Gravitational Collapse for Newtonian Stars

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Abstract

We report on a recent mathematical development in gravitational collapse for Newtonian stars governed by the Euler-Poisson system, which shows the existence of smooth initial data that lead to finite time gravitational collapse, characterised by the blow-up of the star density. We discuss two distinct regimes describing dust-like collapse and self-similar collapse respectively.

Contents

1 Introduction **2**
1.1 Lane-Emden steady stars 2
1.2 Scaling symmetry 3
1.3 Goldreich-Weber (GW) stars 4
1.4 Dispersion via expansion 4
1.5 Gravitational collapse in the mass super-critical regime 5

2 Dust-like collapse for $1 < \gamma < \frac{4}{3}$ **6**
2.1 Rescaling and Lagrangian formulation 6
2.2 Methodology 8

3 Self-similar collapse for $1 \leq \gamma < \frac{4}{3}$ **10**
3.1 Self-similar Euler-Poisson system and the sonic point - the isothermal case . . 11
3.2 Proof ideas 12
3.3 Self-similar Euler-Poisson system and the sonic point - the polytropic case . . 14

4 Some open problems **14**

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1 Introduction

A classical model of a self-gravitating Newtonian star is given by the gravitational Euler-Poisson system:

$$\partial_t \rho + \text{div}\,(\rho \mathbf{u}) = 0, \tag{1.1a}$$

$$\rho\,(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = -\rho\nabla\Phi, \tag{1.1b}$$

$$\Delta\Phi = 4\pi\rho, \quad \lim_{|\mathbf{x}|\to\infty} \Phi(t,\mathbf{x}) = 0, \tag{1.1c}$$

where $\rho(t,\mathbf{x}) \geq 0$ is the density, $\mathbf{u}(t,\mathbf{x}) \in \mathbb{R}^3$ is the velocity, $p(t,\mathbf{x}) \geq 0$ is the pressure of the gas and $\Phi(t,\mathbf{x})$ is the gravitational potential. We consider the polytropic equation of state given by

$$p = \rho^\gamma, \quad 1 \leq \gamma < 2, \tag{1.2}$$

where the value of γ is commonly referred to the adiabatic exponent and $\gamma = 1$ represents the isothermal gas.

The dynamics of stars described by (1.1) is broadly speaking characterised by the antagonism between the expansive nature of the pressure and the contractive nature of the gravitational field. Depending on their relative strength, we expect to encounter different dynamic scenarios.

- *Finite-time gravitational collapse.* This behaviour is characterised by the finite-time blow-up of the gas density starting from "regular" initial data. In the context of compressible fluid mechanics this scenario is sometimes referred to as *implosion*. It is clear that this type of singular behaviour is not a shock singularity. We present new results in this direction in Sections 2 and 3.
- *Global-in-time existence driven by dispersion/expansion.* In this setting the pressure "wins" over the focusing effects of gravity. We give below open classes of initial data that lead to global existence-via-expansion for a large class of polytropic indices γ, see the discussion in Sections 1.3–1.4.
- *Equilibria, periodic/quasiperiodic solutions, coherent structures.* Beyond the existence of steady state equilibria, not much is known about other spatially localised global-in-time structures. We briefly review the known results about the (radially symmetric) steady states in Section 1.1 below.

1.1 Lane-Emden steady stars

The most famous solutions to the Euler-Poisson system (1.1) with (1.2) are the so-called Lane-Emden steady stars [3]. They are static (nonmoving), radially symmetric equilibrium solutions of (1.1)–(1.2) of the form $\rho = \bar{\rho}(r)$, $\mathbf{u} = \mathbf{0}$ where $r = |\mathbf{x}|$ and the enthalpy $w := c\rho^{\gamma-1}$, where c is a normalization constant, satisfies the following Lane-Emden equation

$$w'' + \frac{2}{r}w' + \pi w^{\frac{1}{\gamma-1}} = 0. \tag{1.3}$$

It is well-known [3, 42, 19] that for any $\gamma \in (\frac{6}{5}, 2)$ there exists a compactly supported steady solution to (1.3) with the boundary conditions $w'(0) = w(1) = 0$. The associated steady state density $\bar{\rho}$ has finite mass and compact support and therefore represents a steady star [3, 42]. The dynamic stability question of Lane-Emden stars is a classical topic. Standard linear stability arguments [23] reveal the following dichotomy in the stability behavior of the above family of steady stars:

$$
\begin{aligned}
&\text{if } \frac{6}{5} < \gamma < \frac{4}{3}, \text{ the steady state } (\bar{\rho}, \mathbf{0}) \text{ is linearly unstable;} \\
&\text{if } \frac{4}{3} \leq \gamma < 2, \text{ the steady state } (\bar{\rho}, \mathbf{0}) \text{ is linearly stable.}
\end{aligned}
$$

The case $\gamma = \frac{4}{3}$ admits 0 as the first eigenvalue and it is often referred to as neutrally stable. Under the assumption that a global-in-time solution exists, nonlinear stability of Lane-Emden steady stars in the range $\frac{4}{3} < \gamma < 2$ has been shown by variational arguments in an energy-based topology, see [34]. Nonlinear instability in the range $\frac{6}{5} \leq \gamma < \frac{4}{3}$ has been rigorously established in [18, 19]. In this case, the instability is induced by the existence of a growing mode in the linearized operator, while there are no such modes when $\frac{4}{3} \leq \gamma < 2$. In fact, Lane-Emden stars in the case $\gamma = \frac{4}{3}$ are nonlinearly unstable despite the absence of growing modes in the linearized operator [5, 14].

