. Let E be a smooth, strictly convex and reflexive Banach space. Let $N \in \mathbb{N}$, $x_1, x_2, \dots, x_N \in E$ and $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}$ such that $\sum_{k=1}^N \alpha_k = 1$. Let $J\colon E \to E^*$ and $J^*\colon E^* \to E$ be the normalized duality mappings, respectively. Then, it holds that

$$\phi\left(y, \sum_{k=1}^N \alpha_k Jx_k\right) = \sum_{k=1}^N \alpha_k \phi(y, Jx_k) - \sum_{k=1}^N \alpha_k \phi\left(J^*\left(\sum_{k=1}^N \alpha_k Jx_k\right), Jx_k\right)$$

for any $y \in E$. Therefore, we have

$$\operatorname*{argmin}_{y\in E} \sum_{k=1}^N \alpha_k \phi(y, Jx_k)$$

$$
\begin{aligned}
&= \operatorname*{argmin}_{y\in E} \left\{ \phi\left(y, \sum_{k=1}^N \alpha_k J x_k\right) + \sum_{k=1}^N \alpha_k \phi\left(J^*\left(\sum_{k=1}^N \alpha_k J x_k\right), J x_k\right)\right\} \\
&= \operatorname*{argmin}_{y\in E} \phi\left(y, \sum_{k=1}^N \alpha_k J x_k\right) = J^*\left(\sum_{k=1}^N \alpha_k J x_k\right).
\end{aligned}
$$

It implies

$$
\operatorname*{argmin}_{y\in E} \{\alpha\phi(y, Ju) + (1-\alpha)\phi(y, Jv)\} = J^*(\alpha Ju + (1-\alpha)Jv)
$$

for every $u, v \in E$ and $\alpha \in \mathbb{R}$. On the other hand, it holds that

$$
\phi\left(\sum_{k=1}^N \alpha_k x_k, Jy\right) = \sum_{k=1}^N \alpha_k \phi(x_k, Jy) - \sum_{k=1}^N \alpha_k \phi\left(x_k, J\left(\sum_{k=1}^N \alpha_k x_k\right)\right)
$$

for any $y \in E$. Therefore, we have

$$
\begin{aligned}
&\operatorname*{argmin}_{y\in E} \sum_{k=1}^N \alpha_k \phi(x_k, Jy) \\
&= \operatorname*{argmin}_{y\in E} \left\{ \phi\left(\sum_{k=1}^N \alpha_k x_k, Jy\right) + \sum_{k=1}^N \alpha_k \phi\left(x_k, J\left(\sum_{k=1}^N \alpha_k x_k\right)\right)\right\} \\
&= \operatorname*{argmin}_{y\in E} \phi\left(\sum_{k=1}^N \alpha_k x_k, Jy\right) = \sum_{k=1}^N \alpha_k x_k.
\end{aligned}
$$

It implies

$$
\operatorname*{argmin}_{y\in E} \{\alpha\phi(u, Jy) + (1-\alpha)\phi(v, Jy)\} = \alpha u + (1-\alpha)v
$$

for every $u, v \in E$ and $\alpha \in \mathbb{R}$.

Consequently, we can replace the definition of affine combinations by using minimizers of some functions shown as above.

4 Balanced mapping

In this section, we consider balanced mappings for a finite family of nonexpansive-type mappings in Banach space. A concept of the balanced mappings was introduced by Hasegawa and Kimura [2] for approximation of common fixed points in Hadamard spaces.

Theorem 4.1 (Hasegawa and Kimura [2]). *Let (X, d) be a Hadamard space and $N \in \mathbb{N}$. Let $\{T_k \mid k \in \{1, \dots, N\}\}$ be a finite family of nonexpansive mappings from X into* *itself and* $\{\alpha_k \mid k \in \{1, \dots, N\}\}$ *a finite real sequence of* $]0,1[$ *such that* $\sum_{k=1}^N \alpha_k = 1$. *For each* $x \in X$*, we define a subset* Ux *of* X *by*

$$Ux = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha_k d(y, T_k x)^2.$$

Then, the following hold:

- Ux *is a singleton for every* $x \in X$ *and therefore* U *is defined as a single-valued mapping from* X *into itself;*
- U *is nonexpansive;*
- *if the set* $\bigcap_{k=1}^N \operatorname{Fix} T_k$ *of common fixed points is nonempty, then* $\bigcap_{k=1}^N \operatorname{Fix} T_k = \operatorname{Fix} U$ *holds.*

In what follows, we introduce two balanced mappings, and show the properties of them like Theorem 4.1.

