Ergodic properties of random dynamical systems via natural extensions of noise transformations

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1. Introduction

We start with the definition of random dynamical system (abbr. RDS). Let $(M, \mathscr{M}, m)$ be a Lebesgue space and $(S, \mathscr{S})$ a countably generated measurable space. The former will be called *the state space of RDS* and the latter *the parameter space of RDS* in this article. Consider a family $\{\tau_s\}_{s\in S}$ of m-nonsingular transformations on $(M, \mathscr{M}, m)$ indexed by S such that the map $S \times M \ni (s, x) \mapsto \tau_s x \in M$ is $(\mathscr{S} \times \mathscr{M})/\mathscr{M}$-measurable. Let $(\Omega, \mathscr{F}, P)$ be a Lebesgue space and $\sigma : \Omega \to \Omega$ a P-preserving transformation which is assumed to be ergodic for the sake of simplicity. The measure-preserving dynamical system (σ, P) will be called *the noise transformation or noise system.* Take an S-valued random variable ξ on $(\Omega, \mathscr{F}, P)$ and define an S-valued strictly stationary process $\{\xi_n\}_{n=0}^{\infty}$ by $\xi_n = \xi \circ \sigma^n$ $(n \geq 0)$. For each n the S-valued random variable ξ_n will be called *the (random) choice at time n.* The family $\mathscr{X} = \{X_n\}$ of randomly composed maps $X_n : M \to M$ is called *the random dynamical system given by* $(\{\tau_s\}_{s\in S}, \sigma, \xi)$ if the maps in $\mathscr{X}$ are defined by

$$X_0(\omega)x = x, \ \ X_{n+1}(\omega)x = \tau_{\xi_n(\omega)} X_n(\omega)x \text{ for } (x, \omega) \in M \times \Omega, \quad (n \geq 0).$$

The main interest of this article is the common statistical behavior of random maps $X_n(\omega)$ with respect to the reference measure m for a great majority of samples $\omega \in \Omega$. It is well known that if $\{\xi_n\}_{n\geq 0}$ is independent, the random sequence $\{X_n x\}_{n\geq 0}$ becomes a Markov chain starting at x and the so-called random ergodic theorem is discussed in Kakutani [6]. Following Kakutani [6], we introduce the skew product transformation $T_{\mathscr{X}} = T_1 : M \times \Omega \to M \times \Omega$ associated to $\mathscr{X}$ by

$$T_1(x, \omega) = (X_1(\omega)x, \sigma\omega) \quad \text{ for } (x, \omega) \in M \times \Omega.$$

Clearly,

$$T^{n+k}(x, \omega) = (X_{n+k}(\omega)x, \sigma^{n+k}\omega) = (X_n(\sigma^k\omega)X_k(\omega)x, \sigma^{n+k}\omega)$$

holds for n, $k \geq 0$. In addition, it is easy to see that T_1 is $m \times P$-nonsingular since each τ_s is m-nonsingular. So one may expect that the study of asymptotic behavior of the RDS $\mathscr{X}$ with respect to m is reduced to that of the single transformation T_1 with respect to $m \times P$.

Recall the study of a single m-nonsingular transformation (τ, m) as a prototype. We usually proceed as follows: We first verify whether an m-absolutely continuous invariant measure (abbr. a.c.i.m.) μ exists or not. Unless otherwise stated invariant measures are assumed to be normalized in this article. If it exists, then next we consider the ergodic

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properties of the measure-preserving dynamical system (τ, μ) (ergodicity, weak-mixing, strong-mixing, exactness in noninvertible case, Kolmogorov property in invertible case etc.). Moreover, if strong ergodic properties e.g. mixing, exactness etc. are established, we may try to show the central limit theorem and the other limit theorems. Therefore the study of statistical properties of the single transformation T_1 via the product measure $m \times P$ may give some clues to our problems. But the following fact makes us recognize that it is not enough when we consider a sort of sample-wise (i.e. ω-wise) properties of the system. Let $\varphi : M \times \Omega \to \Omega$; $(x, \omega) \mapsto \omega$ be the natural projection. Then the commutative diagram

$$\begin{array}{ccc} M \times \Omega & \xrightarrow{T_1} & M \times \Omega \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \Omega & \xrightarrow[\sigma]{} & \Omega \end{array}$$

yields that the noise system (σ, P) should be a factor of the skew product system $(T_1, m \times P)$. Thus one can not expect $(T_1, m \times P)$ having ergodic properties stronger than (σ, P).

Keeping the above situation in mind, we introduce the notion of (*direct*) *product* of a RDS $\mathscr{X}$ given by $(\{\tau_s\}_{s \in S}, \sigma, \xi)$ as the RDS given by $(\{\tau_s \times \tau_s\}_{s \in S}, \sigma, \xi)$ and we denote it by $\mathscr{X} \times \mathscr{X}$, or more simply $\mathscr{X}^2$. Clearly, the corresponding skew product transformation $T_2 : M^2 \times \Omega \to M^2 \times \Omega$ can be defined by

$$T_2(x, y, \omega) = (X_1(\omega)x, X_1(\omega)y, \sigma\omega) \quad \text{for } (x, y, \omega) \in M^2 \times \Omega.$$

and T_2 is $m^2 \times P$-nonsingular. On the other hand, in [2] (see also [1]), the sample-wise (quenched) central limit theorem is obtained by showing the sample-averaged (annealed) central limit theorem for the skew product dynamics T_2 corresponding to $\mathscr{X}^2$ for a class of RDSs $\mathscr{X}$ with independent choices. Inspired by these result the author studies a sample-wise central limit theorem with deterministic centering for a class of RDSs whose choices satisfies the strong mixing conditions but not necessarily independent. By working on the problem above, we get a clue to show that some sample-wise (quenched) ergodic properties of RDSs are obtained by investigating sample-averaged (annealed) ergodic behavior of its product RDS i.e. ergodic properties of a single transformation T_2. In addition we also notice that invertibility of noise dynamics plays the important roles in our investigation.

The purpose of this article is to announce the results obtained in the research above and give some idea to show them. Roughly speaking, we shall pull out some quenched ergodic properties of a RDS $\mathscr{X}$ from appropriate annealed ergodic properties of the product RDS $\mathscr{X}^2$. In order to carry out the study of annealed ergodic properties of the product RDS $\mathscr{X}^2$, we may investigate the ergodic behaviors of the skew product transformation T_2 with respect to the reference measure $m^2 \times P$ following the preceding works [9] and [11] (see also [10] and [14]).