1.2 Scaling symmetry

Two values $\gamma = \frac{4}{3}$ and $\gamma = \frac{6}{5}$ have a special role in the existence and stability theory of the steady states. They are intimately tied to the scaling symmetry of the Euler-Poisson system (1.1)–(1.2). If $(\rho, \mathbf{u})$ is a solution of (1.1)–(1.2), so is the pair $(\tilde{\rho}, \tilde{\mathbf{u}})$ defined by

$$
\rho(t, \mathbf{x}) = \lambda^{-\frac{2}{2-\gamma}} \tilde{\rho}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{\mathbf{x}}{\lambda}\Big), \qquad \mathbf{u}(t, \mathbf{x}) = \lambda^{-\frac{\gamma-1}{2-\gamma}} \tilde{\mathbf{u}}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{\mathbf{x}}{\lambda}\Big), \tag{1.4}
$$

for any $\lambda > 0$. The associated pressure $\tilde{p}$ and the gravitational potential $\tilde{\Phi}$ relate to p and Φ via:

$$
p(t, \mathbf{x}) = \lambda^{-\frac{2\gamma}{2-\gamma}} \tilde{p}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{\mathbf{x}}{\lambda}\Big), \qquad \Phi(t, \mathbf{x}) = \lambda^{-\frac{2\gamma-2}{2-\gamma}} \tilde{\Phi}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{\mathbf{x}}{\lambda}\Big).
$$

It is easy to check the changes of the mass and energy under the self-similar rescaling (1.4):

$$
M(\rho) = \lambda^{\frac{4-3\gamma}{2-\gamma}} M(\tilde{\rho}), \qquad E(\rho, \mathbf{u}) = \lambda^{\frac{6-5\gamma}{2-\gamma}} E(\tilde{\rho}, \tilde{\mathbf{u}}), \tag{1.5}
$$

where the total mass is given by

$$
M(\rho) := \int_{\mathbb{R}^3} \rho \, d\mathbf{x}, \tag{1.6}
$$

and energy

$$
E(\rho, \mathbf{u}) := \int_{\mathbb{R}^3} \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}\rho\Phi + \frac{1}{\gamma-1}\rho^\gamma \right] d\mathbf{x}. \tag{1.7}
$$

The total mass M is invariant under the self-similar rescaling (1.4) when $\gamma = \frac{4}{3}$ and the energy E is invariant when $\gamma = \frac{6}{5}$, so we refer to the cases $\gamma = \frac{4}{3}$ and $\gamma = \frac{6}{5}$ as the *mass-critical* and the *energy-critical* case respectively.

As we shall see, the criticality and scaling symmetry play an important role not only in the study of Lane-Emden stars but also in global dynamics of other physically motivated solutions.

1.3 Goldreich-Weber (GW) stars

It turns out that the classical hydrodynamic model (1.1) for Newtonian self-gravitating gases admits a finite-parameter family of time-dependent solutions in the mass-critical case $\gamma = \frac{4}{3}$. The mass-criticality implies that an effective separation-of-variables ansatz is possible, which decouples the spatial- and the time-dynamics. Such solutions were discovered by Goldreich and Weber [9] in 1980 and they take the form:

$$\rho(t,\mathbf{x}) = \lambda(t)^{-3} w_\delta^3(\lambda(t)^{-1}\mathbf{x}), \quad \mathbf{u}(t,\mathbf{x}) = \dot{\lambda}(t)\lambda(t)^{-1}\mathbf{x} \tag{1.8}$$

where

$$\ddot{\lambda}\lambda^2 = \delta, \quad \lambda(-1) = \lambda_0 > 0, \quad \dot{\lambda}(-1) = \lambda_1 \tag{1.9}$$

and $w_\delta : [0,1] \to \mathbb{R}_+$ is a non-negative enthalpy function solving the generalized Lane-Emden equation:

$$w'' + \frac{2}{z}w' + \pi w^3 = -\frac{3}{4}\delta \quad \text{in} \quad [0,1], \;\; w'(0) = 0, \;\; w(1) = 0. \tag{1.10}$$

These special solutions were originally referred to as *homologous* solutions, but it is important to note that they are a particular example of so-called affine motions discussed further in Section 1.4. In the mathematics community, they were independently discovered and rigorously analyzed by Makino [25], Fu and Lin [8] and in the nonisentropic case by Deng, Xiang and Yang [6].

We see from the formulas (1.8)–(1.9) that the fixed density profile $\rho_\delta = w_\delta^3$ is modulated by the time-dependent function $\lambda(t)$. Stellar implosion/collapse then corresponds to the finite-time vanishing of $\lambda(t)$ and the collapse is homologous in the sense that all the gas content shrinks to the singularity instantaneously. By contrast, if $\lambda(t)$ increases indefinitely, the density ρ decays to 0, and we see that the star expansion (the growth of $\lambda(t)$) and dispersion (the decay of ρ) go hand-in-hand. Interestingly, there exist two distinct rates in the ODE dynamics both in the case of expansion and in the case of collapse: self-similar rate $\lambda(t) \sim_{t\to\infty} t^{\frac{2}{3}}$ and linear rate $\lambda(t) \sim_{t\to\infty} t$ in the expansion regime and $\lambda(t) \sim_{t\to 0^-} (-t)^{\frac{2}{3}}$ and $\lambda(t) \sim_{t\to 0^-} (-t)$ in the collapse regime.

Besides the remarkable fact that the above solutions exist, they lead to the physically important question of understanding their stability. It is already suggested in the above discussion that the mechanism for the stability of the expanding GW motions is in some sense already covertly present in the formulas, and we elaborate on this in Section 1.4. The nonlinear stability of the collapsing GW-motions, even under the assumption of radial symmetry, is an open question.

1.4 Dispersion via expansion

The GW stars form a finite-parameter family of solutions, characterized by the parameter $\delta \in \mathbb{R}$ and the initial velocity $\lambda_1 \in \mathbb{R}$. When the total energy E is strictly positive, the result of Makino and Perthame [26] suggests a possibility of globally defined expanding solutions, however the result does not apply to the case $E = 0$. In [14], we have shown the nonlinear radial stability results for the expanding GW-solutions, both the ones expanding at the linear rate $\sim t$ and the ones expanding at the self-similar rate $\sim t^{\frac{2}{3}}$. In the latter case, the perturbations are restricted to have the same energy as the expanding background GW-solution. Therefore, when we work with zero-energy perturbations ($E = 0$) we obtain a co-dimension one stability result, assuming that $\delta < 0$ is sufficiently close to 0. In the case of the linearly expanding GW-stars, these restrictions are not necessary.

