Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and Jordan Quiver Gauge Theories

Hitoshi Konno

Department of Mathematics, Tokyo University of Marine Science and Technology,
Etchujima, Tokyo 135-8533, Japan
hkonno0@kaiyodai.ac.jp

1 Introduction

This article is a review of our recent paper [10]. There we have formulated the elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ associated with the toroidal algebra $\mathfrak{gl}_{1,tor}$ in the same scheme as the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$ associated with the affine Lie algebra $\widehat{\mathfrak{g}}$ [9]. The algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ is an elliptic analogue of the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$ introduced by Miki as a deformation of the $W_{1+\infty}$ algebra [11].

As for the trigonometric algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$, various representations have been studied by many papers such as [1–3, 11] and applied to the 5d and 6d lifts of the 4d $\mathcal{N}=2$ SUSY gauge theories, which are the gauge theories associated with the linear quivers.

In [10], we have shown that the elliptic algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ is a relevant quantum group structure to treat the 5d and 6d lifts of the the 4d $\mathcal{N}=2^*$ SUSY gauge theories associated with the Jordan quiver. The key to this is that $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ gives a realization of the Jordan quiver W-algebra $W_{q,t}(\Gamma(\widehat{A}_0))$ proposed in [6]. In particular, our realization of the generating function $T(u)$ of $W_{q,t}(\Gamma(\widehat{A}_0))$ as a composition of the vertex operators $\Phi(u)$ and $\Phi^*(u)$ leads to an identification of it with a refined topological vertex depicted in Fig.6.1. This vertex is a basic object in calculating Nekrasov instanton partition functions of the 5d and 6d lifts of the 4d $\mathcal{N}=2^*$ $U(M)$ theories, whose instanton moduli spaces are given by the Jordan quiver varieties.

Notations

For $p_1,p_2,p_3\in\mathbb{C}$ with $|p_i|<1$ $(i=1,2,3)$, we set

$$(z;p_1)_\infty=\prod_{n_1=0}^{\infty}(1-zp_1^{n_1}),\qquad (z;p_1,p_2)_\infty=\prod_{n_1,n_2=0}^{\infty}(1-zp_1^{n_1}p_2^{n_2}),$$

$$(z;p_1,p_2,p_3)_\infty = \prod_{n_1,n_2,n_3=0}^{\infty}(1-zp_1^{n_1}p_r^{n_2}p_3^{n_3}),$$

and

$$\theta_{p_1}(z)=(z;p_1)_\infty(p_1/z;p_1)_\infty, \qquad \Gamma(z;p_1,p_2)=\frac{(p_1p_2/z;p_1,p_2)_\infty}{(z;p_1,p_2)_\infty},$$
$$\Gamma_3(z;p_1,p_2,p_3)=(z;p_1,p_2,p_3)_\infty(p_1p_2p_3/z;p_1,p_2,p_3)_\infty.$$

2 Elliptic Quantum Toroidal Algebra $\mathcal{U}_{q,t,p}(\mathfrak{gl}_{1,tor})$

2.1 Definition of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Let p, q,t be generic complex numbers satisfying $|p|,|q|,|t|<1$.

Definition 2.1. *The elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p}=U_{q,t,p}(\mathfrak{gl}_{1,tor})$ is a $\mathbb{C}[[p]]$-algebra generated by $\alpha_m, x^\pm_n$, $(m\in\mathbb{Z}\backslash\{0\}, n\in\mathbb{Z})$ and invertible elements $\psi^+_0,\gamma^{1/2}$. We set $\psi^-_0=(\psi^+_0)^{-1}$. The defining relations can be conveniently expressed in terms of the generating functions, which we call the elliptic currents,*

$$x^\pm(z)=\sum_{n\in\mathbb{Z}}x^\pm_n z^{-n},$$
$$\psi^+(z)=\psi^+_0\exp\left(-\sum_{m>0}\frac{p^m}{1-p^m}\alpha_{-m}(\gamma^{-1/2}z)^m\right)\exp\left(\sum_{m>0}\frac{1}{1-p^m}\alpha_m(\gamma^{-1/2}z)^{-m}\right),$$
$$\psi^-(z)=\psi^-_0\exp\left(-\sum_{m>0}\frac{1}{1-p^m}\alpha_{-m}(\gamma^{1/2}z)^m\right)\exp\left(\sum_{m>0}\frac{p^m}{1-p^m}\alpha_m(\gamma^{1/2}z)^{-m}\right).$$

The defining relations are

$$\psi^+_0,\gamma^{1/2} \ : \ \textit{central}, \tag{2.1}$$
$$\frac{G^+(w/z;p^*)G^-(w/z;p)}{G^-(w/z;p^*)G^+(w/z;p)}\psi^\pm(z)\psi^\pm(w)=\frac{G^+(z/w;p^*)G^-(z/w;p)}{G^-(z/w;p^*)G^+(z/w;p)}\psi^\pm(w)\psi^\pm(z), \tag{2.2}$$
$$\frac{G^+(p^*\gamma^{-1}w/z;p^*)G^-(\gamma w/z;p)}{G^-(p^*\gamma^{-1}w/z;p^*)G^+(\gamma w/z;p)}\psi^+(z)\psi^-(w)=\frac{G^-(p\gamma^{-1}z/w;p)G^+(\gamma z/w;p^*)}{G^+(p\gamma^{-1}z/w;p^*)G^-(\gamma z/w;p^*)}\psi^-(w)\psi^+(z), \tag{2.3}$$
$$z^3\frac{G^+(\gamma^{-1}w/z;p^*)}{G^-(p^*\gamma^{-1}w/z;p^*)}\psi^+(z)x^+(w)=-\gamma^{-3}w^3\frac{G^+(\gamma z/w;p^*)}{G^-(p^*\gamma z/w;p^*)}x^+(w)\psi^+(z), \tag{2.4}$$
$$z^3\frac{G^+(w/z;p^*)}{G^-(p^*w/z;p^*)}\psi^-(z)x^+(w)=-w^3\frac{G^+(z/w;p^*)}{G^-(p^*z/w;p^*)}x^+(w)\psi^-(z), \tag{2.5}$$

$$z^{-3}\frac{G^-(pw/z;p)}{G^+(w/z;p)}\psi^+(z)x^-(w)=-w^{-3}\frac{G^-(pz/w;p)}{G^+(z/w;p)}x^-(w)\psi^+(z), \tag{2.6}$$

$$z^{-3}\frac{G^-(p\gamma^{-1}w/z;p)}{G^+(\gamma^{-1}w/z;p)}\psi^-(z)x^-(w)=\gamma^3 w^{-3}\frac{G^-(p\gamma z/w;p)}{G^+(\gamma z/w;p)}x^-(w)\psi^-(z), \tag{2.7}$$

$$[x^+(z),x^-(w)]=\frac{(1-q)(1-1/t)}{(1-q/t)}\left(\delta(\gamma^{-1}z/w)\psi^+(w)-\delta(\gamma z/w)\psi^-(\gamma^{-1}w)\right), \tag{2.8}$$