2. Preliminaries

First of all, let us recall the definition of the Perron-Frobenius operators and their basic properties on this occasion. Let $(M, \mathscr{M}, m, \tau)$ be an m-nonsingular dynamical system. As usual it is often denoted by (τ, m) if there is no fear of confusion. The Perron-Frobenius operator for τ with respect to m (abbr. PF operator) is defined to be the positive bounded linear operator on $L^1(m)$ satisfying

$$\int_M (f \circ \tau) g \, dm = \int_M f(\mathscr{L}_{\tau,m} g) \, dm \quad \text{for } f \in L^\infty(m) \text{ and } g \in L^1(m).$$

We summarize the basic facts of the Perron-Frobenius operators in the below.

Proposition 2.1. *Let (τ, m) be an m nonsingular dynamical system. Then we have the following:*

(1) *For $h \in L^1(m)$, hm is τ-invariant if and only if $\mathscr{L}_{\tau,m} h = h$ holds, where hm denotes the m-absolutely continuous measure with density h.*

(2) *Let μ be an m-absolutely continuous τ-invariant probability measure. Consider the measure-preserving dynamical system (τ, μ). Then we have:*

(2-1) *(τ, μ) is ergodic if and only if the eigenspace of $\mathscr{L}_{\tau,\mu} : L^1(\mu) \to L^1(\mu)$ belonging to the eigenvalue 1 is one-dimensional subspace of $L^1(m)$ consisting of constant functions.*

(2-2) *(τ, μ) is weak-mixing if and only if it is ergodic and 1 is the only eigenvalue of modulus 1 for $\mathscr{L}_{\tau,\mu} : L^1(\mu) \to L^1(\mu)$.*

(2-3) *(τ, μ) is strong-mixing if and only if*

$$\int_M f(\mathscr{L}^n_{\tau,\mu} g) \, d\mu \to \int_M f \, d\mu \int_M g \, d\mu \quad (n \to \infty)$$

holds for any $f \in L^\infty(\mu)$ and $g \in L^1(\mu)$.

(2-4) *(τ, μ) is exact, i.e. $\bigcap_{n=0}^\infty \tau^{-n} \mathscr{M}$ is trivial μ-a.e. if and only if*

$$\lim_{n\to\infty} \left\| \mathscr{L}^n_{\tau,\mu} g - \int_M g \, d\mu \right\|_{1,\mu} = 0$$

holds for any $g \in L^1(\mu)$.

Let $\mathscr{X}$ be a RDS given by $(\{\tau_s\}_{s\in S}, \sigma, \xi)$ and $\mathscr{X}^2$ its direct product. T_1 and T_2 denote the skew product transformations associated to $\mathscr{X}$ and $\mathscr{X}^2$, respectively. Our first task is to find a reasonable sufficient condition for the existence of an $m \times P$-a.c.i.m for T_1 and an $m^2 \times P$-a.c.i.m. for T_2. It is easy to see that if $H_2 \in L^2(m^2 \times P)$ is a density of $m^2 \times P$-a.c.i.m. for T_2, then $H_1 \in L^1(m \times P)$ defined by

$$H_1(x, \omega) = \int_M H_2(x, y, \omega) \, m(dy) \quad ((x, \omega) \in M \times \Omega)$$

becomes a density of $m \times P$-a.c.i.m for T_1. Furthermore if the noise transformation σ is invertible, we obtain:

PROPOSITION 2.2. *Suppose that the noise system (σ, P) is invertible. Then T_1 has an $m \times P$-a.c.i.m. if and only if T_2 has an $m^2 \times P$-a.c.i.m.*

Sketch of Proof. By virtue of the remark above, it suffices to show the 'only if ' part. Let $H_1 \in L^1(m \times P)$ is an invariant density for T_1 with respect to $m \times P$. Since the invertibility of σ guarantees that the formula

$$\mathscr{L}_{T,m\times P}\Phi(x,\omega) = \mathscr{L}_{X_1(\sigma^{-1}\omega),m}(\Phi(\cdot,\sigma^{-1}\omega))(x) \qquad P\text{-a.e.}(x,\omega)$$

is valid for $\Phi \in L^1(m \times P)$, it is not hard to see that $H_2 \in L^1(m^2 \times P)$ given by $H_2(x, y, \omega) = H_1(x, \omega)H_1(y, \omega)$ for $(x, y, \omega) \in M^2 \times \Omega$ is an invariant density for T_2 with respect to $m^2 \times P$. □

Note that the commutative diagram

$$\begin{array}{ccc} M^2\times\Omega & \xrightarrow{T_2} & M^2\times\Omega \\ \psi\downarrow & & \downarrow\psi \\ M\times\Omega & \xrightarrow[T_1]{} & M\times\Omega \end{array}$$

holds, where ψ is the natural projection given by $\psi(x, y, \omega) = (x, \omega)$ for $(x, y, \omega) \in M^2 \times \Omega$. This implies the following.

PROPOSITION 2.3. *Let Q_2 be an $m^2 \times P$-a.c.i.m. for T_2 and Q_1 the push-forward of Q_2 by the natural projection ψ. Then Q_1 is an $m \times P$-a.c.i.m. for T_1 and the following hold.*

(1) *If (T_2, Q_2) is ergodic, then so is (T_1, Q_1).*
(2) *If (T_2, Q_2) is weak-mixing, then so is (T_1, Q_1).*
(3) *If (T_2, Q_2) is strong-mixing, then so is (T_1, Q_1).*
(4) *If (T_2, Q_2) is exact, then so is (T_1, Q_1).*

3. EXISTENCE OF A.C.I.M.

We use the same notation as in the previous section. We consider the following conditions:

(UI) $\{\mathscr{L}_{X_n,m}1\}_{n\geq 0}$ is uniformly integrable with respect to $m \times P$.

(UI$_2$). $\{\mathscr{L}_{X_n\times X_n,m^2}1\}_{n\geq 0}$ is uniformly integrable with respect to $m^2 \times P$.

In the above $\mathscr{L}_{X_n(\omega),m} : L^1(m) \to L^1(m)$ and $\mathscr{L}_{X_n(\omega)\times X_n(\omega),m^2} : L^1(m^2) \to L^1(m^2)$ are the Perron-Frobenius operatorers for $X_n(\omega) : M \to M$ and $X_n(\omega) \times X_n(\omega) : M^2 \to M^2$ with respect to m and m^2, respectively.

REMARK 3.1. (1) Recall that a family G in $L^1(m)$ is uniformly integrable if

$$\lim_{a\to\infty} \sup_{g\in G} \int_{(|g|\geq a)} |g|\, dm = 0.$$

In general a family in $L^1(m)$ is uniformly integrable if and only if it is sequentially, weak-compact in $L^1(m)$ (cf. [5] Chapter IV 8-9, 8-11, and 13-54).