As mentioned above, the GW-solutions are in fact a special class of the so-called affine solutions, which have a long tradition in the broader context of fluid mechanics. By definition, an affine motion has both the flow map and the velocity field governed by some time-dependent linear map $t \mapsto A(t)$:

$$\zeta_A(t, \mathbf{y}) = A(t)\mathbf{y}, \ \ \mathbf{u}(t, \mathbf{y}) = \dot{A}(t)A^{-1}(t)\mathbf{y}, \ \ A(t) \in \mathrm{GL}^+(3). \tag{1.11}$$

The above introduced GW-motions (1.8)–(1.9) fit into this framework by letting $A(t) = \lambda(t)I_{3\times 3}$. In the absence of gravity, a family of global-in-time affine solutions surrounded by vacuum to compressible Euler system were constructed by Sideris [39], while the ideas of affine motions and ODE solutions can be tracked back to earlier works by Ovsyannikov [30] and Dyson [7].

For the compressible Euler affine flows, A needs not to be diagonal in the case of the Sideris flows [39] - in fact the motion is supported on ellipsoids. However, these affine motions are global-in-time and eventually expand (i.e. the support of the solutions eventually grows "in all directions"), which should be contrasted to GW affine motions. This highlights the special role of gravity, which can cooperate with affine ansatz to generate imploding solutions. Global-in-time stability of Sideris' affine motion has been shown by the authors [15] for $1 < \gamma \leq \frac{5}{3}$, Shkoller and Sideris [37] for $\gamma > \frac{5}{3}$. See also [36, 35] for stability of non-isentropic affine motions.

As it turns out, the presence of exact expanding affine motions is in itself not crucial to the existence of open classes of data leading to global existence, but are instead suggestive of a more fundamental stabilisation mechanism. As the support of the fluid spreads out, this acts as a defocusing mechanism preventing for example shock formation - one of the main enemies of global well-posedness in the context of compressible fluids. Motivated by this observation, it is possible to prove the existence of global-in-time expanding solutions assuming that the density is suitably small and the velocity suitably "outward pointing", and thereby not relying on the existence of exact background affine solutions, see [32]. For the Euler-Poisson system (in both the gravitational and the electrostatic case), small perturbations of Sideris affine solutions can launch global-in-time solutions [16]. See also [31] for the application of these ideas to the existence of globally expanding N body solutions of the the Euler-Poisson system.

1.5 Gravitational collapse in the mass super-critical regime

We now turn our attention to collapsing solutions. The Euler-Poisson system (1.1)-(1.2) admits finite-time blowup solutions such as the Goldreich and Weber collapsing solutions when $\gamma = \frac{4}{3}$ describing gravitational collapse, where the density blows up on approach to the singularity. For $\gamma > \frac{4}{3}$, the mass subcritical regime, it was shown in [5] that the collapse by density concentration cannot occur. See a recent work [4] for global radial weak solutions with finite energy for $\gamma > \frac{4}{3}$.

A natural, interesting question is then the existence or nonexistence of density blowup solutions in the super-critical regime for $1 \leq \gamma < \frac{4}{3}$. The super-criticality is supposed to be even more "encouraging" of the collapse. From the physics point of view, the lower the polytropic index γ, the heavier the gas molecules, so that gravitational forces are greater. Despite its physical importance, the rigorous proof of Newtonian gravitational collapse in the super-critical regime has not been given until recently.

For the rest of the paper, we discuss mathematical construction of gravitationally collapsing radial solutions to (1.1)–(1.2) obtained in [11, 12, 13]. In Section 2, we discuss dust-like collapse in the mass-supercritical regime $1 < \gamma < \frac{4}{3}$ where we can construct an infinite dimensional family of collapsing solutions. In Section 3, we discuss the self-similar collapse in the supercritical regime $1 \leq \gamma < \frac{4}{3}$. Section 4 contains some remarks on future works and related problems.

We point out that in the case of compressible Euler flows, collapsing solutions cannot be realized as affine motions, but recent breakthrough works [27, 28] by Merle, Raphaël, Rodnianski, and Szeftel show that through the self-similar blowup, collapsing (imploding) solutions do exist for compressible Euler and Navier-Stokes equations starting out from smooth initial data.

2 Dust-like collapse for $1 < \gamma < \frac{4}{3}$

The results in this section are motivated by the simple question of trying to find at least some examples of collapsing solutions in the supercritical regime. A simple separation-of-variables like in the mass-critical case simply fails when $\gamma < \frac{4}{3}$. One natural possibility however is to construct solutions that in some sense mimic the behaviour of solutions to an a priori simpler problem - the pressureless (or dust-) Euler-Poisson system, which is well-known to admit collapsing solutions. We see immediately that there is a serious obstacle to such an attempt - the pressure always enter the problem at the highest level of regularity, and it is not clear in what sense we can hope to treat the pressure as a structural "perturbation" away from the pressureless problem. A hint that a result in this direction may be possible comes from a suitable scaling analysis.