$$z^3\frac{G^+(w/z;p^*)}{G^-(p^*w/z;p^*)}x^+(z)x^+(w)=-w^3\frac{G^+(z/w;p^*)}{G^-(p^*z/w;p^*)}x^+(w)x^+(z), \tag{2.9}$$

$$z^3\frac{G^-(w/z;p)}{G^+(pw/z;p)}x^-(z)x^-(w)=-w^3\frac{G^-(z/w;p)}{G^+(pz/w;p)}x^-(w)x^-(z), \tag{2.10}$$

$$\frac{G^+(p^*\frac{w}{z};p^*)}{G^-(p^*\frac{w}{z};p^*)}\frac{G^+(p^*\frac{u}{w};p^*)}{G^-(p^*\frac{u}{w};p^*)}\frac{G^+(p^*\frac{u}{z};p^*)}{G^-(p^*\frac{u}{z};p^*)}\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^+(z)x^+(w)x^+(u)$$
$$+\ \textit{permutations in } z,w,u=0, \tag{2.11}$$

$$\frac{G^-(p\frac{w}{z};p)}{G^+(p\frac{w}{z};p)}\frac{G^-(p\frac{u}{w};p)}{G^+(p\frac{u}{w};p)}\frac{G^-(p\frac{u}{z};p)}{G^+(p\frac{u}{z};p)}\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^-(z)x^-(w)x^-(u)$$
$$+\ \textit{permutations in } z,w,u=0, \tag{2.12}$$

where we set $p^*=p\gamma^{-2}$ *and*

$$G^\pm(z;p)=(q^{\pm1}z;p)_\infty(t^{\mp1}z;p)_\infty((t/q)^{\pm1}z;p)_\infty,$$
$$G^\pm(z;p^*)=G^\pm(z;p)\big|_{p\mapsto p^*}.$$

We treat these relations as formal Laurent series in z,w and u. The coefficients such as $\frac{G^\pm(w/z;p^*)}{G^\mp(w/z;p^*)}$, $\frac{G^\pm(w/z;p)}{G^\mp(w/z;p)}$ should be expanded in w/z. All the coefficients in z,w,u are well defined in the p-adic topology.

Note that the relations (2.2)-(2.7) are equivalent to

$$[\alpha_m,\alpha_n]=-\frac{\kappa_m}{m}(\gamma^m-\gamma^{-m})\gamma^{-m}\frac{1-p^m}{1-p^{*m}}\delta_{m+n,0}, \tag{2.13}$$

$$[\alpha_m,x^+(z)]=-\frac{\kappa_m}{m}\frac{1-p^m}{1-p^{*m}}\gamma^{-3m/2}z^m x^+(z)\quad (m\neq 0). \tag{2.14}$$

$$[\alpha_m,x^-(z)]=\frac{\kappa_m}{m}\gamma^{-m/2}z^m x^-(z)\quad (m\neq 0), \tag{2.15}$$

where we set

$$\kappa_m=(1-q^m)(1-t^{-m})(1-(t/q)^m).$$

It is sometimes convenient to set

$$\alpha'_m=\frac{1-p^{*m}}{1-p^m}\gamma^m\alpha_m\qquad (m\in\mathbb{Z}_{\neq0})$$

which satisfy

$$[\alpha'_m, \alpha'_n] = -\frac{\kappa_m}{m}(\gamma^m - \gamma^{-m})\gamma^m \frac{1-p^{*m}}{1-p^m}\delta_{m+n,0}. \tag{2.16}$$

Remark. On $\mathcal{U}_{q,t,p}$-modules, the central element $\gamma^{1/2}$ takes a complex value. Then assuming $|p| < 1, |p^*| = |p\gamma^{-2}| < 1$, one can rewrite (2.2)-(2.7), (2.9)-(2.10) as follows.

$$\begin{aligned}
\psi^\pm(z)\psi^\pm(w) &= \tfrac{g(w/z;p^*)}{g(w/z;p)}\psi^\pm(w)\psi^\pm(z),\\
\psi^+(z)\psi^-(w) &= \tfrac{g(\gamma^{-1}w/z;p^*)}{g(\gamma w/z;p)}\psi^-(w)\psi^+(z),\\
\psi^+(z)x^+(w) &= g(\gamma^{-1}w/z;p^*)x^+(w)\psi^+(z),\\
\psi^-(z)x^+(w) &= g(w/z;p^*)x^+(w)\psi^-(z),\\
\psi^+(z)x^-(w) &= g(z/w;p)x^-(w)\psi^+(z),\\
\psi^-(z)x^-(w) &= g(\gamma z/w;p)x^-(w)\psi^-(z),\\
x^+(z)x^+(w) &= g(w/z;p^*)x^+(w)x^+(z),\\
x^-(z)x^-(w) &= g(z/w;p)x^-(w)x^-(z),
\end{aligned}$$

where

$$g(z;p) = \frac{\theta_p(q^{-1}z)\theta_p((q/t)z)\theta_p(tz)}{\theta_p(qz)\theta_p((q/t)^{-1}z)\theta_p(t^{-1}z)}, \tag{2.17}$$

$$g(z;p^*) = g(z;p)\big|_{p\mapsto p^*}. \tag{2.18}$$

2.2 Hopf algebroid structure

For $F(z,p) \in \mathbb{C}[[z,z^{-1}]][[p]]$, let $\widetilde{\otimes}$ denote a tensor product with the following extra condition

$$F(z;p^*)a\widetilde{\otimes}b = a\widetilde{\otimes}F(z;p)b. \tag{2.19}$$

Define two moment maps $\mu_l, \mu_r : \mathbb{C}[[z,z^{-1}]][[p]] \to \mathcal{U}_{q,t,p}[[z,z^{-1}]]$ by

$$\mu_l(F(z,p)) = F(z,p), \qquad \mu_r(F(z,p)) = F(z,p^*).$$

Let $\gamma_{(1)} = \gamma\widetilde{\otimes}1, \gamma_{(2)} = 1\widetilde{\otimes}\gamma$. We also define two algebra homomorphisms $\Delta : \mathcal{U}_{q,t,p} \to \mathcal{U}_{q,t,p} \widetilde{\otimes} \mathcal{U}_{q,t,p}$ and $\varepsilon : \mathcal{U}_{q,t,p} \to \mathbb{C}$ by

$$\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}\widetilde{\otimes}\gamma^{\pm 1/2}, \tag{2.20}$$

$$\Delta(\psi^{\pm}(z)) = \psi^{\pm}(\gamma_{(2)}^{\mp 1/2}z)\widetilde{\otimes}\psi^{\pm}(\gamma_{(1)}^{\pm 1/2}z), \tag{2.21}$$

$$\Delta(x^{+}(z)) = 1\widetilde{\otimes}x^{+}(\gamma_{(1)}^{-1/2}z) + x^{+}(\gamma_{(2)}^{1/2}z)\widetilde{\otimes}\psi^{-}(\gamma_{(1)}^{-1/2}z), \tag{2.22}$$

$$\Delta(x^{-}(z)) = x^{-}(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}1 + \psi^{+}(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}x^{-}(\gamma_{(1)}^{1/2}z), \tag{2.23}$$