(2) $\mathscr{L}_{X_n(\omega),m}1(x,\omega)$ is given by

$$\mathscr{L}_{X_n(\omega),m}1(x,\omega) = \mathscr{L}_{\tau_{\xi_{n-1}(\omega)},m}\mathscr{L}_{\tau_{\xi_{n-2}(\omega)},m}\cdot\cdots\cdot\mathscr{L}_{\tau_{\xi_0(\omega)},m}1(x,\omega).$$

The conditions (UI) and (UI$_2$) imply that $\{\mathscr{L}^n_{T_1,m\times P}1\}_{n\geq 0}$ and $\{\mathscr{L}^n_{T_2,m^2\times P}1\}_{n\geq 0}$ are uniformly integrable with respect to $m\times P$ and $m^2\times P$, respectively. Therefore, by virtue of Kakutani-Yosida Ergodic Theorem [16], the conditions (UI) and (UI$_2$) are sufficient to the existence of an $m\times P$-a.c.i.m. for T_1 and an $m^2\times P$-a.c.i.m. for T_2, respectively. Moreover, we can show the following.

Proposition 3.2. *The conditions* (UI) *and* (UI$_2$) *are equivalent.*

If T_2 has an $m^2\times P$-a.c.i.m. Q_2, then its push-forward $Q_1 = \psi_* Q_2$ is thought as a natural $m\times P$-a.c.i.m. for T_1 corresponding to T_2. Then it is natural to ask the converse problem that given an $m\times P$-a.c.i.m. Q_1 for T_1, are there any natural $m^2\times P$-a.c.i.m. Q_2 for T_2 satisfying $Q_1 = \psi_* Q_2$. In the case when the noise system σ is invertible, the answer is obviously true by Proposition 2.2. In the sequel of this section we consider the methods constructing a natural invariant density for T_2 with respect to $m^2\times P$ starting from a given invariant density for T_1 with respect to $m\times P$.

First we introduce the method of natural extension for our later convenience. Let $(\Omega,\mathscr{F},P,\sigma)$ be a measure-preserving system on a Lebesgue space. Then there exists an invertible measure-preserving system $(\tilde\Omega,\tilde{\mathscr{F}},\tilde P,\tilde\sigma)$ called the natural extension of $(\Omega,\mathscr{F},P,\sigma)$ satisfying the following (i) and (ii), which is unique up to isomorphism.

(i) The commutative diagram

$$\begin{array}{ccc} (\bar\Omega,\bar{\mathscr{F}},\bar P) & \xrightarrow{\bar\sigma} & (\bar\Omega,\bar{\mathscr{F}},\bar P) \\ \pi\big\downarrow & & \big\downarrow\pi \\ (\Omega,\mathscr{F},P) & \xrightarrow[\sigma]{} & (\Omega,\mathscr{F},P) \end{array}$$

holds.

(ii) $\bar{\mathscr{F}}$ is generated by $\bar\sigma^n\pi^{-1}\mathscr{F}$ $(n\in\mathbb{Z})$.

Remark 3.3. (1) $\{\bar{\mathscr{F}}_n = \bar\sigma^n\pi^{-1}\mathscr{F}\}$ is a nondecreasing family of σ fields generates $\bar{\mathscr{F}}$.

(2) Let $\mathscr{X}$ be a RDS given $(\{\tau_s\}_{s\in S},\sigma,\xi)$ and $(\bar\Omega,\bar{\mathscr{F}},\bar P,\bar\sigma)$ the natural extension of the system $(\Omega,\mathscr{F},P,\sigma)$. Define $\bar\xi : \bar\Omega\to S$ by

$$\bar\xi(\bar\omega) = \xi(\pi\bar\omega).$$

Then we obtain a RDS $\bar{\mathscr{X}}$ given by $(\{\tau_s\}_{s\in S},\bar\sigma,\bar\xi)$. Denote by T and $\bar T$ the associated skew product transformations to $\mathscr{X}$ and $\bar{\mathscr{X}}$, respectively. Since $\tau_{\bar\xi(\bar\omega)} = \tau_{\xi(\pi\bar\omega)}$, for each nonnegative integer n we have

$$\bar{X}_n(\bar{\omega}) = X_n(\pi\bar{\omega}).$$

THEOREM 3.4. *Suppose the condition* (UI) *is fulfilled. Let* $H \in L^1(m \times P)$ *be a density of* $m \times P$*-a.c.i.m. for* T_1. *Then there exists a unique* $\bar{H} \in L^1(m \times \bar{P})$ *such that it is a density of* $m \times \bar{P}$*-a.c.i.m. for* $\bar{T}_1$ *and satisfies*

$$H(x, \pi\bar{\omega}) = E_{m\times\bar{P}}[\bar{H} \,|\, \mathscr{M} \times \bar{\mathscr{F}}_0](x, \bar{\omega}) \qquad (m \times \bar{P})\text{-a.e.}(x, \bar{\omega}),$$

where $E_{m\times\bar{P}}[\bar{H} \,|\, \mathscr{M} \times \bar{\mathscr{F}}_0]$ *is the conditional expectation of* $\bar{H}$ *given* $\mathscr{M} \times \bar{\mathscr{F}}_0$ *with respect to* $m \times \bar{P}$.

Sketch of Proof. (Existence) For $n \geq 0$ define $\bar{H}_n$ by

$$\bar{H}_n(x, \bar{\omega}) = \mathscr{L}^n_{\bar{T}}(H(\cdot, \pi\cdot))(x, \bar{\omega}) = \mathscr{L}_{X_n(\pi\bar{\sigma}^{-n}\bar{\omega})}(H(\cdot, \pi\bar{\sigma}^{-n}\bar{\omega})(x),$$

where we write as $\mathscr{L}_{\bar{T}} = \mathscr{L}_{\bar{T}, m\times\bar{P}}$, $\mathscr{L}_{X_n(\pi\bar{\sigma}^{-n}\bar{\omega})} = \mathscr{L}_{X_n(\pi\bar{\sigma}^{-n}\bar{\omega}), m}$ for convenience. Then we can show that $\{(\bar{H}_n, \mathscr{M} \times \bar{\mathscr{F}}_n)\}$ is an L^1-bounded martingale. Further, the condition (UI) yields the uniform integrability of $\{\bar{H}_n\}$. Therefore by Doob Convergence Theorem for uniformly integrable martingale, it converges $m \times \bar{P}$-a.e. and in $L^1(m \times \bar{P})$. The limit $\bar{H}$ is the desired element in $L^1(m \times \bar{P})$.