2.1 Rescaling and Lagrangian formulation

For any $\bar{\varepsilon} > 0$ consider the mass preserving rescaling applied to the Euler-Poisson system:

$$\rho = \bar{\varepsilon}^{-3}\tilde{\rho}(s, \mathbf{y}), \quad \mathbf{u} = \bar{\varepsilon}^{-1/2}\tilde{\mathbf{u}}(s, \mathbf{y}), \quad \Phi = \bar{\varepsilon}^{-1}\tilde{\Phi}(s, \mathbf{y}), \tag{2.12}$$

where

$$s = \bar{\varepsilon}^{-3/2}t, \quad \mathbf{y} = \bar{\varepsilon}^{-1}\mathbf{x}.$$

It is easy to see that the above rescaling is mass-critical, i.e. $M[\rho] = M[\tilde{\rho}]$. A simple calculation reveals that if $(\rho, \mathbf{u}, \Phi)$ solve (1.1)–(1.2), then the rescaled quantities $(\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\Phi})$ solve

$$\partial_s\tilde{\rho} + \mathrm{div}\,(\tilde{\rho}\tilde{\mathbf{u}}) = 0, \tag{2.13a}$$

$$\tilde{\rho}\,(\partial_s\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}) + \varepsilon\nabla(\tilde{\rho}^{\gamma}) + \tilde{\rho}\nabla\tilde{\Phi} = 0, \tag{2.13b}$$

$$\Delta\tilde{\Phi} = 4\pi\,\tilde{\rho}, \quad \lim_{|x|\to\infty}\tilde{\Phi}(t, x) = 0, \tag{2.13c}$$

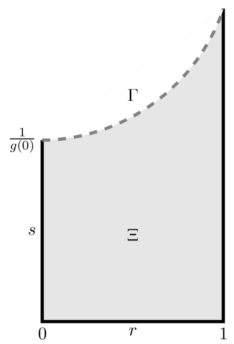

Figure 1: Dust collapse in Lagrangian coordinates

where

$$\varepsilon := \bar{\varepsilon}^{4-3\gamma}.$$

We observe that for $\bar{\varepsilon} \ll 1$ the factor ε in front of the pressure in (2.13b) is small precisely in the supercritical range $1 < \gamma < \frac{4}{3}$. The system obtained by dropping the ε-term in (2.13b) is known as the pressureless- or dust-Euler system.

Let η be the flow map associated with $\tilde{\mathbf{u}}$ of the rescaled system defined on the unit ball, which is a reference domain. For radially symmetric flows, the flow map η takes the form of

$$\eta(s, \mathbf{y}) := \chi(s, r)\mathbf{y}, \quad r = |\mathbf{y}|, \quad r \in [0, 1], \tag{2.14}$$

and the Euler-Poisson system reduces to a nonlinear second order degenerate hyperbolic equation for χ [11]

$$\chi_{ss} + \frac{G(r)}{\chi^2} + \epsilon P[\chi] = 0, \quad P[\chi] = \frac{\chi^2}{w^\alpha r^2}(r\partial_r)\left(w^{1+\alpha}\mathscr{J}[\chi]^{-\gamma}\right) \tag{2.15}$$

where $G(r) = \frac{1}{r^3}\int_0^r 4\pi w^\alpha \tau^2 d\tau$, $w^\alpha = \tilde{\rho}_0(\chi_0(r)r)\mathscr{J}[\chi_0](r)$ and the Jacobian $\mathscr{J}[\chi] = \chi^2(\chi + r\partial_r\chi)$. Here $\alpha = \frac{1}{\gamma-1}$. In this formulation, the collapse is characterised by the vanishing of χ and $\mathscr{J}$, which in turn implies the blowup of the Eulerian density because $\rho(s, \chi(s, r))\mathscr{J}[\chi(s, r)] = w^\alpha$ holds due to the continuity equation. Therefore, our goal is to show that there exists a spacetime point (s^*, r^*) such that $\chi(s^*, r^*) = \mathscr{J}(s^*, r^*) = 0$.

In the absence of the pressure, solutions do collapse. For instance, a pressureless collapse solutions satisfying $\chi_{ss} + \frac{G(r)}{\chi^2} = 0$ is given by

$$\chi_{\text{dust}}(s, r) = (1 - g(r)s)^{\frac{2}{3}}, \quad g(r) = 3\sqrt{\frac{G(r)}{2}} \tag{2.16}$$

which vanishes along the space-time curve $\Gamma := \{(s, r)|1 - g(r)s = 0\}$. Dynamics in the presence of pressure is much more involved as it can be seen from (2.15) the pressure enters the equation at the top order in terms of the derivative count. In [11], we showed the existence of solutions χ to (2.15) whose leading order behavior is described by the dust collapse (2.16).

Theorem 2.1. *For any* $\gamma \in (1, \frac{4}{3})$ *there exist classical solutions* $\chi(s,r)$ *of* (2.15) *defined in* $\Xi = \{(s,r) \,|\, 1 - g(r)s > 0\}$. *The solution behaves qualitatively like the collapsing dust solution* χ_{dust} *and in particular*

$$1 \lesssim \frac{\chi}{\chi_{dust}} \lesssim 1, \quad 1 \lesssim \frac{\mathscr{J}[\chi]}{\mathscr{J}[\chi_{dust}]} \lesssim 1, \quad (s,r) \in \Xi. \tag{2.17}$$

Further, for any $r \in [0,1]$

$$\lim_{s \to \frac{1}{g(r)}} \frac{\chi}{\chi_{dust}} = \lim_{s \to \frac{1}{g(r)}} \frac{\mathscr{J}[\chi]}{\mathscr{J}[\chi_{dust}]} = 1. \tag{2.18}$$

In Theorem 2.1, the enthalpy profile w and therefore the initial density ρ_0 are free to be chosen as long as they satisfy the following geometric conditions: (i) w is smooth and it satisfies the physical vacuum condition [20, 21]: $w \in C^\infty(0,1)$, $w > 0$ on $[0,1)$, $w(1) = 0$ and $w'(1) < 0$, (ii) g is monotonically decreasing on $[0,1]$, and (iii) for a sufficiently large natural number $n \in \mathbb{N}$

$$g(r) = g(0) - \frac{c}{n} r^n + o_{r \to 0}(r^n). \tag{2.19}$$

In particular the expansion (2.19) measures the flatness of the initial density ρ near the centre and, as we shall see below, the choice of a sufficiently high $n = n(\gamma)$ is a key geometric condition to ensure the leading order approximation by the dust solution.

