Geometric R matrices and discrete integrable systems: a study for deriving differential equations

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Abstract

We present two types of systems of differential equations that can be derived from a set of discrete integrable systems which is associated with the geometric R matrices. One is a kind of extended Lotka-Volterra systems, and the other seems to be generally new but reduces to a previously known system in a special case. Both equations are related to Lax equations associated with the loop elementary symmetric functions.

It is a related work done after the talk at RIMS on 19 October 2021, and based on the preprint (arXiv:2203:12325). Interested readers can read the preprint for details and references.

1 Introduction

In a recent paper [1], the author and T. Yoshikawa constructed a new class of discrete integrable systems that can be viewed as a geometric lifting of a class of integrable cellular automata known as the periodic box-ball systems. Since they are related to a realization of type A_{n-1} geometric crystals and geometric R-matrices, we called the new integrable systems *closed geometric crystal chains*.

The purpose of this note is to give an outlook for extending the study given in §2.3.2 of reference [1], which derives a differential equation for $n = 2$ in such a way that respects the integrability of the systems, to that for the case of general n. We present two types of systems of differential equations that can be derived from the closed geometric crystal chains. One is equation (1), which is a kind of extended Lotka-Volterra systems. The other is equation (15) with a function $e^{(\alpha)}_{L-1}$ in (3), which seems to be generally new but reduces to a previously known system by equation (25) in a special case. An important point here is that both equations are related to Lax equations associated with the loop elementary symmetric functions. In this note we restrict ourselves to the former case with $n = 4$ for an explicit derivation of the differential equation.

2 Type I differential equations

2.1 Definitions and the first main result

Let L, n be a pair of coprime integers. Then there is a unique integer $0 < p < n$ such that the condition $Lp \equiv 1 \pmod n$ is satisfied. Let $t \in \mathbb{R}$ be the time variable and $u_i^{(\alpha)}$ be a set of dependent variables labeled by $(\alpha, i) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z})$. Suppose that the system of differential equations

$$\frac{\mathrm{d}u_i^{(\alpha)}}{\mathrm{d}t} = u_i^{(\alpha)} \left(\sum_{j=1}^{\min(Lp-1, L(n-p))} (u_{i-j}^{(\alpha+j)} - u_{i+j}^{(\alpha-j)}) \right), \tag{1}$$

is satisfied by them. This is a kind of the extended Lotka-Volterra systems.

For the set of variables $u_i^{(\alpha)}$ and an integer m, let $e_m^{(\alpha)}$ ($\alpha \in \mathbb{Z}/n\mathbb{Z}$) be the m-th *loop elementary symmetric functions* defined by

$$e_m^{(\alpha)} = \sum_{1 \le j_1 < j_2 < \cdots < j_m \le L} u_{j_1}^{(\alpha+1-j_1)} u_{j_2}^{(\alpha+2-j_2)} \cdots u_{j_m}^{(\alpha+m-j_m)}, \tag{2}$$

and $e_0^{(\alpha)} = 1$, $e_m^{(\alpha)} = 0\,(m < 0 \quad \text{or} \quad m > L)$. In particular, we have $e_1^{(\alpha)} = u_1^{(\alpha)} + u_2^{(\alpha-1)} + \cdots + u_L^{(\alpha-L+1)}$, $e_L^{(\alpha)} = u_1^{(\alpha)} u_2^{(\alpha)} \dots u_L^{(\alpha)}$, and

$$e_{L-1}^{(\alpha)} = \sum_{i=1}^{L} \left(\prod_{j=1}^{i-1} u_j^{(\alpha)} \prod_{k=i+1}^{L} u_k^{(\alpha-1)} \right). \tag{3}$$

Let λ be an indeterminate and $\mathcal{L}$ be the $n \times n$ matrix defined by

$$\mathcal{L} = (\mathcal{L}_{ij})_{1 \le i,j \le n}, \qquad \mathcal{L}_{ij} = \sum_{m \ge 0} e_{j-i+L-mn}^{(i)} \lambda^m, \tag{4}$$

which we call a *Lax matrix*. Let $y^{(\alpha)} = \sum_{j=0}^{p-1} e_1^{(\alpha - jL)} - (p/n) \sum_{r=1}^{n} e_1^{(r)}$, and $\mathcal{Y}$ be the $n \times n$ matrix

$$\mathcal{Y} = \begin{pmatrix} y^{(1)} & & & & \lambda \\ 1 & y^{(2)} & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & y^{(n)} \end{pmatrix}. \tag{5}$$

Through the variables $u_i^{(\alpha)}$, the elements of these matrices are functions of the time variable t.