(Uniqueness) Let $\bar{H}$ and $\bar{K}$ be elements in $(m \times \bar{P})$ satisfying the conditions in the theorem. Then for any $f \in L^1(m)$, $\varphi \in L^\infty(P)$ and $n \geq 0$, we can verify

$$\int_{M\times\bar{\Omega}} f(x)\varphi(\pi\bar{\sigma}^{-n}\bar{\omega})\bar{H}(x, \bar{\omega})\, d(m \times \bar{P}) = \int_{M\times\bar{\Omega}} f(x)\varphi(\pi\bar{\sigma}^{-n}\bar{\omega})\bar{K}(x, \bar{\omega})\, d(m \times \bar{P})$$

by the usual manner. Since $\{\bar{\sigma}^n\pi^{-1}\mathscr{F}\}$ generates $\bar{\mathscr{F}}$, it follows that $\bar{H} = \bar{K}$ $m \times \bar{P}$-.a.e. □

Now by Proposition 2.2, $\bar{H}_2 \in L^1(m^2 \times \bar{P})$ defined by $\bar{H}_2(x, y, \bar{\omega}) = \bar{H}_1(x, \bar{\omega})\bar{H}_1(y, \bar{\omega})$ for $(x, y, \bar{\omega}) \in M^2 \times \bar{\Omega}$ is an invariant density of $m^2 \times \bar{P}$-a.c.i.m. for the skew product transformation $\bar{T}_2$. Consider the conditional expectation of $\bar{H}_2$ given $\mathscr{M}^2 \times \bar{\mathscr{F}}_0 = \mathscr{M}^2 \times \pi^{-1}\mathscr{F}$. Then there exists $H_2 \in L^1(m^2 \times P)$ such that

$$E_{m^2\times\bar{P}}[\bar{H}_2 \,|\, \mathscr{M}^2 \times \bar{\mathscr{F}}_0](\bar{\omega}) = H_2(\cdot, \cdot, \pi\bar{\omega}).$$

We see that $H_2(m^2 \times P)$ is an invariant measure for T_2 such that its push-forward by ψ is $H_1(m \times P)$.

Next, we introduce the method via Kakutani-Yosida Ergodic Theorem, As mentioned in the remark above, if the RDS satisfies the condition (UI), we can apply Kakutani-Yosida Ergodic Theorem to the Perron-Frobenius operator $\mathscr{L}_{T_1, m\times P}$ for T_1 with respect to $m \times P$. Therefore the sequence $(1/n)\sum_{k=0}^{n-1} \mathscr{L}^k_{T_1, m\times P}1$ converges in $L^1(m \times P)$. We denote the limit by H_1. From the basic properties of the Perron-Frobenius operator, H_1 is an invariant probability density of $m \times P$-a.c.i.m. for T_1. Note that any $m \times P$-a.c.i.m. for T_1 is absolutely continuous with respect to the measure $Q_1 = H_1(m \times P)$. In the sequel of this section we construct a natural invariant measure $Q_2 = H_2(m^2 \times P)$ whose push-forward by ψ is Q_1. To this end we consider the element $\tilde{H}_1 \in L^1(m^2 \times P)$ defined by

$$\tilde{H}_1(x, y, \omega) = H_1(x, \omega) H_1(y, \omega) \quad (x, y, \omega) \in M^2 \times \Omega.$$

By Theorem 3.2 we can apply Kakutani-Yosida Ergodic Theorem to $\mathscr{L}_{T_2, m^2 \times P}$. Therefore there exists $H_2 \in L^1(m^2 \times P)$ such that

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \mathscr{L}^k_{T_2, m^2 \times P} \tilde{H}_1 - H_2 \right\|_{1, m^2 \times P} = 0.$$

We can show that $H_1(x, \omega) = \displaystyle\int_M H_2(x, y, \omega)\, m(dy)$ $(m \times P)$-a.e.(x, ω). Moreover, the invariant measure Q_2 is maximal in the following sense.

Theorem 3.5. *Assume that the condition* (UI) *is fulfilled. Let* $Q_1 = H_1(m \times P)$ *with* $H_1 \in L^1(m \times P)$ *be an* $m \times P$*-a.c.i.m.* T_1*. Consider the* $m^2 \times P$*-absolutely continuous measure* $\tilde{Q}_1$ *with density* $\tilde{H}_1$ *given by* $\tilde{H}_1(x, y, \omega) = H_1(x, \omega) H_1(y, \omega)$ *for* $(x, y, \omega) \in M^2 \times \Omega$*. Then* $(1/n) \displaystyle\sum_{k=0}^{n-1} \mathscr{L}_{T_2, m^2 \times P} \tilde{H}_1$ *converges in* $L^1(m^2 \times P)$*. If the limit is denoted by* H_2*,* $Q_2 = H_2(m^2 \times P)$ *is an* $m^2 \times P$*-a.c.i.m. for* T_2 *such that its push-froward by* ψ *is* Q_1 *and any* $\tilde{Q}_1$*-a.c.i.m. for* T_2 *is absolutely continuous with respect to* Q_2*.*

4. Weak-mixing

The notion of weak-mixing plays very important roles in the study of a single measure-preserving transformation. In this section we consider some analogous properties of RDS.

In what follows, $\mathscr{X}$ is a RDS given by $(\{\tau_s\}_{s \in S}, \sigma, \xi,)$ and $\mathscr{X}^2$ is its product RDS defined as RDS given by $(\{\tau_s \times \tau_s\}_{s \in S}, \sigma, \xi,)$. T_1, and T_2 are the skew product transformations corresponding to $\mathscr{X}$ and $\mathscr{X}^2$, respectively. We assume the uniform integrability condition (UI). Given an $m \times P$-a.c.i.m. for T_1 $Q_1 = H_1(m \times P)$, $Q_2 = H_2(m^2 \times P)$ denotes the $m^2 \times P$-a.c.i.m. for T_2 constructed in Theorem 3.5.

For a measure-preserving system (τ, m), it is well known that (τ, m) is weak-mixing if and only if its product system $(\tau \times \tau, m \times m)$ is ergodic. As a trial we compare the ergodic properties of (T_1, Q_1) with that of (T_2, Q_2) although the latter is not the direct product of the former. Let us temporary introduce the notion of conditional weak-mixing. The skew product (T_1, Q_1) said to be *conditionally weak-mixing* if any $F \in L^1(Q_1)$ with

$$\int_M F(x, \omega) H(x, \omega)\, m(dx) = 0$$

satisfies

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left|\int_{M\times\Omega}(F\circ T_1^k)F\,dQ_1\right|=0.$$

Note that $\int_M F(x,\omega)H(x,\omega)\,m(dx)$ is expressed as $E_{Q_1}[F\,|\,\mathrm{proj}_2^{-1}\mathscr{F}](\omega)$ P-a.e. ω by using the conditional expectation. Then we can show the following.