Let E be a smooth, strictly convex and reflexive Banach space with its dual E^* and C a nonempty closed convex subset of E. Let $N \in \mathbb{N}$ and $\{T_k \mid k \in \{1, \dots, N\}\}$ a finite family of mappings from C into E. Let $\{\alpha_k \mid k \in \{1, \dots, N\}\}$ be a finite real sequence of $]0,1[$ such that $\sum_{k=1}^N \alpha_k = 1$. We define a mapping $U \colon C \to E$ as

$$Ux = \operatorname*{argmin}_{y \in E} \sum_{k=1}^{N} \alpha_k \phi(y, JT_k x) = J^* \left(\sum_{k=1}^{N} \alpha_k JT_k x \right)$$

for each $x \in C$, where $J \colon E \to E^*$ and $J^* \colon E^* \to E$ are the normalized duality mappings. We call such a mapping U the *balanced mapping* for $\{\alpha_k\}$ and $\{T_k\}$. We also define a mapping $V \colon C \to E$ as

$$Vx = \operatorname*{argmin}_{y \in E} \sum_{k=1}^{N} \alpha_k \phi(T_k x, Jy) = \sum_{k=1}^{N} \alpha_k T_k x$$

for each $x \in C$, where $J \colon E \to E^*$ is the normalized duality mapping. We call such a mapping V the *dual-balanced mapping* for $\{\alpha_k\}$ and $\{T_k\}$.

Particularly, if such mappings are relatively quasinonexpansive, we obtain the following theorems:

Theorem 4.2. *Let* E *be a smooth, strictly convex and reflexive Banach space with its dual* E^* *and* C *a nonempty closed convex subset of* E*. Let* $N \in \mathbb{N}$ *and let* $\{T_k \mid k \in \{1, \dots, N\}\}$ *be a finite family of relatively quasinonexpansive mappings from* C *into* E*. Let* $\{\alpha_k \mid k \in \{1, \dots, N\}\}$ *be a finite real sequence of* $]0,1[$ *such that* $\sum_{k=1}^N \alpha_k = 1$*. Let* U *be the balanced mapping from* C *into* E *for* $\{\alpha_k\}$ *and* $\{T_k\}$*. If the set* $\bigcap_{k=1}^N \operatorname{Fix} T_k$ *of common fixed pints is nonempty, then* $\bigcap_{k=1}^N \operatorname{Fix} T_k = \operatorname{Fix} U$ *holds.*

***Proof*.** Let $z \in \bigcap_{k=1}^N \operatorname{Fix} T_k$. Then, it follows that

$$\begin{aligned} Uz &= \operatorname*{argmin}_{y \in E} \sum_{k=1}^N \alpha_k \phi(y, JT_k z) \\ &= \operatorname*{argmin}_{y \in E} \sum_{k=1}^N \alpha_k \phi(y, Jz) = \operatorname*{argmin}_{y \in E} \phi(y, Jz) = z. \end{aligned}$$

It implies $Uz = z$ and $z \in \operatorname{Fix} U$. Therefore, we obtain $\bigcap_{k=1}^N \operatorname{Fix} T_k \subset \operatorname{Fix} U$.

Conversely, let $z \in \operatorname{Fix} U$. From Theorem 3.2, for $u \in \bigcap_{k=1}^N \operatorname{Fix} T_k$, we have

$$\begin{aligned} \phi(u, Jz) = \phi(u, JUz) &= \phi\left(u, \sum_{k=1}^N \alpha_k JT_k z\right) \\ &= \sum_{k=1}^N \alpha_k \phi(u, JT_k z) - \sum_{k=1}^N \alpha_k \phi(z, JT_k z) \\ &\leq \phi(u, Jz) - \sum_{k=1}^N \alpha_k \phi(z, JT_k z), \end{aligned}$$

which implies

$$0 \leq \sum_{k=1}^N \alpha_k \phi(z, JT_k z) \leq \phi(u, Jz) - \phi(u, Jz) = 0.$$

Since $\alpha_k \neq 0$ for every $k \in \{1, \ldots, N\}$, $z \in \bigcap_{k=1}^N \operatorname{Fix} T_k$ holds. Therefore, we have $\operatorname{Fix} U \subset \bigcap_{k=1}^N \operatorname{Fix} T_k$.

Consequently, we obtain $\bigcap_{k=1}^N \operatorname{Fix} T_k = \operatorname{Fix} U$. □

Theorem 4.3. *Let E be a smooth, strictly convex and reflexive Banach space with its dual E^* and C a nonempty closed convex subset of E. Let $N \in \mathbb{N}$ and let $\{T_k \mid k \in \{1, \ldots, N\}\}$ be a finite family of relatively quasinonexpansive mappings from C into E such that the set $\bigcap_{k=1}^N \operatorname{Fix} T_k$ of common fixed points is nonempty. Let $\{\alpha_k \mid k \in \{1, \ldots, N\}\}$ be a finite real sequence of $]0,1[$ such that $\sum_{k=1}^N \alpha_k = 1$. Let U be the balanced mapping from C into E for $\{\alpha_k\}$ and $\{T_k\}$. Then, it holds that*

$$\phi(z, JUx) + \sum_{k=1}^N \alpha_k \phi(Ux, JT_k x) \leq \phi(z, Jx)$$

for every $x \in C$ and $z \in \operatorname{Fix} U$, where $J \colon E \to E^*$ is the normalized duality mapping.