$$\Delta(\mu_l(F(z,p))) = \mu_l(F(z,p))\widetilde{\otimes}1, \qquad \Delta(\mu_r(F(z,p))) = 1\widetilde{\otimes}\mu_r(F(z,p)), \tag{2.24}$$

$$\varepsilon(\gamma^{1/2}) = \varepsilon(\psi_0^{+}) = 1, \qquad \varepsilon(\psi^{\pm}(z)) = 1, \qquad \varepsilon(x^{\pm}(z)) = 0, \tag{2.25}$$

$$\varepsilon(\mu_l(F(z,p))) = \varepsilon(\mu_r(F(z,p))) = F(z,p). \tag{2.26}$$

The map Δ is the so-called Drinfeld comultiplication. We have

Proposition 2.2. *The maps ε and Δ satisfy*

$$(\Delta\widetilde{\otimes}\mathrm{id}) \circ \Delta = (\mathrm{id}\widetilde{\otimes}\Delta) \circ \Delta, \tag{2.27}$$

$$(\varepsilon\widetilde{\otimes}\mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id}\widetilde{\otimes}\varepsilon) \circ \Delta. \tag{2.28}$$

One also has an algebra anti-homomorphism $S : \mathcal{U}_{q,t,p} \to \mathcal{U}_{q,t,p}$ and makes $(\mathcal{U}_{q,t,p}, \Delta, \varepsilon, \mu_l, \mu_r, S)$ an Hopf algebroid. See [9, 10] for details.

3 Representations of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Definition 3.1. *Let $\mathcal{V}$ be a $\mathcal{U}_{q,t,p}$-module. For $(k,l) \in \mathbb{Z}^2$, we say that $\mathcal{V}$ has level (k,l), if the central elements $\gamma^{1/2}$ and ψ_0^{+} act as*

$$\gamma^{1/2}v = (t/q)^{k/4}v, \qquad \psi_0^{+}v = (t/q)^{-l/2}v \qquad \forall v \in \mathcal{V}.$$

Hence on the level (k,l) module, $p^{*} = p(q/t)^{k}$. We assume $|p^{*}| = |p(q/t)^{k}| < 1$.

3.1 Level $(1,N)$ representation of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

For $u\in\mathbb{C}^*$, let $\mathcal{F}_u^{(1,N)}=\mathbb{C}[\alpha_{-m}\ (m>0)]1_u^{(N)}$ be a Fock space on which the Heisenberg algebra $\{\alpha_m\ (m\in\mathbb{Z}_{\neq 0})\}$ and the central elements $\gamma^{1/2},\psi_0^+$ act as

$$\begin{aligned}
&\gamma^{1/2}\cdot 1_u^{(N)}=(t/q)^{1/4}1_u^{(N)},\quad \psi_0^+\cdot 1_u^{(N)}=(t/q)^{-N/2}1_u^{(N)},\quad \alpha_m\cdot 1_u^{(N)}=0,\\
&\alpha_{-m}\cdot\xi=\alpha_{-m}\xi,\\
&\alpha_m\cdot\xi=-\frac{\kappa_m}{m}(1-(q/t)^m)\frac{1-p^m}{1-p^{*m}}\frac{\partial}{\partial\alpha_{-m}}\xi
\end{aligned}$$

for $m>0$, $\xi\in\mathcal{F}$. Note that $p^*=pq/t$ on $\mathcal{F}_u^{(1,N)}$.

Theorem 3.2. *The following assignment gives a level $(1,N)$ representation of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u^{(1,N)}$.*

$$x^+(z)=uz^{-N}(t/q)^{3N/4}\exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha_n z^{-n}\right\},\tag{3.1}$$

$$x^-(z)=u^{-1}z^{N}(t/q)^{-3N/4}\exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha'_{-n}z^n\right\}\exp\left\{-\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha'_n z^{-n}\right\},\tag{3.2}$$

$$\psi^+(z)=(t/q)^{-N/2}\exp\left\{-\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_n z^{-n}\right\},\tag{3.3}$$

$$\psi^-(z)=(t/q)^{N/2}\exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_n z^{-n}\right\}.\tag{3.4}$$

3.2 Vector representation and the q-Fock space represenattion

We next consider the elliptic analogue of the level (0,1) representation of $U_{q,t}(\mathfrak{gl}_{1,tor})$ ([3], Corollary 4.4). We call it the q-Fock space representation.

For $u\in\mathbb{C}^*$, let $V(u)$ be a vector space spanned by the symbols $[u]_j$ $(j\in\mathbb{Z})$.

Proposition 3.3. *By the following action, $V(u)$ is a level $(0,0)$ $\mathcal{U}_{q,t,p}$-module. We call $V(u)$ a vector representation.*

$$\begin{aligned}
x^+(z,p)[u]_j&=a^+(p)\delta(q^ju/z)[u]_{j+1},\\
x^-(z,p)[u]_j&=a^-(p)\delta(q^{j-1}u/z)[u]_{j-1},\\
\psi^\pm(z,p)[u]_j&=\left.\frac{\theta_p(q^jt^{-1}u/z)\theta_p(q^{j-1}tu/z)}{\theta_p(q^ju/z)\theta_p(q^{j-1}u/z)}\right|_\pm[u]_j,\\
\psi_0^\pm[u]_j&=[u]_j,
\end{aligned}$$

where we set

$$a^+(p) = (1-t)\frac{(pt/q;p)_\infty (p/t;p)_\infty}{(p;p)_\infty (p/q;p)_\infty}, \quad (3.5)$$

$$a^-(p) = (1-t^{-1})\frac{(pq/t;p)_\infty (pt;p)_\infty}{(p;p)_\infty (pq;p)_\infty}. \quad (3.6)$$

Let us consider a tensor product of the vector representations. Define

$$V^{(N)}(u) = V(u)\widetilde{\otimes}V(u(t/q)^{-1})\widetilde{\otimes}V(u(t/q)^{-2})\widetilde{\otimes}\cdots\widetilde{\otimes}V(u(t/q)^{-N+1}).$$

Set

$$\mathcal{P}^{(N)} = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \},$$

$$|\lambda\rangle_u^{(N)} = [u]_{\lambda_1}\widetilde{\otimes}[u(t/q)^{-1}]_{\lambda_2-1}\widetilde{\otimes}[u(t/q)^{-2}]_{\lambda_3-2}\widetilde{\otimes}\cdots\widetilde{\otimes}[u(t/q)^{-N+1}]_{\lambda_N-N+1}$$

and define $W^{(N)}(u)$ be a subspace of $V^{(N)}(u)$ spanned by $\{|\lambda\rangle_u^{(N)} \mid \lambda \in \mathcal{P}^{(N)}\}$. The action of $\mathcal{U}_{q,t,p}$ on $W^{(N)}(u)$ can be constructed by using the comultiplication Δ repeatedly.