THEOREM 4.1. *Under the condition* (UI), *if* (T_2, Q_2) *is ergodic, then* (T_1, Q_1) *is conditionally weak-mixing.*

Sketch of Proof. Suppose that (T_2, Q_2) is ergodic and $F \in L^\infty(Q_1)$ satisfies

$$\int_M F(x,\omega)H(x,\omega)\,m(dx)=0.$$

First we see that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=0}^{n-1}\left|\int_{M\times\Omega}(F\circ T_1^k)F\,dQ_1\right|^2\\
\leq&\frac{1}{n}\sum_{k=0}^{n-1}\int_\Omega\left|\int_M(F\circ T_1^k)FH_1\,dm\right|^2\,dP\\
=&\frac{1}{n}\sum_{k=0}^{n-1}\int_\Omega\int_{M^2}F(X_k(\omega)x,\sigma^k\omega)\overline{F(X_k(\omega)y,\sigma^k\omega)}\cdot\\
&\qquad\cdot F(x,\omega)\overline{F(y,\omega)}H_1(x,\omega)H_1(y,\omega)\,dm^2dP\\
\to&\int_{M^2\times\Omega}F(x,\omega)\overline{F(y,\omega)}H_2(x,y,\omega)\,d(m^2\times P)\cdot\\
&\qquad\cdot\int_\Omega\int_{M^2}F(x)\overline{F(y)}H_1(x,\omega)H_1(y,\omega)\,dm^2dP\\
=&\int_{M^2\times\Omega}F(x,\omega)\overline{F(y,\omega)}H_2(x,y,\omega)\,d(m^2\times P)\int_\Omega\left|\int_M F(x,\omega)H_1(x,\omega)\,dm\right|^2\,dP\\
=&0.
\end{aligned}$$

In the above, we need the maximality of the measure Q_2 in Theorem 3.5 to justify the convergence in the fifth line. For instance, we divide the argument into two parts according as $(x,y,\omega)\in(H_2>0)$ or $(x,y,\omega)\in(\tilde{H}_1>0)\setminus(H_2>0)$. It is not so hard but slightly long. So we omit it. Now noticing that for $\{a_n\}_{n\geq 0}$, $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} a_k = 0$ if and only if $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} a_k^2 = 0$, the argument above leads us to the desired result. □

Theorem 4.1 has the following corollaries.

Corollary 4.2. *Assume the condition* (UI) *is fulfilled. Let ρ denote the probability measure on M with density $\int_\Omega H_1(\cdot,\omega)\,dP$ with respect to m. If (T_2, Q_2) is ergodic, then for any $f \in L^2(\rho)$, we have*

$$\lim_{n\to\infty}\int_\Omega\left[\frac{1}{n}\sum_{k=0}^{n-1}\left|\int_M\left(f(X_n(\omega)x)-\int_M f(y)H_1(y,\sigma^k\omega)\,dm\right)\cdot\right.\right.$$
$$\left.\left.\cdot\left(f(x)-\int_M f(y)H_1(y,\omega)\,dm\right)H_1(x,\omega)\,dm\right|^2\right]dP = 0.$$

Corollary 4.3. *Assume that m is $\tau_{\xi(\omega)}$-invariant for P-a.e.ω. If $(T_2, m^2\times P)$ is ergodic, then for any $f\in L^2(m)$ we have*

$$\lim_{n\to\infty}\int_\Omega\left[\frac{1}{n}\sum_{k=0}^{n-1}\left|\int_M\left(f(X_n(\omega)x)-\int_M f(y)\,dm\right)\cdot\right.\right.$$
$$\left.\left.\cdot\left(f(x)-\int_M f(y)\,dm\right)dm\right|^2\right]dP = 0.$$

Remark 4.4. Corollary 4.2 and Corollary 4.3 may be regarded as quenched results on random maps $X_n(\omega)$ in the very weak sense. We might say that an annealed condition on the product $\mathscr{X}^2$ (ergodicity of (T_2, Q_2) in this case) yields a sort of quenched weak-mixing property (not P-a.e. but in the sense of $L^2(P)$-convergence).

5. Strong-mixing

In this section $\mathscr{X}$, $\mathscr{X}^2$, T_1, and T_2 are the same as the previous section. Our present concern is the case when there exists a unique $m\times P$-a.c.i.m. Q_1 and the system (T_1, Q_1) is mixing. We first introduce the notion of weak-asymptotic stability. Let (τ, m) be an m-nonsingular system. Let $h\in L^1(m)$ be a probability density. The PF operator $\mathscr{L}_{\tau,m}$ for τ with respect to m is called w-asymptotically stable at $h\in L^1(m)$ if for any $g\in L^1(m)$

$$\lim_{n\to\infty}\mathscr{L}^n_{\tau,m}g = \int_M g\,dm\cdot h \quad \text{weakly in } L^1(m)$$

holds.

Let $Q_1 = H_1(m \times P)$ and $Q_2 = H_2(m^2 \times P)$ be invariant measures for T_1 and T_2, respectively. In addition Q_1 and Q_2 satisfy $\psi_* Q_2 = Q_1$. Now we consider the following conditions.

(MX) $\mathscr{L}_{T_1, m\times P}$ is w-asymptotically stable at $H_1 \in L^1(m \times P)$.

(MX$_2$) $\mathscr{L}_{T_2, m^2\times P}$ is w-asymptotically stable at $H_2 \in L^1(m^2 \times P)$.

One can easily see that the condition (MX$_2$) yields the condition (MX).

We have the following proposition which illustrates that the condition (MX) implies a weak version of quenched mixing property of the RDS.

Proposition 5.1. *Assume that $\mathscr{L}_{T_1, m\times P}$ satisfies* (MX). *Then, for any $f \in L^\infty(m)$ and $g \in L^1(m)$ we have*

$$\int_M f(X_n(\omega)x)g(x)\,dm \to \int_M f\,d\rho \int_M g\,dm \text{ weakly in } L^1(P),$$

where ρ is a probability measure on M with density $h(\cdot) = \int_M H_1(\cdot, \omega)\,P(d\omega)$ with respect to m.

Therefore, we obtain a quenched mixing result of the RDS in the weak L^1 sense if the Perron-Frobenius operator for T_1 is w-asymptotically stable. But the next theorem tells us that except for the trivial case, we can hardly expect the corresponding result in the strong L^1 sense even if the Perron-Frobenius operator for T_2 is w-asymptotically stable.