The fragmented collapse described by (2.16) implies that a particle with a label r is absorbed into the singularity at time $t^*(r) = \frac{1}{g(r)}$. Since g is a decreasing function, particles that start out closer to the boundary of the star take longer to vanish into the singularity. So by letting the particle density be space-inhomogeneous on the support of the fluid, we allow for the physically realistic inhomogeneous collapse.

2.2 Methodology

The proof of Theorem 2.1 involves new ideas and several steps. Here we only discuss the main ideas. We first introduce the foliation by the level sets of χ_{dust} by the change of variables as we want to build a solution to (2.15) around the dust profile (2.16):

$$\tau := 1 - g(r)s, \quad \phi(\tau, r) := \chi(s,r), \tag{2.20}$$

so that $0 \le \tau \le 1$, and $\tau = 0$ corresponds to the space-time blowup curve Γ and $\tau = 1$ corresponds to the initial time. The new unknown ϕ solves

$$\phi_{\tau\tau} + \frac{2}{9\phi^2} + \varepsilon P[\phi] = 0, \quad P[\phi] := \frac{\phi^2}{g^2(r) w^\alpha r^2} \Lambda \left(w^{1+\alpha} \mathscr{J}[\phi]^{-\gamma} \right), \tag{2.21}$$

where $\Lambda := -\frac{r g'(r)(1-\tau)}{g(r)} \partial_\tau + r \partial_r$ and $\mathscr{J}[\phi] := \phi^2 (\phi + \Lambda \phi)$. In (τ, r) coordinates, the dust collapse solution is explicitly given by $\phi_0(\tau, r) = \tau^{\frac{2}{3}}$. A simple computation shows that

$$\mathscr{J}[\phi_0] = \phi_0^2 (\phi_0 + \Lambda \phi_0) \sim \tau^2 \left(1 + \frac{r^n}{\tau}\right) \tag{2.22}$$

wherefrom the scale $\frac{r^n}{\tau}$ naturally emerges. The goal is to construct ϕ solving (2.21) and satisfying $1 \lesssim \frac{\mathscr{J}[\phi]}{\mathscr{J}[\phi_0]} \lesssim 1$. To that end, we seek a special solution ϕ of the form

$$\phi = \phi_{\text{app}} + \frac{\tau^m}{r} H \tag{2.23}$$

where ϕ_{app} is chosen as a good approximate solution of (2.21) and the remainder H is suitably controlled in some function space.

To construct ϕ_{app}, we set up the ansatz $\phi_{\text{app}} = \phi_0 + \epsilon\phi_1 + \cdots + \epsilon^M \phi_M$, $M \gg 1$, Taylor expand the pressure term $\epsilon P[\phi_0 + \epsilon\phi_1 + \cdots]$ into the powers of ϵ, and compare the coefficients in ϵ in (2.21) to obtain a hierarchy of ODEs satisfied by the ϕ_j:

$$\partial_{\tau\tau}\phi_{j+1} - \frac{4}{9\tau^2}\phi_{j+1} = f_{j+1}[\phi_0, \phi_1, \ldots, \phi_j], \;\; j = 0, 1, \ldots, M. \tag{2.24}$$

Functions f_{j+1}, $j = 0, 1, \ldots, M$ are explicit and generally depend nonlinearly on ϕ_k, $0 \le k \le j$, and their spatial derivatives (up to the second order). The system of ODEs (2.24) can be solved iteratively as the right-hand side f_{j+1} is always known as a function of the first j iterates. To show that ϕ_{app} are good approximate solutions of (2.21), we must prove that the iterates ϕ_j, $j \ge 1$, are effectively small with respect to ϕ_0. The mechanism by which this is indeed true is one of the key ingredients, in both the conceptual and the technical sense. In particular we select special solutions of (2.24), as they are in general not unique (the two general solutions of the homogeneous problem are $\tau^{4/3}$ and $\tau^{-1/3}$), which will allow us to see the *gain*. Recalling n is the flatness index introduced for g, we now fix a sufficiently large n so that

$$\delta = 2\left(\frac{4}{3} - \gamma - \frac{1}{n}\right) > 0 \tag{2.25}$$

which is possible thanks to the super-criticality of $\gamma < \frac{4}{3}$. Then we can construct ϕ_j's such that

$$|\partial_\tau^m (r\partial_r)^\ell \phi_j| \le C_{jkm} \tau^{\frac{2}{3} + j\delta - m} \frac{(\frac{r^n}{\tau})^{\lambda - \frac{2}{n}}}{(1 + \frac{r^n}{\tau})^\lambda}. \tag{2.26}$$

for some $\lambda > \frac{2}{n}$, which in turn leads to

$$\mathscr{J}[\phi_{\text{app}}] \lesssim \varepsilon^{2M+1} \tau^{-\frac{4}{3} + (M+1)\delta - 1}. \tag{2.27}$$

With ϕ_{app} at hand, thanks to the crucial gain of $\tau^{j\delta}$ and in the presence of τ^m factor, now the remainder H satisfies the following quasilinear wave-like equation:

$$g^{00}\partial_{\tau\tau}H + 2g^{01}\partial_r\partial_\tau H + \frac{2m}{\tau}\partial_\tau H + \left[\frac{m(m-1)}{\tau^2} - \frac{4}{9\phi_{\text{app}}^3}\right] H - \varepsilon\gamma c[\phi] \frac{1}{w^\alpha}\partial_r\left(\frac{w^{1+\alpha}}{r^2}\partial_r[r^2 H]\right) = F, \tag{2.28}$$

where at the leading order

$$g^{00} = g^{00}[\phi] \approx 1, \; g^{01} = g^{01}[\phi] \approx \frac{\varepsilon}{\tau}, \; c[\phi] \approx c[\phi_{\text{app}}] \approx \frac{\tau^{\frac{5}{3} - \gamma}}{\tau + r^n}.$$

We note that

$$\frac{m(m-1)}{\tau^2} > \frac{4}{9\phi_{\text{app}}^3} \backsim \frac{1}{\tau^2},$$

for m sufficiently large, which gives a coercive positive definite control of the solution at the singular surface $\{\tau = 0\}$.