We then take the inductive limit $N \to \infty$ in the same way as in the trigonometric case [3]. Let

$$\mathcal{P}^{(N),+} = \{\lambda \in \mathcal{P}^{(N)} \mid \lambda_N \geq 0 \}.$$

and define $W^{(N),+}(u)$ be the subspace of $W^{(N)}(u)$ spanned by $\{|\lambda\rangle_u^{(N)},\ \lambda \in \mathcal{P}^{(N),+}\}$. Let us define $\tau_N : \mathcal{P}^{(N),+} \to \mathcal{P}^{(N+1),+}$ by

$$\tau_N(\lambda) = (\lambda_1, \lambda_2, \cdots, \lambda_N, 0).$$

This induces the embedding $W^{(N),+} \hookrightarrow W^{(N+1),+}$. We then define a semi-infinite tensor product space $\mathcal{F}_u$ by the inductive limit

$$\mathcal{F}_u = \lim_{N\to\infty} W^{(N),+}(u).$$

The space $\mathcal{F}_u$ is spanned by the vectors $|\lambda\rangle_u$ ($\lambda \in \mathcal{P}^+$), where

$$\mathcal{P}^+ = \{\lambda = (\lambda_1, \lambda_2, \cdots) \mid \lambda_i \geq \lambda_{i+1},\ \lambda_i \in \mathbb{Z},\ \lambda_l = 0 \text{ for sufficiently large } l \}. \quad (3.7)$$

We denote by $\ell(\lambda)$ the length of $\lambda \in \mathcal{P}^+$ i.e. $\lambda_{\ell(\lambda)} > 0$ and $\lambda_{\ell(\lambda)+1} = 0$. We also set $|\lambda| = \sum_{i\geq 1}\lambda_i$ and denote by λ' the conjugate of λ. The action of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u$ is defined inductively as follows.

Define

$$x^{+[N]}(z) = (q/t)^{1/2}\frac{\theta_p(q^{-1}t^{-N+1}u/z)}{\theta_p(t^{-N}u/z)}x^+(z), \qquad x^{-[N]}(z) = x^-(z),$$
$$\psi^{+[N]}(z) = (q/t)^{1/2}\frac{\theta_p(q^{-1}t^{-N+1}u/z)}{\theta_p(t^{N}u/z)}\psi^+(z), \qquad \psi^{-[N]}(z) = (q/t)^{-1/2}\frac{\theta_p(qt^{N-1}z/u)}{\theta_p(t^{-N}z/u)}\psi^+(z).$$

Then we have

Lemma 3.4. *For $x = x^\pm, \psi^\pm$, we have*

$$\tau_N(x^{[N]}(z)|\lambda\rangle_u^{(N)}) = x^{[N+1]}(z)\tau_N(|\lambda\rangle_u^{(N)}).$$

Thanks to this Lemma, one can define the action of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u$ by

$$x(z)|\lambda\rangle_u = \lim_{N\to\infty} x^{[N]}(z)|(\lambda_1,\lambda_2,\cdots,\lambda_N)\rangle^{(N)}.$$

Theorem 3.5. *The following action gives a level (0,1) representation of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u$. We denote this representation by $\mathcal{F}_u^{(0,1)}$.*

$$\gamma^{1/2}|\lambda\rangle_u = |\lambda\rangle_u, \tag{3.8}$$
$$x^+(z)|\lambda\rangle_u = a^+(p)\sum_{i=1}^{\ell(\lambda)(+1)} A^{+'}_{\lambda,i}(p)\delta(u_i/z)|\lambda+\mathbf{1}_i\rangle_u, \tag{3.9}$$
$$x^-(z)|\lambda\rangle_u = (q/t)^{1/2}a^-(p)\sum_{i=1}^{\ell(\lambda)} A^{-'}_{\lambda,i}(p)\delta(q^{-1}u_i/z)|\lambda-\mathbf{1}_i\rangle_u, \tag{3.10}$$
$$\psi^+(z)|\lambda\rangle_u = (q/t)^{1/2}B^+_\lambda(u/z;p)|\lambda\rangle_u, \tag{3.11}$$
$$\psi^-(z)|\lambda\rangle_u = (q/t)^{-1/2}B^-_\lambda(z/u;p)|\lambda\rangle_u. \tag{3.12}$$

where

$$A^{+'}_{\lambda,i}(p) = \prod_{j=i+1}^{\ell(\lambda)}\frac{\theta_p(tu_i/u_j)}{\theta_p(qu_i/u_j)}\prod_{j=i+1}^{\ell(\lambda)+1}\frac{\theta_p(qu_i/tu_j)}{\theta_p(u_i/u_j)}, \tag{3.13}$$
$$A^{-'}_{\lambda,i}(p) = \prod_{j=1}^{i-1}\frac{\theta_p(tu_j/u_i)\theta_p(qu_j/tu_i)}{\theta_p(qu_j/u_i)\theta_p(u_j/u_i)}. \tag{3.14}$$

Remark 1. For a representation with $\gamma^{1/2} = 1$, there is an opposite comultiplication [10] and it yields the same level (0,1) representation as Theorem 3.5 except for replacing the coefficients

$A^{\pm'}_{\lambda,i}(p)$ with the non-primed ones given by

$$A^{+}_{\lambda,i}(p) = \frac{c'_\lambda(p)c_{\lambda+\mathbf{1}_i}(p)}{c_\lambda(p)c'_{\lambda+\mathbf{1}_i}(p)}A^{+'}_{\lambda,i}(p) \tag{3.15}$$

$$A^{-}_{\lambda,i}(p) = \frac{c'_\lambda(p)c_{\lambda-\mathbf{1}_i}(p)}{c_\lambda(p)c'_{\lambda-\mathbf{1}_i}(p)}A^{-'}_{\lambda,i}(p), \tag{3.16}$$

where $c_\lambda(p), c'_\lambda(p)$ are given by

$$c_\lambda(p) = \prod_{\square\in\lambda}\theta_p(q^{a(\square)}t^{\ell(\square)+1}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{\Gamma(q^{\lambda_i-\lambda_{j+1}}t^{j-i+1};q,p)}{\Gamma(q^{\lambda_i-\lambda_j}t^{j-i+1};q,p)}, \tag{3.17}$$

$$c'_\lambda(p) = \prod_{\square\in\lambda}\theta_p(q^{a(\square)+1}t^{\ell(\square)}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{\Gamma(q^{\lambda_i-\lambda_{j+1}+1}t^{j-i};q,p)}{\Gamma(q^{\lambda_i-\lambda_j+1}t^{j-i};q,p)}. \tag{3.18}$$

Here $a(\square) \equiv a_\lambda(\square) = \lambda_i - j$, $\ell(\square) \equiv \ell_\lambda(\square) = \lambda'_j - i$ for $\square = (i,j) \in \lambda$. The resultant representation is a direct elliptic analogue of the one in [3].

These two level $(0,1)$ representations are related by the gauge transformation

$$|\lambda\rangle_u = \frac{c_\lambda(p)}{c'_\lambda(p)}|\lambda\rangle'_u. \tag{3.19}$$

Namely, changing the basis from $\{|\lambda\rangle_u\}$ to $\{|\lambda\rangle'_u\}$ in Theorem 3.5, one gets the 2nd representation.