Theorem 1 *Suppose that the variables $u_i^{(\alpha)}$ are satisfying the system of differential equations* (1). *Then the Lax matrix satisfies the equation*

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = [\mathcal{L}, \mathcal{Y}]. \tag{6}$$

This result implies that conserved quantities of the dynamical system represented by equation (1) can be given by the coefficients of the characteristic polynomial of the Lax matrix $\mathcal{L}$, or equivalently by $\mathrm{Tr}\,\mathcal{L}^m/m\,(m=1,\ldots,n)$.

2.2 Proof of Theorem 1

We define

$$\mathbf{u}_i=(u_i^{(1)},\ldots,u_i^{(n)}),\quad U=(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_L), \tag{7}$$

and $\sigma U=(\mathbf{u}_2,\ldots,\mathbf{u}_L,\mathbf{u}_1)$, where σ denotes the cyclic shift to the left. For the above defined $y^{(\alpha)}$, we write its dependence on the variable U as $y^{(\alpha)}(U)$, and define $y_i^{(\alpha)}=y^{(\alpha)}(\sigma^{i-1}U)$ for any $i\in\mathbb{Z}/L\mathbb{Z}$. Let $\mathcal{Y}_i,\mathcal{M}_i$ denote the $n\times n$ matrices defined by

$$\mathcal{Y}_i=\begin{pmatrix} y_i^{(1)} & & & & \lambda \\ 1 & y_i^{(2)} & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & y_i^{(n)} \end{pmatrix},\qquad \mathcal{M}_i=\begin{pmatrix} u_i^{(1)} & & & & \lambda \\ 1 & u_i^{(2)} & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & u_i^{(n)} \end{pmatrix}. \tag{8}$$

Using the identities $\mathcal{L}=\mathcal{M}_1\cdots\mathcal{M}_L$ and $\mathcal{Y}=\mathcal{Y}_1$, one sees that the assertion of Theorem 1 follows from:

Proposition 2 *The system of differential equations* (1) *is equivalent to*

$$\frac{\mathrm{d}\mathcal{M}_i}{\mathrm{d}t}=\mathcal{M}_i\mathcal{Y}_{i+1}-\mathcal{Y}_i\mathcal{M}_i. \tag{9}$$

Proof. Equation (9) is equivalent to the equations

$$y_{i+1}^{(\alpha)}-y_i^{(\alpha+1)}=u_i^{(\alpha)}-u_i^{(\alpha+1)}, \tag{10}$$

$$\frac{\mathrm{d}u_i^{(\alpha)}}{\mathrm{d}t}=u_i^{(\alpha)}(y_{i+1}^{(\alpha)}-y_i^{(\alpha)}). \tag{11}$$

Thus the assertion of the proposition is a consequence of the following two lemmas. □

Lemma 3 *The relation* (10) *holds for any* $(\alpha,i)\in(\mathbb{Z}/n\mathbb{Z})\times(\mathbb{Z}/L\mathbb{Z})$.

Lemma 4 *The following relation holds:*

$$y_{i+1}^{(\alpha)}-y_i^{(\alpha)}=\sum_{j=1}^{\min(Lp-1,L(n-p))}\left(u_{i-j}^{(\alpha+j)}-u_{i+j}^{(\alpha-j)}\right). \tag{12}$$

Remark 5 *Let the loop elementary symmetric functions* $e_i^{(\alpha)}$ *be denoted by* $e_i^{(\alpha)}(U)$ *for showing their dependence on the variable* U. *Suppose* $p = 1$ *or the condition* $L \equiv 1 (\mod n)$ *is satisfied. Then the set of differential equations* (1) *is written as*

$$\frac{\mathrm{d}u_i^{(\alpha)}}{\mathrm{d}t} = u_i^{(\alpha)} \left(e_1^{(\alpha)}(\sigma^i U) - e_1^{(\alpha)}(\sigma^{i-1} U) \right). \tag{13}$$