The analysis of the solutions to (2.28) proceeds by using suitable weighted energy methods. The difficulty in producing energy estimates for (2.28) comes from two different singularities present in the equation. At $\tau = 0$, the coefficient $c[\phi_{\text{app}}]$ and various others formally blow up to infinity. This is the singularity associated with the collapse at the singular surface $\tau = 0$ and it is important to work with carefully chosen τ weighted functionals. At $r = 1$, we have $w = 0$ and therefore the elliptic part of the quasilinear operator on the left-hand side of (2.28) does not scale like the Laplacian as $r \to 1$. This is a well-known degeneracy associated with the presence of the vacuum boundary and we adapt the well-posedness framework developed in [21, 19, 14, 15]. We refer to [11] for more details.

We emphasise that the assumption of compact support for the star is in fact not necessary in the analysis, and Theorem 2.1 can easily be relaxed to include densities with infinite extent, that decay sufficiently fast to allow for finite mass. The presence of the vacuum boundary makes the analysis strictly harder and our approach highlights the flexibility of the well-posedness framework introduced by Jang-Masmoudi [21] to handle the presence of vacuum.

Remark 2.2 (Open questions). *It is a priori not clear if the near-dust collapse above is in any sense stable. A careful choice of the high-order "flatness" encoded in the assumptions* (2.19) *and* (2.25) *suggests that the initial density profiles that lead to the near-dust collapse are not generic in the space of initial data. It is important to clarify the answers to both questions.*

3 Self-similar collapse for $1 \leq \gamma < \frac{4}{3}$

In the study of the near-dust collapse from Section 2, one could say that we have prepared our initial data to effectively put us in a situation where the inertial forces balance out the gravity, and the pressure remains "under" control. It is however natural, in fact necessary in the light of Remark 2.2, to look for examples of collapse where all three participating forces - inertia, pressure, and gravity - balance each other out. This leads us naturally to the study of self-similar solutions to the Euler-Poisson system.

There have been numerous studies of self-similar collapse in the astrophysics literature and self-simlar solutions play an important role in the study of so-called critical phenomena [10]. Among others, using numerical integration, in 1969 Penston [33] and Larson [22] independently discovered a self-similar solution to (1.1) describing the collapse of a self-gravitating, asymptotically flat, isothermal sphere ($\gamma = 1$). In 1977 Hunter [17] numerically discovered a further (discrete) family of smooth self-similar solutions, commonly referred to as Hunter solutions, see also [38, 40]. In 1988 Ori and Piran [29] gave numerical evidence that the Larson-Penston (LP) collapse is the only stable self-similar solution in the above family of solutions, and therefore physically the most relevant. Brenner and Witelski [1], Maeda and Harada [24] reached the same conclusion after careful numerical analysis of the collapse. In 1983 Yahil [41] found polytropic analogues of LP solutions to (1.1)-(1.2) for $\gamma < \frac{4}{3}$.

In this section, we discuss the mathematical construction of Larson-Penston solution and Yahil solutions, exact self-similar solutions to the Euler-Poisson system (1.1)-(1.2) in the

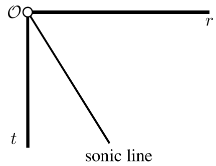

Figure 2: Schematic depiction of the sonic line, across which the solution has to be smooth. As we approach the origin $\mathcal{O} = (0,0)$ the density blows up.

super-critical regime obtained in [12, 13]. We start with the LP solution.

3.1 Self-similar Euler-Poisson system and the sonic point - the isothermal case

Motived by the self-similar scaling invariance (1.4) for $\gamma = 1$, we seek a self-similar solution of (1.1) of the form:

$$\rho(t,r) = \frac{1}{(\sqrt{2\pi}\, t)^2}\tilde{\rho}(y), \quad u(t,r) = \tilde{u}(y), \quad y := \frac{r}{-t} \tag{3.29}$$

where $u(t,r)$ is the radial velocity. By introducing the relative velocity $\tilde{\omega} := \frac{\tilde{u}(y)+y}{y}$, the Euler-Poisson system (1.1)–(1.2) with $\gamma = 1$ (i.e. $p = \rho$) reduces to the non-autonomous system of ODE

$$\tilde{\rho}' = -\frac{2y\tilde{\omega}\tilde{\rho}}{1-y^2\tilde{\omega}^2}(\tilde{\rho}-\tilde{\omega}), \tag{3.30}$$

$$\tilde{\omega}' = \frac{1-3\tilde{\omega}}{y} + \frac{2y\tilde{\omega}^2}{1-y^2\tilde{\omega}^2}(\tilde{\rho}-\tilde{\omega}), \tag{3.31}$$

where the derivative notation $'$ is short for ∂_y. We are interested in smooth solutions to (3.30)-(3.31). Notice that the assumption of smoothness on the self-similar profiles $\tilde{\rho}$ and $\tilde{u}$, by (3.29) implies the blow-up of the density $\rho(t,r)$ as $t \to 0^-$. A simple Taylor expansion at the origin $y = 0$ and the asymptotic infinity $y \to +\infty$ shows that in order for a solution $(\tilde{\rho}, \tilde{\omega})$ to (3.30)–(3.31) to be smooth and decaying at infinity, we must have

$$\tilde{\omega}(0) = \frac{1}{3}, \;\; \tilde{\rho}(0) > 0, \tag{3.32}$$

$$\tilde{\rho}(y) \sim_{y\to\infty} y^{-2}, \;\; \lim_{y\to\infty} \tilde{\omega}(y) = 1. \tag{3.33}$$

By continuity, for any continuous solution satisfying (3.32)–(3.33) there must exist at least one point y_* such that $1 - y_*^2\tilde{\omega}^2(y_*) = 0$. At such a point the system (3.30)–(3.31) is in general singular. This leads us to one of the central notions in the construction of self-similar solutions.