Note also that $c_\lambda(p), c'_\lambda(p)$ are elliptic analogues of the combinatorial factors c_λ, c'_λ, respectively, appearing in the inner product of the Macdonald symmetric functions as

$$\langle P_\lambda, P_\lambda\rangle_{q,t} = \frac{c'_\lambda}{c_\lambda}, \tag{3.20}$$

$$c_\lambda = \prod_{\square\in\lambda}(1-q^{a(\square)}t^{\ell(\square)+1}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty}{(q^{\lambda_i-\lambda_{j+1}}t^{j-i+1};q)_\infty},$$

$$c'_\lambda = \prod_{\square\in\lambda}(1-q^{a(\square)+1}t^{\ell(\square)}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_{j+1}+1}t^{j-i};q)_\infty}. \tag{3.21}$$

Remark 2. In the trigonometric case, the level (0,1) representation of $U_{q,t}(\mathfrak{gl}_{1,tor})$ with $u=1$ is identified with the geometric representation of the same algebra on $\bigoplus_N \mathrm{K}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$, the $T=\mathbb{C}^*\times\mathbb{C}^*$ equivariant K-theory of the Hilbert scheme of N points on $\mathbb{C}^2$ [3]. There the basis $\{|\lambda\rangle_1\}$ in $\mathcal{F}_1$ is identified with the fixed point classes $\{[\lambda]\}$ in $\mathrm{K}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$. We conjecture that the same is true in the elliptic case. Namely, if one could properly formulate a geometric action of $\mathcal{U}_{q,t,p}$ on the equivariant elliptic cohomology $\bigoplus_N \mathrm{E}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$, it should be identified with the level (0,1) representation in Theorem 3.5 by identifying $|\lambda\rangle_1$ with the fixed point class $[\lambda]$ in $\bigoplus_N \mathrm{E}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$ in the similar way to the case on the equivariant elliptic cohomology of the partial flag variety [8].

4 The Vertex Operators

We summarize a construction of the type I vertex operator $\Phi(u)$ and its shifted inverse $\Phi^*(u)$. These vertex operators are used to realize the affine quiver W algebra and to calculate instanton partition functions.

4.1 The type I vertex operator

The type I vertex operator is the intertwining operator

$$\Phi(u) : \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(0,1)}_{u} \widetilde{\otimes} \mathcal{F}^{(1,N)}_{v}$$

satisfying

$$\Delta(x)\Phi(u) = \Phi(u)x \qquad \forall x \in \mathcal{U}_{q,t,p}. \tag{4.1}$$

We define the components of $\Phi(u)$ by

$$\Phi(u)|\xi\rangle = \sum_{\lambda\in\mathcal{P}^+} |\lambda\rangle_u \widetilde{\otimes} \Phi_\lambda(u)|\xi\rangle \qquad \forall|\xi\rangle \in \mathcal{F}^{(1,N+1)}_{-uv}. \tag{4.2}$$

Lemma 4.1. *The intertwining relation* (4.1) *reads*

$$\Phi_\lambda(u)\psi^+((t/q)^{1/4}z) = (q/t)^{1/2} B^+_\lambda(u/z;p)\psi^+((t/q)^{1/4}z)\Phi_\lambda(u), \tag{4.3}$$

$$\Phi_\lambda(u)\psi^-((t/q)^{-1/4}z) = (q/t)^{-1/2} B^-_\lambda(z/u;p)\psi^-((t/q)^{-1/4}z)\Phi_\lambda(u), \tag{4.4}$$

$$\begin{aligned}\Phi_\lambda(u)x^+((t/q)^{-1/4}z) &= x^+((t/q)^{-1/4}z)\Phi_\lambda(u) \\ &+ qf(1;p)^{-1}\psi^-((t/q)^{-1/4}z) \sum_{i=1}^{\ell(\lambda)+1} a^-(p)A^-_{\lambda,i}(p)\delta(q^{-1}u_i/z)\Phi_{\lambda-\mathbf{1}_i}(u),\end{aligned} \tag{4.5}$$

$$\begin{aligned}\Phi_\lambda(u)x^-((t/q)^{1/4}z) &= (q/t)^{1/2} B^+_\lambda(u/z;p)x^-((t/q)^{1/4}z)\Phi_\lambda(u) \\ &+ q^{-1}f(1;p)(q/t)^{1/2} \sum_{i=1}^{\ell(\lambda)+1} a^+(p)A^+_{\lambda,i}(p)\delta(u_i/z)\Phi_{\lambda+\mathbf{1}_i}(u).\end{aligned} \tag{4.6}$$

By using the representations in Theorem 3.2 and 3.5, one can solve these relations, and obtain the following result.

Theorem 4.2.

$$\Phi_\lambda(u) = \frac{q^{n(\lambda')} N_\lambda(p) t^*(\lambda,u,v,N)}{c_\lambda} \widetilde{\Phi}_\lambda(u),$$

$$\widetilde{\Phi}_\lambda(u) =: \Phi_\emptyset(u) \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \widetilde{x}^-((t/q)^{1/4} q^{j-1} t^{-i+1} u) :,$$

$$\Phi_\emptyset(u) = \exp\left\{ -\sum_{m>0} \frac{1}{\kappa_m} \alpha'_{-m} ((t/q)^{1/2} u)^m \right\} \exp\left\{ \sum_{m>0} \frac{1}{\kappa_m} \alpha'_m ((t/q)^{1/2} u)^{-m} \right\}$$

where $x^-(z) = u^{-1} z^N (t/q)^{-3N/4} \widetilde{x}^-(z)$ *on* $\mathcal{F}_u^{(1,N)}$ *and*

$$n(\lambda) = \sum_{i\geq 1} (i-1)\lambda_i, \qquad n(\lambda') = \sum_{i\geq 1}(i-1)\lambda'_i = \sum_{i\geq 1} \frac{\lambda_i(\lambda_i - 1)}{2}, \tag{4.7}$$

$$t^*(\lambda,u,v,N) = (q^{-1}v)^{-|\lambda|} (-u)^{N|\lambda|} f_\lambda(q,t)^N, \tag{4.8}$$

$$f_\lambda(q,t) = (-1)^{|\lambda|} q^{n(\lambda')+|\lambda|/2} t^{-n(\lambda)-|\lambda|/2}. \tag{4.9}$$

The factor $t^*(\lambda,u,v,N)$ was introduced in [1]. We also need new factors $N_\lambda(p)$ and $N'_\lambda(p)$ characterized by $N_\lambda(0) = 1 = N'_\lambda(0)$ and

$$\frac{N_\lambda(p)}{N_{\lambda+\mathbf{1}_i}(p)} = \prod_{j=1}^{i-1} \frac{(pu_j/tu_i;p)_\infty (ptu_j/qu_i;p)_\infty}{(pu_j/qu_i;p)_\infty (pu_j/u_i;p)_\infty} \prod_{j=i+1}^{\ell(\lambda)} \frac{(pqu_i/u_j;p)_\infty}{(ptu_i/u_j;p)_\infty} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{(pu_i/u_j;p)_\infty}{(pqu_i/tu_j;p)_\infty}, \tag{4.10}$$

$$\frac{N'_\lambda(p)}{N'_{\lambda+\mathbf{1}_i}(p)} = \prod_{j=1}^{i-1} \frac{(pu_i/u_j;p)_\infty (pqu_i/u_j;p)_\infty}{(pqu_i/tu_j;p)_\infty (ptu_i/u_j;p)_\infty} \prod_{j=i+1}^{\ell(\lambda)} \frac{(pu_j/tu_i;p)_\infty}{(pu_j/qu_i;p)_\infty} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{(ptu_j/qu_i;p)_\infty}{(pu_j/u_i;p)_\infty}. \tag{4.11}$$

In later sections the following formula is useful.