In particular, consider the case of $n = 2$. *By the reason that can be shown easily, we can set* $u_i^{(1)} u_i^{(2)} = 1$. *So if we define* $u_i = u_i^{(1)}$, *then* $u_i^{(2)} = 1/u_i$. *In this case one has* $e_1^{(1)}(\sigma^{i-1}U) = \sum_{j=0}^{L-1} (u_{i+j})^{(-1)^j} = u_i + \sum_{j=1}^{L-1} (1/u_{i+j})^{(-1)^{j-1}}$ *and* $e_1^{(1)}(\sigma^i U) = \sum_{j=0}^{L-1} (u_{i+j+1})^{(-1)^j} = \sum_{j=1}^{L} (u_{i+j})^{(-1)^{j-1}} = u_i + \sum_{j=1}^{L-1} (u_{i+j})^{(-1)^{j-1}}$, *because* L *is odd. Therefore equation* (13) *is written as*

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = u_i \sum_{j=1}^{L-1} (-1)^{j-1} \left(u_{i+j} - \frac{1}{u_{i+j}} \right). \tag{14}$$

This is the system of differential equations that we obtained in §2.3.2 of reference [1].

3 Type II differential equations

3.1 Definitions and the second main result

As in §2.1, let $t \in \mathbb{R}$ be the time variable and $u_i^{(\alpha)}$ be a set of dependent variables labeled by $(\alpha, i) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z})$, but now the integers L and n are not necessarily coprime. Suppose that the system of differential equations

$$\frac{\mathrm{d}u_i^{(\alpha)}}{\mathrm{d}t} = u_i^{(\alpha)} \left(\frac{1}{e_{L-1}^{(\alpha)}} \prod_{l=1}^{i-1} u_l^{(\alpha)} \prod_{k=i+1}^{L} u_k^{(\alpha-1)} - \frac{1}{e_{L-1}^{(\alpha+1)}} \prod_{l=1}^{i-1} u_l^{(\alpha+1)} \prod_{k=i+1}^{L} u_k^{(\alpha)} \right), \tag{15}$$

is satisfied by them.

It is easy to see that $e_L^{(\alpha)}$ is a conserved quantity for any α. Recall that the Lax matrix $\mathcal{L}$ was defined by equation (4), and let $\mathcal{Z}$ be the $n \times n$ matrix

$$\mathcal{Z} = \begin{pmatrix} 0 & z^{(2)} & & & \\ & 0 & z^{(3)} & & \\ & & \ddots & \ddots & \\ & & & \ddots & z^{(n)} \\ z^{(1)}/\lambda & & & & 0 \end{pmatrix}, \tag{16}$$

where $z^{(\alpha)} = 1/e_{L-1}^{(\alpha)}$.

Theorem 6 *Suppose that the variables* $u_i^{(\alpha)}$ *are satisfying the system of differential equations* (15), *and that the condition* $e_L^{(\alpha)} = 1$ *is satisfied for any* α. *Then the Lax matrix* $\mathcal{L}$ *satisfies the equation*

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = [\mathcal{L}, \mathcal{Z}]. \tag{17}$$

By Theorems 1 and 6, we see that the dynamical systems represented by equations (1) and (15) share a common Lax matrix defined by (4) in some cases. As a result, they share a common set of conserved quantities in such cases.

3.2 Proof of Theorem 6

For the above defined $z^{(\alpha)}$, we write its dependence on the variable U in (7) as $z^{(\alpha)}(U)$, and define $z_i^{(\alpha)} = z^{(\alpha)}(\sigma^{i-1}U)$ for any $i \in \mathbb{Z}/L\mathbb{Z}$. Let $\mathcal{M}_i$ be the matrix in (8) and define the matrix $\mathcal{Z}_i$ as

$$\mathcal{Z}_i = \begin{pmatrix} 0 & z_i^{(2)} & & & \\ & 0 & z_i^{(3)} & & \\ & & \ddots & \ddots & \\ & & & \ddots & z_i^{(n)} \\ z_i^{(1)}/\lambda & & & & 0 \end{pmatrix}. \tag{18}$$

Proposition 7 *The system of differential equations* (15) *with the condition* $e_L^{(\alpha)} = 1$ *is equivalent to*

$$\frac{\mathrm{d}\mathcal{M}_i}{\mathrm{d}t} = \mathcal{M}_i \mathcal{Z}_{i+1} - \mathcal{Z}_i \mathcal{M}_i. \tag{19}$$