Proof. Let $x \in C$, $z \in \operatorname{Fix} U = \bigcap_{k=1}^{N} \operatorname{Fix} T_k$. From Theorem 3.2, we have

$$\begin{aligned}\phi(z, JUx) = \phi\left(z, \sum_{k=1}^{N} \alpha_k JT_k x\right)\\ &= \sum_{k=1}^{N} \alpha_k \phi(z, JT_k x) - \sum_{k=1}^{N} \alpha_k \phi\left(J^*\left(\sum_{k=1}^{N} \alpha_k JT_k x\right), JT_k z\right)\\ &= \sum_{k=1}^{N} \alpha_k \phi(z, JT_k x) - \sum_{k=1}^{N} \alpha_k \phi(Ux, JT_k z),\end{aligned}$$

where $J^* \colon E^* \to E$ is the nomalized duality mapping. Since T_k is relatively quasinonexpansive for every $k \in \{1, \dots, N\}$, we obtain

$$\phi(z, JUx) \leq \phi(z, Jx) - \sum_{k=1}^{N} \alpha_k \phi(Ux, JT_k z)$$

and this is the desired result. □

As direct consequence of Theorem 4.3, we know that the balanced mapping for relatively quasinonexpansive mappings is also relatively quasinonexpansive.

In the same way as Theorem 4.2 and Theorem 4.3, we obtain the following results for generalized quasinonexpansive mappings:

Theorem 4.4. *Let E be a smooth, strictly convex and reflexive Banach space with its dual E^* and C a nonempty closed convex subset of E. Let $N \in \mathbb{N}$ and let $\{T_k \mid k \in \{1, \dots, N\}\}$ be a finite family of generalized quasinonexpansive mappings from C into E. Let $\{\alpha_k \mid k \in \{1, \dots, N\}\}$ be a finite real sequence of $]0, 1[$ such that $\sum_{k=1}^{N} \alpha_k = 1$. Let V be the dual-balanced mapping from C into E for $\{\alpha_k\}$ and $\{T_k\}$. If the set $\bigcap_{k=1}^{N} \operatorname{Fix} T_k$ of common fixed points is nonempty, then $\bigcap_{k=1}^{N} \operatorname{Fix} T_k = \operatorname{Fix} V$ holds.*

Theorem 4.5. *Let E be a smooth, strictly convex and reflexive Banach space with its dual E^* and C a nonempty closed convex subset of E. Let $N \in \mathbb{N}$ and let $\{T_k \mid k \in \{1, \dots, N\}\}$ be a finite family of generalized quasinonexpansive mappings from C into E such that the set $\bigcap_{k=1}^{N} \operatorname{Fix} T_k$ of common fixed points is nonempty. Let $\{\alpha_k \mid k \in \{1, \dots, N\}\}$ be a finite real sequence of $]0, 1[$ such that $\sum_{k=1}^{N} \alpha_k = 1$. Let V be the dual-balanced mapping from C into E for $\{\alpha_k\}$ and $\{T_k\}$. Then, it holds that*

$$\phi(Vx, Jz) + \sum_{k=1}^{N} \alpha_k \phi(T_k x, JVx) \leq \phi(x, Jz)$$

for every $x \in C$ and $z \in \operatorname{Fix} V$, where $J \colon E \to E^*$ is the normalized duality mapping.

As direct consequence of Theorem 4.5, we know that the dual-balanced mapping for generalized quasinonexpansive mappings is also generalized quasinonexpansive.

In a Hadamard space, it is only defined that a convex combination for two points. Balanced mappings enable us to consider combinations for more points. Moreover, the set of fixed points of balanced mapping coincides with common fixed point set of considered mappings. We can use this fact for common fixed points approximation. Balanced mappings are also defined in CAT(1) spaces and CAT(-1) spaces, respectively, and common fixed points approximation theorems are showed by using them. For more details about the balanced mappings in CAT(1) spaces and CAT(-1) spaces, see, for instance, [4, 6].

Acknowlegement. This work was partially supported by JSPS KAKENHI Grant Number JP21K03316.

References

[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, 15–50.

[2] T. Hasegawa and Y. Kimura, *Convergence to a fixed point of a balanced mapping by the Mann algorithm in a Hadamard space*, Linear and Nonlinear Anal. **4** (2018) 405–412.

[3] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[4] T. Kajimura, K. Kasahara, Y. Kimura and K. Nakagawa, *Balanced mappings and an iterative scheme in complete geodesic space*, J. Nonlinear Anal. Optim. **12** (2021), 1–11.

[5] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.

[6] K. Sasaki, *Halpern type and Mann type fixed point approximation methods using multiple anchor points or balanced mappings on geodesic spaces*, Master Thesis, Toho University, 2021.

[7] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.