Proposition 4.3.

$$\widetilde{\Phi}_\lambda(u) =: \exp\left(\sum_{m\neq 0} \frac{1-t^m}{\kappa_m} \mathcal{E}_{\lambda,m} \alpha'_m ((t/q)^{1/2} u)^{-m} \right) :, \tag{4.12}$$

where we set

$$\mathcal{E}_{\lambda,m} = \frac{1}{1-t^m} + \sum_{j=1}^{\ell(\lambda)} (q^{-m\lambda_j} - 1) t^{m(j-1)} \qquad (m \in \mathbb{Z}_{\neq 0}).$$

4.2 The shifted inverse $\Phi^*(u)$ of $\Phi(u)$

Let us consider the linear map

$$\Phi^*(u) : \mathcal{F}_u^{(0,1)} \widetilde{\otimes} \mathcal{F}_v^{(1,N)} \to \mathcal{F}_{-uv}^{(1,N+1)}, \tag{4.13}$$

whose components are defined by

$$\Phi^*(u) \left(|\lambda\rangle_u \widetilde{\otimes} |\xi\rangle \right) = \Phi_\lambda^*(u)|\xi\rangle, \qquad \forall |\xi\rangle \in \mathcal{F}_v^{(1,N)}, \tag{4.14}$$

$$\Phi_\lambda^*(u) = \frac{q^{n(\lambda')} N'_\lambda(p) t(\lambda, v, up^{-1}, N)}{c'_\lambda} : \widetilde{\Phi}_\lambda(p^{-1}u)^{-1} : . \tag{4.15}$$

Proposition 4.4. *The vertex operator $\Phi_\lambda^*(u)$ satisfies the following relations.*

$$\psi^+((t/q)^{1/4}z)\Phi_\lambda^*(u) = (t/q)^{-1/2} B_\lambda^+(p^{-1}u/z;p)\Phi_\lambda^*(u)\psi^+((t/q)^{1/4}z), \tag{4.16}$$

$$\psi^-((t/q)^{-1/4}z)\Phi_\lambda^*(u) = (t/q)^{1/2} B_\lambda^-(pz/u;p)\Phi_\lambda^*(u)\psi^-((t/q)^{-1/4}z), \tag{4.17}$$

$$\begin{aligned} x^+((t/q)^{-1/4}z)\Phi_\lambda^*(u) &= \Phi_\lambda^*(u)x^+((t/q)^{-1/4}z) \\ &+ (t/q)^{-1/2}a^+(p) \sum_{i=1}^{\ell(\lambda)+1} A_{\lambda,i}^{+'}(p)\delta(p^{-1}tu_i/qz)\Phi_{\lambda+\mathbf{1}_i}^*(u)\psi^+((t/q)^{1/4}qz/t), \end{aligned} \tag{4.18}$$

$$\begin{aligned} x^-((t/q)^{1/4}z)\Phi_\lambda^*(u) &= (t/q)^{-1/2}B_\lambda^+(p^{-1}u/z;p)\Phi_\lambda^*(u)x^-((t/q)^{1/4}z) \\ &+ (t/q)a^-(p) \sum_{i=1}^{\ell(\lambda)} A_{\lambda,i}^{-'}(p)\delta(p^{-1}q^{-1}u_i/z)\Phi_{\lambda-\mathbf{1}_i}^*(u). \end{aligned} \tag{4.19}$$

These relations are quite similar to those derived from a naively expected intertwining relation given by

$$\Phi^*(u)\Delta(x) = x\Phi^*(u) \qquad\qquad \forall x \in \mathcal{U}_{q,t,p}.$$

But they are not exactly the same. This discrepancy is probably due to a lack of understanding the dual representation to $\mathcal{F}_u^{(0,1)}$. In this sense we have not yet found a representation theoretical meaning of $\Phi^*(u)$.

5 Jordan quiver W-algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$

One of the import feature of the elliptic quantum group is that it gives a realization of the deformed W algebras and provides an algebraic structure i.e. a co-algebra structure, which enables us to construct vertex operators as intertwining operators [9]. In this section, we realize the deformed W algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$ associated with the Jordan quiver $\widehat{A}_0$ [6] by using the level $(1,N)$ representation of $\mathcal{U}_{q,t,p}$ given in §3.1 in the same way as $U_{q,p}(\widehat{\mathfrak{g}})$ realizes $W_{p,p^*}(\mathfrak{g})$ [7,9].

5.1 Screening currents

Let us set

$$s^+_m = \frac{(t/q)^{m/2}}{1-(t/q)^m}\alpha_m, \qquad s^-_m = \frac{(t/q)^{m/2}}{1-(t/q)^m}\alpha'_m.$$

Then from (2.13) and (2.16), one obtains the following commutation relations

$$\begin{aligned} [s^+_m, s^+_n] &= -\frac{1}{m}\frac{1-p^m}{1-p^{*m}}(1-q^m)(1-t^{-m})\delta_{m+n,0}, \\ [s^-_m, s^-_n] &= -\frac{1}{m}\frac{1-p^{*-m}}{1-p^{-m}}(1-q^m)(1-t^{-m})\delta_{m+n,0}. \end{aligned}$$

Moreover one can rewrite the elliptic currents $x^{\pm}(z)$ in Theorem 3.2 as

$$x^{\pm}((t/q)^{1/4}z) = ((t/q)^{N/2}\,u/z^N)^{\pm 1} : \exp\left\{\pm\sum_{m\neq 0} s^{\pm}_m\; z^{-m}\right\} : .$$

Hence one of $x^{\pm}((t/q)^{1/4}z)$ coincides with the screening currents of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ [6][1] with the $SU(4)$ Ω-deformation parameters p, p^*, q, t [12] satisfying

$$p/p^* = t/q.$$

5.2 Generating function

To obtain the generating function of $W_{p,p^*}(\Gamma(\widehat{A}_0))$, we apply the same scheme as used in [7]. Namely we compose the vertex operators $\Phi(u)$ and $\Phi^*(u)$ constructed in § 4 as follows.

$$T(u) \;=\; \Phi^*(u)\Phi(u) = \sum_{\lambda\in\mathcal{P}^+}\Phi^*_\lambda(u)\Phi_\lambda(u) \quad : \; \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(1,N+1)}_{-uv}.$$

Note that $v \in \mathbb{C}^*$ and $N \in \mathbb{Z}$ can be chosen arbitrarily. In the summand, taking the normal ordering one obtains

$$\Phi^*_\lambda(u)\Phi_\lambda(u) \;=\; \mathcal{C}_\lambda(q,t,p) : \widetilde{\Phi}_\lambda(u)\widetilde{\Phi}^*_\lambda(u) : .$$

