Furstenberg measure and Iterated Function Systems with inverses

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ABSTRACT. Motivated by the study of the Furstenberg measure, in [1] the author introduced Iterated Function Systems with inverses (i.e. IFS that contain inverse maps). In this note we present a conjecture.

1. Introduction

Let $\mathcal{F} = \{f_a\}_{a \in \Lambda}$ be a finite collection of linear fractional transformations. Assume that the associated $SL(2, \mathbb{R})$ matrices generate a noncompact and totally irreducible subgroup. Let $p = (p_a)_{a \in \Lambda}$ be a probability vector. It is well-known that there exists a unique Borel probability measure ν on $\mathbb{R}_* = \mathbb{R} \cup \{\infty\}$, called the *Furstenberg measure*, such that

$$\nu = \sum_{a \in \Lambda} p_a f_a \nu,$$

where $f_a \nu$ is the push-forward of ν under the action of $f_a : \mathbb{R}_* \to \mathbb{R}_*$. Furstenberg measure arises naturally in the study of random products of matrices.

We say that $\mathcal{F}$ is *uniformly hyperbolic* if there exists an open set $U \subsetneq \mathbb{R}_*$ with finitely many components having disjoint closures that satisfies $\overline{f_a(U)} \subset U$ for all $a \in \Lambda$. It is known that if $\mathcal{F}$ is uniformly hyperbolic then $\mathcal{F}$ is (essentially) an Iterated Function System (IFS).

From the viewpoint of the random walks on groups, it is natural to consider the case that $\mathcal{F}$ is symmetric ($\mathcal{F}$ is symmetric is $\mathcal{F} = \mathcal{F}^{-1}$). However, it is easy to see that if we have $f, f^{-1} \in \mathcal{F}$ for some $f \neq \mathrm{id}$ then $\mathcal{F}$ is not uniformly hyperbolic.

In [1] the author introduced IFS with inverse. IFS with inverse is in a sense very "close" to IFS. By relying on the transversality argument, in [1] the author showed that for a one-parameter family of IFS with inverse, if the random walk entropy divided the Lyapunov exponent is greater than one then the associated invariant measure is absolutely continuous for a.e. parameter.

In section 2 we discuss IFS with inverse, and in section 3 we pose a conjecture.

2. IFS with inverse

2.1. Settings. In [**1**], IFS with inverse is introduced. In this section we briefly discuss the definition.

Let G be the free group of rank $r \geqslant 2$, and let S be a free generating set of G. Let $\Lambda \subset G$ be such that $S \subset \Lambda \subset S \cup S^{-1}$. Let $\Omega^* = \bigcup_{n \geqslant 1} \Lambda^n$ and $\Omega = \Lambda^{\mathbb{N}}$. For $\omega = \omega_0 \omega_1 \cdots$ we denote $\omega|_n = \omega_0 \cdots \omega_n$. Let $p = (p_a)_{a \in \Lambda}$ be a probability vector, and let μ be the associated Bernoulli measure on Ω. We assume that $p_a \neq 0$ for all $a \in S$.

We say that a (finite or infinite) sequence $\omega \in \Omega^* \cup \Omega$ is *reduced* if $\omega_i \omega_{i+1} \neq aa^{-1}$ for all $i \geqslant 0$ and $a \in \Lambda$. Let Ω^*_{red} (resp. Ω_{red}) be the set of all finite (resp. infinite) reduced sequences. Define the map

$$\mathrm{red} : \Omega^* \to \Omega^*_{\mathrm{red}}$$

in the obvious way. Let $\overline{\Omega} \subset \Omega$ be the set of all ω such that the limit

$$\tag{2.1} \lim_{n \to \infty} \mathrm{red}(\omega|_n)$$

exists. By abuse of notation, for $\omega \in \overline{\Omega}$ we denote the limit (2.1) by $\mathrm{red}(\omega)$. Denote $\Lambda^\star = \left\{ (a,b) \in \Lambda^2 : a \neq b^{-1} \right\}$.