This together with the relations $\mathcal{L} = \mathcal{M}_1 \cdots \mathcal{M}_L$ and $\mathcal{Z} = \mathcal{Z}_1$ yields the assertion of Theorem 6.
Proof. The differential equation (19) is equivalent to

$$u_i^{(\alpha-1)} z_{i+1}^{(\alpha)} = u_i^{(\alpha)} z_i^{(\alpha)}, \tag{20}$$

$$\frac{\mathrm{d}u_i^{(\alpha)}}{\mathrm{d}t} = z_{i+1}^{(\alpha)} - z_i^{(\alpha+1)}. \tag{21}$$

Thus the assertion of the proposition is a consequence of the following two lemmas. □

Lemma 8 *If* $e_L^{(\alpha)}$ *does not depend on* α*, then the relation* (20) *or equivalently the following relation holds:*

$$e_{L-1}^{(\alpha)}(\sigma^i U) \frac{u_i^{(\alpha)}}{u_i^{(\alpha-1)}} = e_{L-1}^{(\alpha)}(\sigma^{i-1} U). \tag{22}$$

Lemma 9 *Suppose* $e_L^{(\alpha)} = 1$ *for any* α*. Then*

$$z_{i+1}^{(\alpha)} - z_i^{(\alpha+1)} = u_i^{(\alpha)} \left(\frac{1}{e_{L-1}^{(\alpha)}} \prod_{l=1}^{i-1} u_l^{(\alpha)} \prod_{k=i+1}^{L} u_k^{(\alpha-1)} - \frac{1}{e_{L-1}^{(\alpha+1)}} \prod_{l=1}^{i-1} u_l^{(\alpha+1)} \prod_{k=i+1}^{L} u_k^{(\alpha)} \right). \tag{23}$$

Remark 10 *There is a simple expression for the system of differential equations* (15) *that can be compared with equation* (13) *in the type I case. In fact, we see that equation* (15) *is written as*

$$\frac{\mathrm{d}u_i^{(\alpha)}}{\mathrm{d}t} = \frac{1}{e_{L-1}^{(\alpha)}(\sigma^i U)} - \frac{1}{e_{L-1}^{(\alpha+1)}(\sigma^{i-1} U)}. \tag{24}$$

In particular, consider the case of $L = 2$. *If we define* $u^{(\alpha)} = u_1^{(\alpha)}$, *then we have* $u_2^{(\alpha)} = 1/u^{(\alpha)}$ *because we set* $e_L^{(\alpha)} = 1$. *Therefore one has*

$$\frac{\mathrm{d}u^{(\alpha)}}{\mathrm{d}t} = \frac{u^{(\alpha)}}{u^{(\alpha)}u^{(\alpha-1)}+1} - \frac{u^{(\alpha)}}{u^{(\alpha)}u^{(\alpha+1)}+1}. \tag{25}$$

This is a previously known differential equation, whose discretization was referred to as a lattice KdV equation.

4 Connection to the closed geometric crystal chains: A case study for n = 4

4.1 A review of the closed geometric crystal chains

We briefly review on the closed geometric crystal chains for the totally one-row tableaux case ([1], §3. 1). We introduce the set $\mathbb{Y}_1 = (\mathbb{R}_{>0})^{n-1} \times \mathbb{R}_{>0}$. Let $(\mathbf{x}, s)$ denote an element of $\mathbb{Y}_1$ with $\mathbf{x} = (x^{(1)}, \dots, x^{(n-1)})$, and let $x^{(n)} := s/(x^{(1)} \cdots x^{(n-1)})$. Furthermore, we define $x^{(i)}$ for arbitrary $i \in \mathbb{Z}$ to be a variable determined from $\mathbf{x}$ by the relation $x^{(i)} = x^{(i+n)}$. In what follows, we set $n = 4$. Given $(\mathbf{x}, s) \in (\mathbb{R}_{>0})^3 \times \mathbb{R}_{>0}$, we define the matrices g^* and g by

$$g^*(\mathbf{x}, s; \lambda) = \begin{pmatrix} x^{(1)}x^{(2)}x^{(3)} & \lambda & \lambda x^{(3)} & \lambda x^{(2)}x^{(3)} \\ x^{(1)}x^{(2)} & x^{(4)}x^{(1)}x^{(2)} & \lambda & \lambda x^{(2)} \\ x^{(1)} & x^{(4)}x^{(1)} & x^{(3)}x^{(4)}x^{(1)} & \lambda \\ 1 & x^{(4)} & x^{(3)}x^{(4)} & x^{(2)}x^{(3)}x^{(4)} \end{pmatrix}, \tag{26}$$

$$g(\mathbf{x}, s; \lambda) = \begin{pmatrix} x^{(1)} & 0 & 0 & \lambda \\ 1 & x^{(2)} & 0 & 0 \\ 0 & 1 & x^{(3)} & 0 \\ 0 & 0 & 1 & x^{(4)} \end{pmatrix}. \tag{27}$$