Theorem 5.2. *Let $\rho = hm$ be the same as in Proposition* 5.1. *Under the condition* (MX$_2$) *the conditions* (1), (2), (3) *below are equivalent.*

(1) *The probability measure ρ on M is L^1-asymptotically invariant in the following sense.*

For any $f \in L^\infty(m)$ we have

$$E\left|\int_M f(X_n(\cdot)x)\,\rho(dx) - \int_M f(x)\,\rho(dx)\right| \to 0.$$

(2) *The RDS $\mathscr{X}$ is mixing in mean in the following sense.*

For any $f \in L^\infty(m)$ and $g \in L^1(m)$ we have

$$E\left|\int_M f(X_n(\cdot)x)g(x)\,m(dx) - \int_M f(x)\,\rho(dx)\int_M g(x)\,dm\right| \to 0.$$

(3) *H_2, H_1, and h satisfy the following.*

$$H_2(x, y, \omega) - H_1(x, \omega)h(y) - H_1(y, \omega)h(x) + h(x)h(y) = 0 \quad (m^2 \times P)\text{-a.e.}(x, y, \omega).$$

In the case when the sequence of choices $\{\xi_n\}_{n\geq 0}$ is independent, Theorem 5.2 has the following corollary.

Corollary 5.3. *In addition to the assumptions in Theorem* 5.2*, we assume the choice* $\{\xi_n\}_{n\geq 0}$ *of the RDS is independent. Then the condition* (3) *in Theorem* 5.2 *is replaced by the condition* $(3)^*$ *below. As a consequence each of* (1)*,* (2)*, and* (3) *in Theorem* 5.2 *is equivalent to* (4) *below.*

$(3)^*$
$$H_1(x,\omega) = h(x) \quad (m\times P)\text{-a.e.}(x,\omega), \textit{ and}$$
$$H_2(x,y,\omega) = h(x)h(y) \quad (m^2\times P)\text{-a.e.}(x,y,\omega).$$

(4) *For any* $f\in L^\infty(m)$ *we have*

$$\int_M f(X_1(\omega)x)\,\rho(dx) = \int_M f(x)\,\rho(dx) \quad P\text{-a.e.}\omega.$$

Sketch of Proof. We just give the idea of proving the equivalence of (3) and $(3)^*$ under the condition that $\{\xi_n\}_{n\geq 0}$ is independent. In such a case the deterministic version lemma in [11] implies that H_1 and H_2 have deterministic versions, i.e. versions free from ω. Thus $H_1(x,\omega)=h(x)$ $m\times P$-a.e. (x,ω). Therefore, (3) yields $(3)^*$. The converse is obvious. □

The assumption of independence can be removed if the condition of uniform integrability is fulfilled.

Theorem 5.4. *In addition to* (MX_2)*, we assume* (UI)*.Then the conditions* (1)*,* (2)*,* (3)*,* $(3)^*$*, and* (4) *in Theorem* 5.2 *and Corollary* 5.3 *are equivalent.*

Sketch of Proof. We restrict ourselves just explain about how to get $(3)^*$ form (3).

We make use of the natural extension $(\bar\sigma,\bar P)$ of the noise system (σ,P). Let $\bar T_1$ and $\bar T_2$ be the skew product transformations on $M\times\bar\Omega$ and $M^2\times\bar\Omega$ associated to RDSs $\mathscr{X}$ and $\mathscr{X}^2$, respectively. By virtue of Theorem 3.4 $\bar T_1$ and $\bar T_2$ have invariant measures $\bar Q_1 = \bar H_1(m\times P)$ and $\bar Q_2=\bar H_2(m^2\times P)$ such that

$$(5.1)\qquad \begin{aligned} H_1(x,\pi\bar\omega) &= E_{m\times\bar P}[\bar H_1\,|\,\mathscr{M}\times\bar{\mathscr{F}}_0](x,\bar\omega) \quad (m\times P)\text{-a.e.}(x,\bar\omega)\\ H_2(x,y,\pi\bar\omega) &= E_{m^2\times\bar P}[\bar H_2\,|\,\mathscr{M}^2\times\bar{\mathscr{F}}_0](x,y,\bar\omega) \quad (m^2\times P)\text{-a.e.}(x,y,\bar\omega),\end{aligned}$$

where $\pi:\bar\Omega\to\Omega$ is the natural projection. Combining the condition (MX_2) with the fact that $\bar{\mathscr{F}}_n=\bar\sigma^n\pi^{-1}\mathscr{F}$ $(n\geq 0)$ generates $\bar{\mathscr{F}}$, we can show that (3) holds if one replaces H_2 and H_1 with $\bar H_2$ and $\bar H_1$. Since $\bar H_2(x,y,\bar\omega)=\bar H_1(x,\bar\omega)\bar H_1(y,\bar\omega)$ $m^2\times\bar P$-a.e. $(x,y,\bar\omega)$ holds in this case, (3) yields

$$\bar H_1(x,\bar\omega)\bar H_1(y,\bar\omega) - \bar H_1(x,\bar\omega)h(y) - \bar H_1(y,\bar\omega)h(x) + h(x)h(y) = 0 \quad (m^2\times\bar P)\text{-a.e.}(x,y,\bar\omega).$$

Therefore we have $\bar H_1(x,\bar\omega)=h(x)$ $(m\times\bar P)$-a.e.$(x,\bar\omega)$. Thus by (5.1) we arrive at $(3)^*$. □

6. Central limit theorem

In this section $\mathscr{X}$, $\mathscr{X}^2$, T_1, T_2, $\mathscr{L}_{T_1}=\mathscr{L}_{T_1,m\times P}$, $\mathscr{L}_{T_2}=\mathscr{L}_{T_2,m^2\times P}$ are the same as in the previous section.

We need notions and results in [13]. First we recall the asymptotic stability of the PF operator. Let $(M, \mathscr{M}, m, \tau)$ be a m-nonsingular dynamical system. The PF operator $\mathscr{L}_{\tau,m}$ for τ is called to be *asymptotically stable at* $h \in L^1(m)$ if there exists a probability density $h \in L^1(m)$ such that for any $g \in L^1(m)$

$$\lim_{n\to\infty} \int_M \left| \mathscr{L}_{\tau,m}^n g - \left(\int_M g\, dm \right) h \right| dm = 0$$

holds (see [8] Chapter 5). We consider the following conditions on $\mathscr{L}_{T_1}$ and $\mathscr{L}_{T_2}$.

(AS) The PF operator for T_1 with respect to $m \times P$ is asymptotically stable at H_1.

(AS$_2$) The PF operator for T_2 with respect to $m^2 \times P$ is asymptotically stable at H_2.