Definition 3.1 (Sonic point). *A point $y_* > 0$ is called a* sonic point *for the flow $(\tilde{\rho}(\cdot), \tilde{\omega}(\cdot))$ if*

$$1 - y_*^2\tilde{\omega}^2(y_*) = 0. \tag{3.34}$$

For a solution to be smooth through the sonic point y_*, it has to be the case that the sonic point is a removable singularity. Assuming smoothness, we can formally compute the Taylor coefficients of $(\tilde{\rho}, \tilde{\omega})$ around y_*. Two possibilities emerge (see e.g. [1]) - either

$$\tilde{\rho}(y) = \frac{1}{y_*} - \frac{1}{y_*^2}(y - y_*) + \frac{-y_*^2 + 6y_* - 7}{2y_*^3(2y_* - 3)}(y - y_*)^2 + O(|y - y_*|^3) \tag{3.35}$$

$$\tilde{\omega}(y) = \frac{1}{y_*} + \frac{1}{y_*}(1 - \frac{2}{y_*})(y - y_*) + \frac{-5y_*^2 + 19y_* - 17}{2y_*^3(2y_* - 3)}(y - y_*)^2 + O(|y - y_*|^3); \tag{3.36}$$

we refer to this as the Larson-Penston (LP)-type solution, or

$$\tilde{\rho}(y) = \frac{1}{y_*} + \frac{1}{y_*}\left(1 - \frac{3}{y_*}\right)(y - y_*) + O((y - y_*)^2), \tag{3.37}$$

$$\tilde{\omega}(y) = \frac{1}{y_*} + O((y - y_*)^2), \tag{3.38}$$

which we refer to as the Hunter-type solution. The main result of [12] is the following:

Theorem 3.2 (Existence of a Larson-Penston self-similar collapsing solution)**.** *There exists a $y_* \in (2, 3)$ such that* (3.30)–(3.33) *possesses a real-analytic solution $(\tilde{\rho}, \tilde{\omega})$ with a single sonic point at y_*. Moreover the solution satisfies the Larson-Penston expansion* (3.35)–(3.36) *at $y = y_*$ and*

$$\tilde{\rho}(y) > 0, \quad \frac{2}{3}y \le \tilde{u}(y) < 0, \quad y \in [0, \infty). \tag{3.39}$$

There are two known explicit solutions to (3.30)–(3.33). One of them is the Friedmann solution

$$\tilde{\rho}_F(y) = \tilde{\omega}_F(y) \equiv \frac{1}{3} \tag{3.40}$$

and the other one is the far-field solution

$$\tilde{\rho}_\infty(y) = \frac{1}{y^2}, \quad \tilde{\omega}_\infty(y) \equiv 1. \tag{3.41}$$

The Friedman solution (3.40) is the Newtonian analogue of the classical cosmological Friedmann solution - it satisfies the boundary condition (3.32), but the density does not decay to 0 at spatial infinity. On the other hand, the far-field solution (3.41) does decay as $y \to \infty$, but blows up at the origin $y = 0$. The LP solution constructed in Theorem 3.2 provides a smooth solution connecting these two solutions at two boundaries $y = 0$ and $y = \infty$. Although not exactly true, it is useful to think of it as a heteroclinic orbit in the language of dynamical systems.

3.2 Proof ideas

The main difficulty of Theorem 3.2 is the presence of an a priori unknown sonic point y_*, since we cannot use any standard ODE theory to construct a real analytic (or a C^∞) solution. The subtlety of sonic points of compressible Euler system is well illustrated in a recent pioneering work [27], where C^∞ self-similar imploding solutions to the compressible Euler

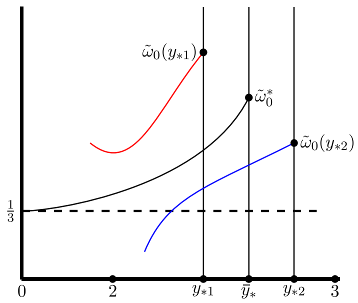

Figure 3: Schematic depiction of the shooting argument. The critical point $\bar{y}_*$ is obtained by sliding to the left inside Y starting from $y_* = 3$, until we hit the topological boundary of Y.

system were constructed. We note that in the Euler case the associated ODE-system is autonomous. Our system (3.30)–(3.33) is non-autonomous and we develop a different strategy. We first identify a "good" guess for the location of the sonic point - the so-called sonic window $[2, 3]$. We then try to extend the local smooth solution to the left and to the right. These are two separate arguments. The somewhat more "regular" direction is the construction of the global solution to the right. The solution to the left, which connects the sonic point to the origin $y = 0$ and the corresponding Friedmann-like boundary conditions (3.32) is more difficult. We design a shooting method to construct this Friedmann connection based on dynamic invariances to the flow.

Step 1. Analysis around the sonic point. The sonic point, or more precisely the sonic hypersurface, corresponds to the boundary of the backward sound cone emanating from the singularity $(0, 0)$. It separates the semi axis $y \geq 0$ into an inner region $[0, y_*]$ and an outer region $[y_*, \infty)$. The starting point of our analysis is to look for the local analytic solutions around $y = y_*$ of the form $\tilde{\rho} = \sum_{N=0}^{\infty} \tilde{\rho}_N (y - y_*)^N$ and $\tilde{\omega} = \sum_{N=0}^{\infty} \tilde{\omega}_N (y - y_*)^N$. The sonic condition determines the zero order coefficient $\tilde{\rho}_0 = \tilde{\omega}_0 = \frac{1}{y_*}$, and we demand $\tilde{\rho}_1 = -\frac{1}{y_*^2}$ and $\tilde{\omega}_1 = \frac{1}{y_*}(1 - \frac{2}{y_*})$ so that the LP type condition (3.35) is satisfied. This LP type condition uniquely specifies a real analytic solution in a neighborhood of $y = y_*$ for any $y_* > \frac{3}{2}$.