Actually, any element of the matrix $g^*(\mathbf{x}, s; \lambda)$ is so defined as to be an order 3 minor of the matrix $g(\mathbf{x}, s; \lambda)$. For instance, the top-left element of the former is equal to the determinant of the top-left 3×3 submatrix of the latter.

By Theorem 16 of [1], we see that for any $s, l \in \mathbb{R}_{>0}$ and $(\mathbf{b}_1, \dots, \mathbf{b}_L) \in (\mathbb{R}_{>0})^{3L}$, there is a unique positive real solution $(\mathbf{v}, \mathbf{b}'_1, \dots, \mathbf{b}'_L) \in (\mathbb{R}_{>0})^{3(L+1)}$ to the equation

$$g(\mathbf{b}_1, s; \lambda) \cdots g(\mathbf{b}_L, s; \lambda) g(\mathbf{v}, l; \lambda) = g(\mathbf{v}, l; \lambda) g(\mathbf{b}'_1, s; \lambda) \cdots g(\mathbf{b}'_L, s; \lambda). \tag{28}$$

For any $|\mathbf{b}\rangle = (\mathbf{b}_1, \dots, \mathbf{b}_L)$, let $\mathcal{L}(|\mathbf{b}\rangle; \lambda)$ be the matrix

$$\mathcal{L}(|\mathbf{b}\rangle; \lambda) = g(\mathbf{b}_1, s; \lambda) \cdots g(\mathbf{b}_L, s; \lambda), \tag{29}$$

which is called a Lax matrix, and let $\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle)$ be the matrix

$$\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle) = g^*(\mathbf{b}_1, s; l) \cdots g^*(\mathbf{b}_L, s; l), \tag{30}$$

which we call the monodromy matrix of the Lax matrix $\mathcal{L}(|\mathbf{b}\rangle; \lambda)$ with $\lambda = l$. Note that every matrix element of $\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle)$ is an order 3 minor of $\mathcal{L}(|\mathbf{b}\rangle; l)$.

Let E be the largest eigenvalue in absolute value of matrix $\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle)$, and $\vec{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, 1)^t$ be an eigenvector corresponding to E. By the Perron-Frobenius theorem, E is real positive, $\vec{P}$ is uniquely determined, and $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are all positive. Then, the solution $\mathbf{v} \in (\mathbb{R}_{>0})^3$ of equation (28) is given by $\mathbf{v} = (\mathcal{P}_3, \mathcal{P}_2/\mathcal{P}_3, \mathcal{P}_1/\mathcal{P}_2)$ (Proposition 15 of [1]). This unique solution $\mathbf{v}$ allows us to define $T_l^{(1)} : (\mathbb{R}_{>0})^{3L} \to (\mathbb{R}_{>0})^{3L}$ to be the map given by

$$T_l^{(1)}(\mathbf{b}_1, \dots, \mathbf{b}_L) = (\mathbf{b}'_1, \dots, \mathbf{b}'_L), \tag{31}$$

which is called a time evolution. Due to equation (28), this time evolution (31) is described by a discrete time analogue of the Lax equation

$$\mathcal{L}(T_l^{(1)}|\mathbf{b}\rangle; \lambda) = g(\mathbf{v}, l; \lambda)^{-1} \mathcal{L}(|\mathbf{b}\rangle; \lambda) g(\mathbf{v}, l; \lambda). \tag{32}$$

In what follows, we are going to show that this equation reduces to the continuous time Lax equation (6) in the limit $l \to \infty$ by making several reasonable assumptions.