Remark 6.1. (1) Clearly, the condition (AS$_2$) yields the condition (AS).

(2) If the condition (AS) is satisfied, the measure-preserving system (T_1, Q_1) with $Q_1 = H_1(m \times P)$ is exact. Therefore, so is the noise system (σ, P). Consequently, it is noninvertible.

Before going to the body of this section, we prepare some notation. Let $(M, \mathscr{M}, m, \tau)$ be an m-nonsingular dynamical system, f a function on M, and n a nonnegative integer. Put

$$S_n(\tau)f = \sum_{k=0}^{n-1} f \circ \tau^k.$$

Now if the condition (AS) is fulfilled, for any $\Phi \in L^1(m \times P)$ we obtain

$$\lim_{n\to\infty} \int_{M\times\Omega} \left| \mathscr{L}_{T_1}^n \Phi - \left(\int_{M\times\Omega} \Phi\, d(m\times P) \right) H_1 \right| d(m\times P) = 0.$$

From this fact it follows that for P-a.e.ω and any observable $f \in L^\infty(m)$ on M, we see that

$$\frac{1}{n} S_n(T_1) f(x,\omega) = \frac{1}{n} \sum_{k=0}^{n-1} f(X_k(\omega)x) \to \int_M f\, d\rho \quad m\text{-a.e.}x \tag{6.1}$$

holds, where ρ is a probability measure on M with density

$$h(\cdot) = \int_\Omega H_1(\cdot,\omega)\, P(d\omega).$$

Therefore we may say that quenched (i.e. sample-wise) strong law of large numbers is valid for the RDS $\mathscr{X}$. For an observable $f \in L^\infty(m)$ we consider the following condition

(DC) $$\int_M f\, d\rho \quad \left(= \int_M fh\, dm \right) \quad = \quad 0$$

and say that the observable f satisfies *the deterministic centering condition or non random centering condition.* As we just have obtained a sort of sample-wise law of large numbers (6.1), we are now in a position to consider the central limit theorem for $(1/\sqrt{n})S_n(T_1)f$ under the condition (DC). For the annealed case we have the following.

THEOREM 6.2. *Assume that the PF operator $\mathcal{L}_{T_1}$ for the skew product transformation T_1 associated to $\mathscr{X}$ satisfies the condition* (AS). *Let $v \geq 0$ and $f \in L^\infty(m)$ an observable satisfying the condition* (DC). *Then* (1) $\sim$ (6) *below are equivalent.*

(1) *There exists an $m \times P$-absolutely continuous probability measure Q such that the distribution of $S_n(T_1)f/\sqrt{n}$ with respect to Q converges in distribution to the normal distribution $N(0,v)$.*

(2) *For any $m \times P$-absolutely continuous probability measure Q, the distribution of $S_n(T_1)f/\sqrt{n}$ with respect to Q converges in distribution to the normal distribution $N(0,v)$.*

(3) *There exists a probability density $g \in L^1(m)$ such that for any bounded continuous function u on $\mathbb{R}$, the sequence of random variables $\int_M u(S_n(T_1)f(x,\cdot)/\sqrt{n})g(x)\,m(dx)$ converges weakly to $\int_{\mathbb{R}} u(t)\,N(0,v)(dt)$ in $L^1(P)$.*

(4) *For any bounded continuous function u on $\mathbb{R}$ and for any probability density $g \in L^1(m)$, the sequence of random variables $\int_M u(S_n(T_1)f(x,\cdot)/\sqrt{n})g(x)\,m(dx)$ converges weakly to $\int_{\mathbb{R}} u(t)\,N(0,v)(dt)$ in $L^1(P)$.*

(5) *There exists a probability density $g \in L^1(m)$ such that for any $t \in \mathbb{R}$ the sequence of random variables $\int_M e^{\sqrt{-1}t(S_n(T_1)f(x,\cdot)/\sqrt{n})}g(x)\,m(dx)$ converges weakly to $e^{-vt^2/2}$ in $L^1(P)$.*

(6) *For any probability density $g \in L^1(m)$ and $t \in \mathbb{R}$ the sequence of random variables $\int_M e^{\sqrt{-1}t(S_n(T_1)f(x,\cdot)/\sqrt{n})}g(x)\,m(dx)$ converges weakly to $e^{-vt^2/2}$ in $L^1(P)$.*

□

From Theorem 6.2 we see that for an observable $f \in L^\infty(m)$ with the condition (DC) the distribution of $S_n(T_1)f/\sqrt{n}$ with respect to $m \times P$ satisfies the central limit theorem if and only if $\int_M u(S_n(T_1)f(x,\cdot)/\sqrt{n})g(x)\,m(dx)$ converges weakly to $\int_{\mathbb{R}} u(t)\,N(0,v)(dt)$ in $L^1(P)$ for any bounded continuous function u on $\mathbb{R}$. So it is natural to ask when the convergence of $\int_M u(S_n(T_1)f(x,\cdot)/\sqrt{n})g(x)\,m(dx)$ strong-L^1 or more.

In what follows we assume the validity of 'annealed' type central limit theorem for T_1 and proceed to arguments about 'quenched' type results. To this end we impose the conditions on T_1 and T_2 sufficient for that Gordin's theorem holds (for Gordin's theorem, consult the book [4]).

For $f \in L^1(m)$, F_f and $\tilde{f}$ are members of $L^1(m^2)$ defined by

$$F_f(x,y) = f(x) - f(y), \quad \tilde{f}(x,y) = f(x)f(y) \quad ((x,y) \in M^2).$$

For a $\mathscr{M}$-measurable function f and $\mathscr{M}^2$-measurable function F, we briefly write as

$$S_n f(x,\omega) = S_n(T)f(x,\omega), \quad S_n F(x,y,\omega) = S_n(T_2)F(x,y,\omega).$$

Note that whether $f \in L^\infty(m)$ satisfies the condition (DC), i.e. $\displaystyle\int_M fh\,dm = 0$ for T_1 or not, F_f satisfies the condition (DC) i.e. $\displaystyle\int_{M^2} F_f h_2\,dm^2 = 0$ for T_2, where

$$h_2(x,y) = \int_\Omega H_2(x,y,\omega)\,P(d\omega).$$

Indeed, since H_2 is the limit of $\mathscr{L}_{T_2}^n \tilde{H}_1$ in $L^1(m^2 \times P)$, it is symmetric in the variables x, y and

$$\begin{aligned}\int_{M^2} F_f h_2\,dm^2 &= \int_{M^2} (f(x) - f(y)) \left(\int_\Omega H_2(x,y,\omega)\,dP \right) dm^2 \\ &= \int_{M^2\times\Omega} (f(x) - f(y)) H_2(x,y,\omega)\,d(m^2 \times P) = 0\end{aligned}$$

holds.