The operator part turns out to be given by

$$: \widetilde{\Phi}^*_\lambda(u)\widetilde{\Phi}_\lambda(u) :=: \prod_{\square\in A(\lambda)} Y(u/q^{\square}) \prod_{\blacksquare\in R(\lambda)} Y((q/t)u/q^{\blacksquare})^{-1} : \tag{5.1}$$

[1] In [6] only one type of screening current is given. However it is natural for the (deformed) W algebras that there are two types of screening currents. In this sense our realization completes the construction.

with

$$Y(u) =: \exp\left\{\sum_{m\neq 0} y_m u^{-m}\right\} : . \tag{5.2}$$

Here we set $q^{\square} \equiv t^{i-1}q^{-j+1}$ for $\square = (i,j) \in \lambda$ etc., and $y_m = \frac{1-p^m}{\kappa_m}(t/q)^{-m/2}\alpha'_m$. The symbols $R(\lambda)$ and $A(\lambda)$ denote the set of removable and addable boxes in the Young diagram λ, respectively. The combinatorial structure of (5.1) is due to Proposition 4.3, which yields

$$: \widetilde{\Phi}^*_\lambda(u)\widetilde{\Phi}_\lambda(u) :=: \exp\left(\sum_{m\neq 0}\frac{(1-t^m)(1-p^m)}{\kappa_m}\mathcal{E}_{\lambda,m}\alpha'_m((t/q)^{1/2}u)^{-m}\right) :, \tag{5.3}$$

and the following formula.

Proposition 5.1.

$$\mathcal{E}_{\lambda,m} = \frac{1}{1-t^m}\left(\sum_{\square\in A(\lambda)} q^{m\square} - (t/q)^m \sum_{\blacksquare\in R(\lambda)} q^{m\blacksquare}\right) \tag{5.4}$$

Moreover from (2.16) with $\gamma = (t/q)^{1/2}$, one finds the following commutation relation.

$$[y_m, y_n] = -\frac{1}{m}\frac{(1-p^{*m})(1-p^{-m})}{(1-q^m)(1-t^{-m})}\delta_{m+n,0}.$$

This agrees with the one in [6].

The coefficient part in $\Phi^*_\lambda(u)\Phi_\lambda(u)$ can be calculated by combining the normalization factors of the vertex operators and the OPE coefficient. The calculation of the latter coefficient is essentially due to the following formula.

Proposition 5.2.

$$\begin{aligned} &-\frac{1-t^m}{1-q^m}\mathcal{E}_{\lambda,-m}\mathcal{E}_{\mu,m} \\ &= \frac{t^m}{(1-q^m)(1-t^m)} + \sum_{\square\in\mu} q^{m a_\lambda(\square)} t^{m(\ell_\mu(\square)+1)} + \sum_{\blacksquare\in\lambda} q^{-m(a_\mu(\blacksquare)+1)} t^{-m\ell_\lambda(\blacksquare)}. \end{aligned} \tag{5.5}$$

Then one finds

$$\mathcal{C}_\lambda(q,t,p) = \mathcal{C}\mathfrak{q}^{|\lambda|}\mathcal{Z}^{\widehat{A}_0}_\lambda(t,q^{-1},p),$$

where

$$
\begin{aligned}
&\mathfrak{q} = p^{*-1}p^{N-1}(t/q)^{1/2}, && (5.6)\\
&\mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p) = \prod_{\square\in\lambda} \frac{(1-p\,q^{a(\square)+1}t^{\ell(\square)})(1-pq^{-a(\square)}t^{-\ell(\square)-1})}{(1-q^{a(\square)+1}t^{\ell(\square)})(1-q^{-a(\square)}t^{-\ell(\square)-1})}, && (5.7)\\
&\mathcal{C} = \frac{(p^{-1}t;q,t,p)_\infty}{(q;q,t,p)_\infty}. && (5.8)
\end{aligned}
$$

Note that the sum $\sum_{\lambda,|\lambda|=n} \mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p)$ coincides with the equivariant χ_y-genus of the Hilbert scheme of n points on $\mathbb{C}^2$, $\mathrm{Hilb}_n(\mathbb{C}^2)$ with $y=p$. The space $\mathrm{Hilb}_n(\mathbb{C}^2)$ is isomorphic to the moduli space of the rank 1 instantons with charge n.

Note also that one can rewrite (5.7) as

$$
\mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p) = \frac{N_{\lambda\lambda}(pq/t)}{N_{\lambda\lambda}(q/t)} \tag{5.9}
$$

in terms of the 5d analogue of the Nekrasov function given by

$$
N_{\lambda\mu}(x) = \prod_{\square\in\lambda}(1-xq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)}) \prod_{\blacksquare\in\mu}(1-xq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1}). \tag{5.10}
$$

Hence the whole operator

$$
T(u) \;=\; \mathcal{C}\sum_\lambda \mathfrak{q}^{|\lambda|}\mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p) : \prod_{\square\in A(\lambda)} Y(u/q^\square) \prod_{\blacksquare\in R(\lambda)} Y((q/t)u/q^\blacksquare)^{-1} : \tag{5.11}
$$

agrees with the generating function of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ in [6] up to an over all constant factor.

5.3 The higher rank extension

To extend $W_{p,p^*}(\Gamma(\widehat{A}_0))$ to the one associated with the higher rank instantons, one needs to take a composition of $T(u)$'s. By using (5.3) and Propositions 5.1 and 5.2, one obtains the following expression.

$$
\begin{aligned}
T(u_1)\cdots T(u_M) \;=\; & \mathcal{C}_M \sum_{k=0}^{\infty} \mathfrak{q}_M^k \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)} \\ \sum_j |\lambda^{(j)}|=k}} \prod_{i,j=1}^{M} \frac{N_{\lambda^{(i)}\lambda^{(j)}}(pqu_{i,j}/t)}{N_{\lambda^{(i)}\lambda^{(j)}}(qu_{i,j}/t)} \\
& \times : \prod_{l=1}^{M} \prod_{\square\in A(\lambda^{(l)})} Y(u_l/q^\square) \prod_{\blacksquare\in R(\lambda^{(l)})} Y((q/t)u_l/q^\blacksquare)^{-1} :, \quad (5.12)
\end{aligned}
$$

where we set $u_{j,i} = u_j/u_i$, and

$$\mathfrak{q}_M = \mathfrak{q}p^{-(M-1)} = p^{*-1}p^{M+N}(t/q)^{1/2}, \tag{5.13}$$

$$\mathcal{C}_M = \left(\frac{(p^{-1}t;q,t,p)_\infty}{(q;q,t,p)_\infty}\right)^M \prod_{1\le i<j\le M} \frac{(p^{-1}tu_{j,i};q,t)_\infty (pqu_{j,i};q,t)_\infty}{(tu_{j,i};q,t)_\infty (qu_{j,i};q,t)_\infty}. \tag{5.14}$$

In fact, the sum

$$\chi_p(\mathfrak{M}_{k,M}) = \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)} \\ \sum_j |\lambda^{(j)}|=k}} \prod_{i,j=1}^{M} \frac{N_{\lambda^{(i)}\lambda^{(j)}}(pqu_{i,j}/t)}{N_{\lambda^{(i)}\lambda^{(j)}}(qu_{i,j}/t)} \tag{5.15}$$

coincides with the equivariant χ_y $(y = p)$ genus of the moduli space of rank M instantons with charge k.