4.2 Limit for the type I Lax equation

We assume $L \equiv 1(\mod 4)$, and set $L = 4\kappa + 1$. Recall that $|\mathbf{b}\rangle = (\mathbf{b}_1, \dots, \mathbf{b}_L)$ and $\mathbf{b}_i = (b_i^{(1)}, b_i^{(2)}, b_i^{(3)})$. By the reason explained above Proposition 2, the Lax matrix (29) can be identified with the matrix $\mathcal{L}$ defined by (4), in which the loop elementary symmetric functions are defined by (2) but with the substitution $u_i^{(\alpha)} = b_i^{(\alpha)}$. Using the explicit expression (4) and the condition $L = 4\kappa + 1$, we can obtain the asymptotic form of the Lax matrix $\mathcal{L}(|\mathbf{b}\rangle; l)$ under the limit $l \to \infty$ as

$$\mathcal{L}(|\mathbf{b}\rangle; l) \approx \begin{pmatrix} e_1^{(1)} l^\kappa & e_2^{(1)} l^\kappa & e_3^{(1)} l^\kappa & l^{\kappa+1} \\ l^\kappa & e_1^{(2)} l^\kappa & e_2^{(2)} l^\kappa & e_3^{(2)} l^\kappa \\ e_3^{(3)} l^{\kappa-1} & l^\kappa & e_1^{(3)} l^\kappa & e_2^{(3)} l^\kappa \\ e_2^{(4)} l^{\kappa-1} & e_3^{(4)} l^{\kappa-1} & l^\kappa & e_1^{(4)} l^\kappa \end{pmatrix}. \tag{33}$$

We assume that the eigenvalues η_q ($q \in \mathbb{Z}/4\mathbb{Z}$) of the Lax matrix $\mathcal{L}(|\mathbf{b}\rangle; l)$ for sufficiently large l's are given by the Puiseux series expansion

$$\eta_q = l^\kappa \sum_{m=-1}^{\infty} c_m \exp\left(\frac{\pi\sqrt{-1}mq}{2}\right) l^{-m/4}, \tag{34}$$

where $c_{-1}=1$ and $c_0=(\sum_{\alpha=1}^4 e_1^{(\alpha)})/4$. This assumption is consistent with the relations

$$(s-l)^L = \det \mathcal{L}(|\mathbf{b}\rangle;l) = \prod_{q=1}^{4} \eta_q = -l^{4\kappa+1} + \mathcal{O}(l^{4\kappa}),$$

$$(\sum_{\alpha=1}^{4} e_1^{(\alpha)}) l^{\kappa} + \mathcal{O}(l^{\kappa-1}) = \mathrm{Tr}\,\mathcal{L}(|\mathbf{b}\rangle;l) = \sum_{q=1}^{4} \eta_q = 4l^{\kappa}c_0 + \mathcal{O}(l^{\kappa-1}),$$

where $\mathcal{O}$ denotes Landau's symbol, and we used the identity $\det g(\mathbf{b}_i,s;l)=s-l$ for $1\le i\le L$ and the asymptotic form (33). Then, the asymptotic form of the largest eigenvalue of the monodromy matrix $\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle)$ is given by

$$E = \eta_1\eta_3\eta_4 = l^{3\kappa+\frac{3}{4}} + c_0 l^{3\kappa+\frac{2}{4}} + \cdots . \tag{35}$$

In view of (33), we see that the asymptotic form of the monodromy matrix $\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle)$ under the limit $l\to\infty$ is given by

$$\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle) \approx \begin{pmatrix} \mathcal{O}(l^{3\kappa}) & l^{3\kappa+1} & e_1^{(3)} l^{3\kappa+1} & \mathcal{O}(l^{3\kappa+1}) \\ \mathcal{O}(l^{3\kappa}) & \mathcal{O}(l^{3\kappa}) & l^{3\kappa+1} & e_1^{(2)} l^{3\kappa+1} \\ e_1^{(1)} l^{3\kappa} & \mathcal{O}(l^{3\kappa}) & \mathcal{O}(l^{3\kappa}) & l^{3\kappa+1} \\ * & * & * & * \end{pmatrix}, \tag{36}$$

where the last row is omitted because we do not need it. Let M_{ij} denote the ij element of $\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle)$. Then, the eigenvector $\vec{P}=(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3,1)^t$ corresponding to the eigenvalue E is determined by the linear equation

$$\begin{pmatrix} \mathsf{M}_{11}-E & \mathsf{M}_{12} & \mathsf{M}_{13} \\ \mathsf{M}_{21} & \mathsf{M}_{22}-E & \mathsf{M}_{23} \\ \mathsf{M}_{31} & \mathsf{M}_{32} & \mathsf{M}_{33}-E \end{pmatrix} \begin{pmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \\ \mathcal{P}_3 \end{pmatrix} = - \begin{pmatrix} \mathsf{M}_{14} \\ \mathsf{M}_{24} \\ \mathsf{M}_{34} \end{pmatrix}. \tag{37}$$