Step 2. Analysis to the left. With local analytic solutions at hand, we continue to solve the system (3.30)–(3.33) to the left of each sonic point y_*. The goal is to find y_* for which the solution extends all the way to $y = 0$ and satisfies $\lim_{y \to 0} \tilde{\omega}(y; y_*) = \frac{1}{3}$. Invariances of our ODE system (3.30)–(3.33) allow us to limit the values of y_* to $y_* \in [2, 3]$. In particular, for y_* close enough to 3, the solution $\tilde{\omega}(y; y_*)$ to the left meets with the Friedmann curve $\tilde{\omega}_F(y) = \frac{1}{3}$ for some $y < y_*$, and it never comes back above the curve. This motivates us to consider

$$Y := \left\{ y_* \in [2, 3] \;\middle|\; \exists\, y \;\text{ such that } \tilde{\omega}(y; \tilde{y}_*) = \frac{1}{3} \;\text{ for all }\; \tilde{y}_* \in [y_*, 3] \right\}, \tag{3.42}$$

and set $\bar{y}_* := \inf Y$. The idea is that $\tilde{\omega}(\cdot; \bar{y}_*)$ will achieve the value $\frac{1}{3}$ exactly at $y = 0$ and this will lead to an LP-solution, see Figure 3. Using the minimality of $\bar{y}_*$ it is indeed possible to show that the solution exists on $(0, \bar{y}_*]$ and satisfies $\liminf_{y \to 0} \tilde{\omega}(y; \bar{y}_*) \geq \frac{1}{3}$. To show that the limit is $\frac{1}{3}$, we use the intermediate value theorem to match the solution emanating from the origin via upper and lower solutions.

Step 3. Analysis to the right. In the outer region, we can show that a unique LP-type solution exists to the right on $[y_*, \infty)$ for all $y_* \in [2,3]$, thereby satisfying the necessary far-field conditions (3.33). Therefore, we obtain the desired global smooth solution for $y = \bar{y}_*$.

3.3 Self-similar Euler-Poisson system and the sonic point - the polytropic case

The strategy developed in [12] provides a general recipe to construct a solution connecting a sonic point and a singular point, such as the origin $y = 0$ in this case. Recently, this method has been further developed in [13] to construct the polytropic analogue of the LP solution, also called Yahil solutions, which are self-similar solutions to (1.1)-(1.2) in the full super-critical range of $1 < \gamma < \frac{4}{3}$. The polytropic self-similar problem is considerably more complex than the isothermal case. The part of the strategy shared with the isothermal case $\gamma = 1$ is to solve away from the sonic point and connect it to $y = 0$ (to the left) and $y = \infty$ (to the right) in two separate steps. However, some completely new ideas are introduced to handle serious new difficulties that arise, most notably due to the complicated nonlinear structure of the ODE analogous to (3.30)–(3.31):

$$\rho' = \frac{y\rho\left(2\omega^2 + (\gamma-1)\omega - \frac{4\pi\rho\omega}{4-3\gamma} + (\gamma-1)(2-\gamma)\right)}{\gamma\rho^{\gamma-1} - y^2\omega^2}, \tag{3.43}$$

$$\omega' = \frac{4-3\gamma-3\omega}{y} - \frac{y\omega\left(2\omega^2 + (\gamma-1)\omega - \frac{4\pi\rho\omega}{4-3\gamma} + (\gamma-1)(2-\gamma)\right)}{\gamma\rho^{\gamma-1} - y^2\omega^2}. \tag{3.44}$$

The sonic denominator $\gamma\rho^{\gamma-1} - y^2\omega^2$ has a strong nonlinear dependence on the self-similar density ρ and the numerators in (3.43)–(3.44) are considerably more complicated than their isothermal analogues in (3.30)–(3.31). Even if one could hope to use techniques very similar to the ones from [12] in the regime where $0 < \gamma - 1 \ll 1$, as γ approaches $\frac{4}{3}$ we expect to see completely new phenomenology as the problem becomes less and less supercritical. This is indeed the case, and we provide a self-contained and new approach to identify various dynamically invariant (trapped) regions. These include the most important new realisation: we show that for the corresponding LP-type solution - we call it the Yahil solution - the relative velocity ω is monotonically decreasing. This completely removes the need to use the upper and lower solutions strategy from [12], and instead provides a conceptually clean and elegant way of connecting the sonic point to the origin $y = 0$.

The details of the proof of the existence of Yahil solutions to (3.43)–(3.44) is given in [13].

4 Some open problems

A somewhat vague, but useful conjecture toward the understanding of gravitational collapse in astrophysics, aptly termed *similarity hypothesis* (see e.g. [2]), states that there should exist open (and therefore "generic") classes of data that lead to solutions whose long-term evolution is approximately self-similar. From the rigorous point of view, the works [12, 13] are the first step towards a precise resolution of such a conjecture, as they identify possible attractors for the collapsing dynamics in the framework of self-gravitating Newtonian compressible gas dynamics.

Perhaps the most important open question in this context is

1. to prove the nonlinear radial stability of the LP and the Yahil solutions. There do exist numerical indications that the LP solution is stable under suitable radial perturbations [24], but a rigorous proof is yet to be given.

The Goldreich-Weber collapsing solutions are an important object in the description of the mass-critical dynamics, as discussed in Section 1.3. They form an essential part of the understanding of the nonlinear dynamics near the mass-critical Lane-Emden stars. In this context, one of the key open questions is to understand

2. the stability of the collapsing GW-stars.

Finally, in both the mass-critical and the mass-supercritical case, nothing is rigorously known when we leave the realm of radial symmetry.

3. Non-radial collapse is a wide open playground for interesting dynamics. It is of particular interest to see how different characteristic spatial scales can "form" in the collapse process and thus generate asymmetries.

Many other interesting problems arise in the presence of important physical effects such as the viscosity, or by including relativistic effects. We hope that the methods described in this paper will prove itself useful to other physically important models pertaining to the dynamics of collapse and imploding behaviour.

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