Note that $T(u)$ gives a non-commutative 5d-analogue of Nekrasov's qq-character of the $\mathcal{N} = 2^*$ $U(1)$ theory [6]. We expect that $T(u_1)\cdots T(u_M)$ gives a non-commutative 5d-analogue of Nekrasov's qq-character of the $\mathcal{N} = 2^*$ $U(M)$ theory.

6 Instanton calculus in the Jordan quiver gauge theories

By using the generating function $T(u)$ of $W_{p,p^*}(\Gamma(\widehat{A}_0))$, one can derive various instanton partition functions of the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ SUSY gauge theories.

6.1 The 5d and 6d lifts of the $\mathcal{N} = 2^*$ $U(1)$ theory

From (5.11) it is immediate to obtain the rank 1 instanton partition function of the 5d lift of the 4d $\mathcal{N} = 2^*$ theory [4] by taking the vacuum expectation value :

$$\langle 0|T(u)|0\rangle = \mathcal{C} \sum_\lambda \mathfrak{q}^{|\lambda|} \mathcal{Z}_\lambda^{\widehat{A}_0}(t, q^{-1}, p). \tag{6.1}$$

For further calculation, it is important to recognize that $T(u)$ can be identified with a basic refined topological vertex depicted in Fig.6.1, which was introduced in [4,5]. This is due to the result (6.1) and our realization $T(u) = \sum_\lambda \Phi_\lambda^*(u)\Phi_\lambda(u)$. Then one can apply $T(u)$ to various instanton calculus.

An immediate application is to take a trace of $T(u)$. Let d be the degree counting operator satisfying

$$[d, \alpha'_m] = m\alpha'_m \qquad m \in \mathbb{Z}_{\neq 0}.$$

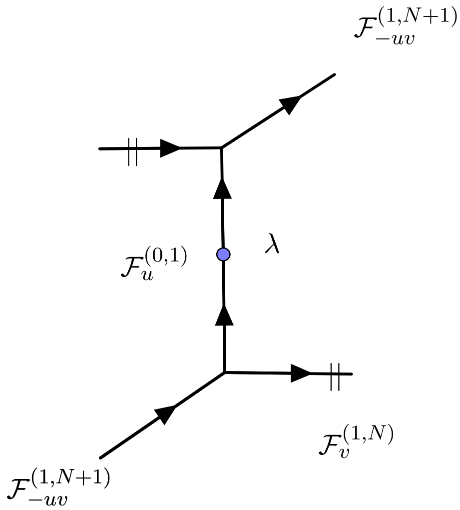

Figure 6.1: Graphical expression of $\sum_\lambda \Phi^*_\lambda(u)\Phi_\lambda(u)$. The two horizontal lines with || are glued together.

Then the following trace yields the 6d version of the partition function of the rank 1 instantions.

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-uv}} Q^d\, T(u) = \mathcal{C}_Q \sum_\lambda \mathfrak{q}^{|\lambda|} \mathcal{Z}^{\widehat{A}_0}_\lambda(t, q^{-1}, p; Q), \tag{6.2}$$

where

$$\mathcal{C}_Q = \frac{1}{(Q;Q)_\infty} \frac{(p^{-1}t; q, t, p)_\infty}{(q; q, t, p)_\infty} \frac{(p^{-1}tQ; q, t, Q)_\infty (pqQ; q, t, Q)_\infty}{(tQ; q, t, Q)_\infty (qQ; q, t, Q)_\infty}, \tag{6.3}$$

$$\mathcal{Z}^{\widehat{A}_0}_\lambda(t, q^{-1}, p; Q) = \frac{N^\theta_{\lambda\lambda}(pq/t; Q)}{N^\theta_{\lambda\lambda}(q/t; Q)}. \tag{6.4}$$

Here $N^\theta_{\lambda\mu}(x;Q)$ denotes the theta function analogue of the Nekrasov function given by

$$N^\theta_{\lambda\mu}(x;Q) = \prod_{\square\in\lambda} \theta_Q(x q^{-a_\mu(\square)-1} t^{-\ell_\lambda(\square)}) \prod_{\blacksquare\in\mu} \theta_Q(x q^{a_\lambda(\blacksquare)} t^{\ell_\mu(\blacksquare)+1}). \tag{6.5}$$

In fact the sum $\sum_{\lambda,|\lambda|=n} \mathcal{Z}^{\widehat{A}_0}_\lambda(t, q^{-1}, p; Q)$ gives the equivariant elliptic genus of $\mathrm{Hilb}_n(\mathbb{C}^2)$.

6.2 The 5d and 6d lifts of the $\mathcal{N}=2^*$ $U(M)$ theory

The higher rank instanton partition functions can be obtained from the composition $T(u_1)\cdots T(u_M)$ in (5.12). The vacuum expectation value gives the instanton partition function of the 5d lift of the 4d $\mathcal{N}=2^*$ $U(M)$ theory.

$$\langle 0|T(u_1)\cdots T(u_M)|0\rangle \quad = \quad \mathcal{C}_M\sum_{k=0}^{\infty}\mathfrak{q}_M^k\,\chi_p(\mathfrak{M}_{k,M}), \tag{6.6}$$

where $\chi_p(\mathfrak{M}_{k,M})$ is given by (5.15).

Furthermore taking the trace of (5.12), one obtains

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-u_1v_1}}Q^d\,T(u_1)\cdots T(u_M) \quad = \quad \mathcal{C}_{Q,M}\sum_{k=0}^{\infty}\mathfrak{q}_M^k\,\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}), \tag{6.7}$$

where $u_1v_1=u_2v_2=\cdots=u_Mv_M$ with arbitrary $v_1,\cdots,v_M\in\mathbb{C}^*$. We here also set

$$\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}) = \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)}\\ \sum_j|\lambda^{(j)}|=k}}\prod_{i,j=1}^{M}\frac{N^{\theta}_{\lambda^{(i)}\lambda^{(j)}}(pqu_{i,j}/t;Q)}{N^{\theta}_{\lambda^{(i)}\lambda^{(j)}}(qu_{i,j}/t;Q)}, \tag{6.8}$$

$$\mathcal{C}_{Q,M} = \frac{1}{(Q;Q)_\infty}\left(\frac{(t;q,t)_\infty\Gamma_3(p^{-1}t;q,t,Q)}{(p^{-1}t;q,t)_\infty\Gamma_3(t;q,t,Q)}\right)^M\prod_{1\le i<j\le M}\frac{\Gamma_3(p^{-1}tu_{j,i};q,t,Q)\Gamma_3(pqu_{j,i};q,t,Q)}{\Gamma_3(tu_{j,i};q,t,Q)\Gamma_3(qu_{j,i};q,t,Q)}. \tag{6.9}$$

The sum $\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M})$ gives the equivariant elliptic genus of the moduli space of rank M instantons with charge k. Hence (6.7) gives the instanton partition function of the 6d lift of the $\mathcal{N}=2^*$ $U(M)$ theory.

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