Let $l=1/\delta^4$. By expressing the solution of this equation by the Cramer formula and then substituting the asymptotic forms (35) and (36) into it, we obtain the expressions

$$\begin{aligned} \mathcal{P}_1 &= \frac{1}{\delta^3} + (e_1^{(1)}+e_1^{(2)}+e_1^{(3)}-3c_0)\frac{1}{\delta^2} + \mathcal{O}\left(\frac{1}{\delta}\right), \\ \mathcal{P}_2 &= \frac{1}{\delta^2} + (e_1^{(1)}+e_1^{(2)}-2c_0)\frac{1}{\delta} + \mathcal{O}\left(1\right), \\ \mathcal{P}_3 &= \frac{1}{\delta} + (e_1^{(1)}-c_0) + \mathcal{O}\left(\delta\right). \end{aligned}$$

Therefore, the asymptotic form of the matrix $g(\mathbf{v},l;\lambda)$ with $\mathbf{v}=(\mathcal{P}_3,\mathcal{P}_2/\mathcal{P}_3,\mathcal{P}_1/\mathcal{P}_2)$ and $l=1/\delta^4$ is expressed as

$$\delta\cdot g(\mathbf{v},1/\delta^4;\lambda) = \delta\cdot \begin{pmatrix} \mathcal{P}_3 & & & \lambda \\ 1 & \mathcal{P}_2/\mathcal{P}_3 & & \\ & 1 & \mathcal{P}_1/\mathcal{P}_2 & \\ & & 1 & 1/(\delta^4\mathcal{P}_1) \end{pmatrix} = \mathbb{I}_4 + \delta\cdot\mathcal{Y} + \mathcal{O}\left(\delta^2\right), \tag{38}$$

where $\mathcal{Y}$ is a matrix of the form (5) with $n = 4, p = 1$.

Now, recall the discrete time Lax equation (32), and let $\mathcal{L}(|\mathbf{b}\rangle;\lambda)$ and $\mathcal{L}(T_l^{(1)}|\mathbf{b}\rangle;\lambda)$ be denoted by $\mathcal{L}(t)$ and $\mathcal{L}(t+\delta)$, respectively. Then by using the expression (38), we obtain the continuous time Lax equation (6) in the limit $\delta \to 0$.

4.3 Limit for the type I differential equation

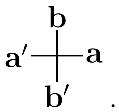
The discussions for deriving the Lax equations in the previous subsection can be generalized to for derivations of equations (1). Consider the following matrix equation

$$g(\mathbf{b}, s;\lambda)g(\mathbf{a}, l;\lambda) = g(\mathbf{a}', l;\lambda)g(\mathbf{b}', s;\lambda). \tag{39}$$

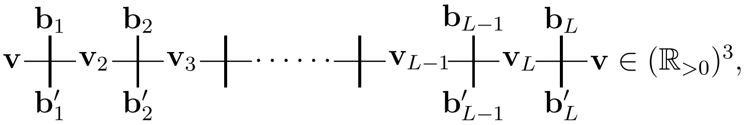
For any $s, l \in \mathbb{R}_{>0}$ and $(\mathbf{a}, \mathbf{b}) \in (\mathbb{R}_{>0})^6$, there is a unique solution $(\mathbf{a}', \mathbf{b}') \in (\mathbb{R}_{>0})^6$ to this matrix equation. Let $R^{(s,l)} : (\mathbb{R}_{>0})^6 \to (\mathbb{R}_{>0})^6$ be a rational map given by $R^{(s,l)} : (\mathbf{b}, \mathbf{a}) \mapsto (\mathbf{a}', \mathbf{b}')$. This is the geometric R-matrix in the present case. An explicit expression for the rational map is written as

