Weak and strong convergence theorems for monotone nonexpansive mappings

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Abstract

In this paper, we show nonlinear mean convergence theorems for two monotone nonexpansive mappings in ordered uniformly convex Banach spaces. We also show some convergence theorems for the mappings.

1 Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping T of C into E, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$. Let T be a mapping of C into itself. A mapping T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 1975, Baillon [2] proved the following first nonlinear mean convergence theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$,

$$\{S_n x\} = \left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\}$$

converges weakly to a fixed point of T (see also [13]).

Ran and Reurings [11] proved an analogue of the classical Banach contraction principle in a partially ordered metric space. Dehaish and Khamsi [8] proved a convergence theorem by Mann type iteration [9] for a monotone nonexpansive mapping in an ordered Banach

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spaces. Shukla and Wiśnicki [12] proved nonlinear ergodic theorems for a monotone nonexpansive mapping in an ordered Banach space.

In this paper, we show nonlinear mean convergence theorems for two monotone nonexpansive mappings in ordered uniformly convex Banach spaces. We also show some convergence theorems for the mappings.

2 Preliminaries and notations

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x+z \preceq y+z,$$

$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x,y,z \in E$ and $\lambda \geq 0$.
As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x, \rightarrow] = \{z \in E : x \preceq z\}$ and $[\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x, \rightarrow]$ and $[\leftarrow, y]$ are closed. Recall that an order interval is any of the subsets

$$[a, \rightarrow] = \{x \in X : a \preceq x\} \quad \text{or} \quad [\leftarrow, a] = \{x \in X : x \preceq a\}.$$

for any $a \in E$. As a direct consequence of this, the subset $[a,b] = \{x \in X : a \preceq x \preceq b\} = [a, \rightarrow] \cap [\leftarrow, b]$ is also closed and convex for each $a,b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E. Let C be a nonempty subset of E. Let T be a mapping of C into itself. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x,y \in C$. A mapping T is called *monotone* if

$$Tx \preceq Ty$$

for each $x,y \in C$ such that $x \preceq y$. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \; : \; \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Let $S_E = \{x \in E : \|x\| = 1\}$ be a unit sphere in a Banach space E.

3 Monotone and approximating fixed point sequences

In this section, we deal with approximating fixed point sequences and monotone sequences. Let C be a nonempty subset of E and let T be a mapping of C into itself. The mapping T is said to be *demiclosed* if for any sequence $\{x_n\}$ in C the following implication hold:

$$\text{w-}\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \|Tx_n - y\| = 0$$

imply that

$$Tx = y$$

(see [6]).

Theorem 3.1 ([6])**.** *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E. Let T be a nonexpansive mapping of C into itself and let I be the identity mapping. Then, $I - T$ is demiclosed at* 0*, that is,*

$$\text{w-}\lim_{n\to\infty} x_n = x \textit{ and } \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

imply that

$$Tx = x.$$

A sequence $\{x_n\}$ in C is said to be an *approximating fixed point sequence* of a mapping T if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(see also [10, 13]). A sequence $\{x_n\}$ in E is said to be monotone if

$$x_1 \preceq x_2 \preceq x_3 \preceq, \dots$$

(see also [8]).

Lemma 3.2. *Let C be a bounded convex subset of an ordered uniformly convex Banach space E. Let T be a monotone nonexpansive mapping of C into itself. Let $\{u_n\}$ and $\{v_n\}$ be approximate fixed point sequences of T such that $u_n \preceq v_n$ for each $n \in \mathbb{N}$. Let $w_n = \frac{1}{2}(u_n + v_n)$ for each $n \in \mathbb{N}$. Then, the sequence $\{w_n\}$ is an approximate fixed point sequence too.*

(see also [13]).

The following result plays an important role in the proof of Theorem 4.1 (see [1]).

Theorem 3.3 ([1]). *Let C be a closed convex subset of an ordered uniformly convex Banach space E and let S and T be monotone nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a monotone approximating fixed point sequence for T and S, i.e.,*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Then, then the sequence $\{x_n\}$ converges weakly to $z_0 \in F(S) \cap F(T)$.

4 Nonlinear mean convergence theorems

In this section, we show weak and strong mean convergence theorems for monotone nonexpansive mappings. Using Lemma 3.3, we can prove a weak mean convergence theorem for monotone nonexpansive mappings in an ordered uniformly convex Banach space (see [1]).

Theorem 4.1 ([1]). *Let C be a closed convex subset of an ordered uniformly convex Banach space E and let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Then,*

$$\{S_n x\} = \left\{ \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x \right\}$$

converges weakly to $z_0 \in F(S) \cap F(T)$.

We can prove a strong mean convergence theorem for monotone nonexpansive mappings with compact domains (see [1]).

Theorem 4.2 ([1]). *Let C be a compact convex subset of an ordered strictly convex space E and let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Then,*

$$\{S_n x\} = \left\{ \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x \right\}$$

converges strongly to $z_0 \in F(S) \cap F(T)$.

Using Theorem 4.1, we get some convergence theorems for monotone nonexpansive mappings in ordered uniformly convex Banach spaces (see [12]).

Theorem 4.3 ([12]). *Let C be a closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{S_n x\} = \{\frac{1}{n} \sum_{k=0}^{n-1} T^k x\}$ converges weakly to $z_0 \in F(T)$.*

Theorem 4.4 ([12]). *Let C be a closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{T^n x\}$ converges weakly to $z_0 \in F(T)$.*

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