DEFINITION 2.1 (IFS with inverse). Let $\mathcal{X} = \{X_a\}_{a \in \Lambda}$ be a collection of (not necessarily mutually disjoint) open intervals and $\theta \in (0,1]$. Write $X = \bigcup_{a \in \Lambda} X_a$. Suppose that there exists $0 < \gamma < 1$ such that the following holds: for any $(a,b) \in \Lambda^\star$ the map $\varphi_{ab} : X_b \to X_a$ is $C^{1+\theta}$ and satisfies

  (i) $\overline{\varphi_{ab}(X_b)} \subset X_a$;
  (ii) $0 < |\varphi'_{ab}(x)| < \gamma$ for all $x \in X_b$;
  (iii) $\varphi_{ab}^{-1} : \varphi_a(X_b) \to X_b$ is $C^{1+\theta}$.

We call $\Phi = \{\varphi_{ab}\}_{(a,b) \in \Lambda^\star}$ an *IFS with inverse*, and denote $\Phi \in \Gamma_{\mathcal{X}}(\theta)$.

For $\omega = \omega_0 \cdots \omega_n \in \Omega^*_{\mathrm{red}}$, we denote

$$\varphi_\omega = \varphi_{\omega_0 \omega_1} \circ \cdots \circ \varphi_{\omega_{n-1} \omega_n}.$$

Let $\Pi : \overline{\Omega} \to X$ be the natural projection map, i.e.,

$$\Pi(\omega) = \bigcap_{n \geqslant 1} \varphi_{\mathrm{red}(\omega)|_n} (\overline{X_{\mathrm{red}(\omega)_n}}).$$

Define the measure ν by $\nu = \Pi \mu$, where $\Pi \mu$ is the push-forward of the measure μ under the map Π.

2.2. Transversality condition and the main result in [1]. Let $U \subset \mathbb{R}^d$ be an open set. Consider a family of IFS with inverse

$$\Phi^{\mathbf{t}} = \left\{ \varphi_{ab}^{\mathbf{t}} \right\}_{(a,b) \in \Lambda^\star} \in \Gamma_{\mathcal{X}}(\theta), \quad \mathbf{t} \in \overline{U}.$$

Denote by $\Pi_{\mathbf{t}} : \overline{\Omega} \to X$ the natural projection map. Let $\nu_{\mathbf{t}} = \Pi_{\mathbf{t}} \mu$. Assume that for any $(a,b) \in \Lambda^\star$ the maps $\mathbf{t} \mapsto \varphi_{ab}^{\mathbf{t}}$ and $\mathbf{t} \mapsto (\varphi_{ab}^{\mathbf{t}})^{-1}$ are continuous, where $\varphi_{ab}^{\mathbf{t}}$ and $(\varphi_{ab}^{\mathbf{t}})^{-1}$ are equipped with $C^{1+\theta}$ norm. Denote the Lyapunov exponent of $\Phi^{\mathbf{t}}$ by $\chi_{\mathbf{t}}$ and the d-dimensional Lebesgue measure by $\mathcal{L}_d$.

DEFINITION 2.2. We say that $\Phi^{\mathbf{t}}$ satisfies the *transversality condition* if the following holds: there exists a constant $C_1 > 0$ such that for all $\omega, \tau \in \Omega_{\text{red}}$ with $\omega_0 = \tau_0$ and $\omega_1 \neq \tau_1$, we have

$$\mathcal{L}_d \left\{ \mathbf{t} \in U : |\Pi_{\mathbf{t}}(\omega) - \Pi_{\mathbf{t}}(\tau)| < r \right\} < C_1 r \quad \text{for all } r > 0.$$

THEOREM 2.1 (Theorem 2.1 in [1]). *Assume that the transversality condition is satisfied. Then the measure $\nu_{\mathbf{t}}$ is absolutely continuous for a.e. $\mathbf{t}$ in*

$$U' = \left\{ \mathbf{t} \in U : \frac{h_{RW}}{\chi_{\mathbf{t}}} > 1 \right\}.$$

3. Conjectures and open problems

In [1] the author showed that in some cases Furstenberg measure is an IFS with inverse.

CONJECTURE 3.1. If $\mathcal{F} = \{f_a\}_{a \in \Lambda}$ is symmetric, then the associated Furstenberg measure is not an invariant measure for any IFS with inverse.

Therefore, we believe that for showing absolute continuity for a.e. parameter for symmetric generators one needs completely need ideas and techniques.

References

[1] Y. Takahashi, Invariant measures for Iterated Function Systems with inverses, to appear in *J. Fractal. Geom.*

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