$$\begin{aligned}
a'^{(1)} &= a^{(1)}\frac{a^{(2)}a^{(3)}a^{(4)} + a^{(2)}a^{(3)}b^{(1)} + a^{(2)}b^{(4)}b^{(1)} + b^{(3)}b^{(4)}b^{(1)}}{a^{(1)}a^{(2)}a^{(3)} + a^{(1)}a^{(2)}b^{(4)} + a^{(1)}b^{(3)}b^{(4)} + b^{(2)}b^{(3)}b^{(4)}},\\
a'^{(2)} &= a^{(2)}\frac{a^{(3)}a^{(4)}a^{(1)} + a^{(3)}a^{(4)}b^{(2)} + a^{(3)}b^{(1)}b^{(2)} + b^{(4)}b^{(1)}b^{(2)}}{a^{(2)}a^{(3)}a^{(4)} + a^{(2)}a^{(3)}b^{(1)} + a^{(2)}b^{(4)}b^{(1)} + b^{(3)}b^{(4)}b^{(1)}},\\
a'^{(3)} &= a^{(3)}\frac{a^{(4)}a^{(1)}a^{(2)} + a^{(4)}a^{(1)}b^{(3)} + a^{(4)}b^{(2)}b^{(3)} + b^{(1)}b^{(2)}b^{(3)}}{a^{(3)}a^{(4)}a^{(1)} + a^{(3)}a^{(4)}b^{(2)} + a^{(3)}b^{(1)}b^{(2)} + b^{(4)}b^{(1)}b^{(2)}},
\end{aligned}$$

and $b'^{(i)} = a^{(i)}b^{(i)}/a'^{(i)}$. Let the map $R^{(s,l)} : (\mathbf{b}, \mathbf{a}) \mapsto (\mathbf{a}', \mathbf{b}')$ be depicted as



Then the solution of equation (28) must satisfy the diagram



(40)

where $\mathbf{v}_i$'s are defined by the downward recursion relation $R^{(s,l)}(\mathbf{b}_i, \mathbf{v}_{i+1}) = (\mathbf{v}_i, \mathbf{b}'_i)$ with the initial condition $\mathbf{v}_{L+1} = \mathbf{v}$. As in §2.2 let σ denote the cyclic shift to the left, so we have $\sigma|\mathbf{b}\rangle = (\mathbf{b}_2, \ldots, \mathbf{b}_L, \mathbf{b}_1)$. It is easy to see that an obvious generalization of equation (32) is

$$\mathcal{L}(T_l^{(1)}(\sigma^{i-1}|\mathbf{b}\rangle);\lambda) = g(\mathbf{v}_i, l;\lambda)^{-1}\mathcal{L}(\sigma^{i-1}|\mathbf{b}\rangle;\lambda)g(\mathbf{v}_i, l;\lambda). \tag{41}$$

This equation implies that $\mathbf{v}_i$ can also be obtained in the same way as for $\mathbf{v}$ in §4.1, by simply replacing $\mathsf{M}_l^{(1)}(|\mathbf{b}\rangle)$ by $\mathsf{M}_l^{(1)}(\sigma^{i-1}|\mathbf{b}\rangle)$. Based on this fact, one can generalize equation (38) as

$$\delta \cdot g(\mathbf{v}_i, 1/\delta^4;\lambda) = \mathbb{I}_4 + \delta \cdot \mathcal{Y}_i + \mathcal{O}\left(\delta^2\right), \tag{42}$$

where $\mathcal{Y}_i$ is a matrix of the form (8) with $n = 4, p = 1$.

We consider type I case, where $l = 1/\delta^4$. As a vertex in the diagram (40) we have the relation

$$g(\mathbf{b}_i', s; \lambda) = (\delta \cdot g(\mathbf{v}_i, l; \lambda))^{-1} g(\mathbf{b}_i, s; \lambda)(\delta \cdot g(\mathbf{v}_{i+1}, l; \lambda)).$$

Then by setting

$$\mathcal{M}_i(t + \delta) = g(\mathbf{b}_i', s; \lambda), \qquad \mathcal{M}_i(t) = g(\mathbf{b}_i, s; \lambda),$$

and using (42), we can derive equation (9) in the limit $\delta \to 0$. Therefore, equation (1) for $p = 1$ is derived from the closed geometric crystal chains by the above discussions and using Propositions 2.

5 Concluding remarks

We restricted ourselves to the case of type I, $n = 4$ for deriving the differential equations. Type II case for equations (15) and (17) can be treated similarly. It seems that the above derivations can also be applied to the case of general n. It also seems that the derivation is not restricted to the totally one-row tableaux case but can be generalized to the rectangular tableaux cases [1]. We hope that we can report a result for such cases in the near future.

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References

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