In the sequel, we assume the condition (AS$_2$) i.e. $\mathcal{L}_{T_2}$ is asymptotically stable at H_2. As noted above, this yields the condition (AS), i.e.$\mathcal{L}_{T_1}$ is asymptotically stable at H_1. We need to introduce some quantities and the conditions on them.

For $\Phi \in L^1(m \times P)$, , $\Psi \in L^1(m^2 \times P)$, and nonnegative integer n, put

$$\Delta(T_1, \Phi, n) = \mathcal{L}_{T_1}^n \Phi - \int_{M\times\Omega} \Phi\,d(m \times P) \cdot H_1,$$

$$\Delta(T_2, \Psi, n) = \mathcal{L}_{T_2}^n \Psi - \int_{M^2\times\Omega} \Psi\,d(m^2 \times P) \cdot H_2.$$

For a real-valued observable $f \in L^\infty(m)$, consider the autocorrelation coefficient

$$C(T_1, f, n) = \int_{M\times\Omega} (f \circ T_1^n) f H_1\,d(m \times P) - \left(\int_{M\times\Omega} f H_1\,d(m \times \Omega) \right)^2,$$

$$C(T_2, F_f, n) = \int_{M^2\times\Omega} (F_f \circ T_2^n) F_f H_2\,d(m^2 \times P) - \left(\int_{M^2\times\Omega} F_f H_2\,d(m^2 \times \Omega) \right)^2.$$

and the condition

$$(\Sigma_2) \qquad \sum_{n=0}^{\infty} \|\Delta(T_2, fH_2, n)\|_{1,m^2\times P} < \infty,$$

where fH_2 stands for the function defined by

$$(fH_2)(x,y,\omega) = f(x)H_2(x,y,\omega) \qquad ((x,y,\omega)\in M^2\times\Omega).$$

We note that if the condition (Σ_2) is satisfied, we can show that

$$(\Sigma) \qquad \sum_{n=0}^{\infty} \|\Delta(T_1, fH_1, n)\|_{1,m\times P} < \infty$$

holds, where fH_1 stands for the function defined by

$$(fH_1)(x,\omega) = f(x)H_2(x,\omega) \qquad ((x,\omega)\in M\times\Omega).$$

Furthermore, since $H_2(x,y,\omega) = H_2(y,x,\omega)$ holds true, it follows that

$$\sum_{n=0}^{\infty} \|\Delta(T_2, F_f H_2, n)\|_{1,m^2\times P} < \infty.$$

By virtu of the basic properties of the PF operator, we see that

$$\|\Delta(T_1, fH_1, n)\|_{1,m\times P} = \|E_{Q_1}\big[f - E_{Q_1}[f]\,|\,T_1^{-n}(\mathscr{M}\times\mathscr{F})\big]\|_{1,Q_1}$$
$$\|\Delta(T_2, F_f H_2, n)\|_{1,m^2\times P} = \|E_{Q_2}\big[F_f\,|\,T_2^{-n}(\mathscr{M}^2\times\mathscr{F})\big]\|_{1,Q_2}$$

hold, where $Q_1 = H_1(m\times P)$, $Q_2 = H_2(m^2\times P)$.

Therefore we can apply Gordin's theorem to $(S_n(T_1)f - n\int_{M\times P} fH_1 d(m\times P))/\sqrt{n}$ and $S_n(T_2)F_f/\sqrt{n}$ with respect to Q_1 and Q_2 with limiting variances

$$v(f) = v(T_1, f) = C(T_1, f, 0) + 2\sum_{n=1}^{\infty} C(T_1, f, n),$$

$$v(F_f) = v(T_2, F_f) = C(T_2, F_f, 0) + 2\sum_{n=1}^{\infty} C(T_2, F_f, n),$$

respectively. Namely, the annealed type central limit holds.

In the following theorem for a function Φ on $M\times\Omega$, $\bar\Phi$ denotes the function on $M^2\times\Omega$ defined by $\bar\Phi(x,y,\omega) = \Phi(x,\omega)\Phi(y,\omega)$.

Theorem 6.3 ([13], cf. [1]). *Assume that the PF operator $\mathscr{L}_{T_2}$ for T_2 with respect to $m^2\times P$ satisfies the condition* (AS$_2$) *and an observable $f\in L^\infty(m)$ satisfies the condition* (DC). *In addition, we assume that the condition (Σ_2). Then* (1)$\sim$(9) *below are equivalent.*

(1) *There exists a probability density $g\in L^1(m)$ such that the distribution of $S_nF_f/\sqrt{n}$ with respect to the $m^2\times P$-absolutely continuous probability with density $\tilde g$ converges in distribution to the normal distribution $N(0, 2v(f))$.*

(2) *For any probability density* $g \in L^1(m)$*, the distribution of* $S_n F_f/\sqrt{n}$ *with respect to the* $m^2 \times P$*-absolutely continuous probability with density* $\tilde{g}$ *converges in distribution to the normal distribution* $N(0, 2v(f))$*.*

(3) *There exists probability density* $g \in L^1(m)$ *such that for any bounded continuous function* u *on* $\mathbb{R}$*, the sequence of random variables* $\int_M u(S_n f/\sqrt{n}) g\, dm$ *converges strongly to* $\int_{\mathbb{R}} u\, dN(0, v(f))$ *in* $L^1(P)$*.*

(4) *For any probability density and for any bounded continuous function* u *on* $\mathbb{R}$*, the sequence of random variables* $\int_M u(S_n f/\sqrt{n}) g\, dm$ *converges strongly to* $\int_{\mathbb{R}} u\, dN(0, v(f))$ *in* $L^1(P)$*.*

(5) *There exists a probability density* $g \in L^1(m)$ *such that for any* $t \in \mathbb{R}$*, the sequence of random variables* $\int_M e^{\sqrt{-1}t(S_n f/\sqrt{n})} g\, dm$ *converges strongly to* $e^{-v(f)t^2/2}$ *in* $L^1(P)$*.*

(6) *For any probability density* $g \in L^1(m)$ *and* $t \in \mathbb{R}$*, the sequence of random variables* $\int_M e^{\sqrt{-1}t(S_n f/\sqrt{n})} g\, dm$ *converges strongly to* $e^{-v(f)t^2/2}$ *in* $L^1(P)$*.*

(7)
$$v(F_f) = 2v(f).$$

(8)
$$\int_{M^2\times\Omega} f(x)f(y)H_2\, d(m^2 \times P) + 2\sum_{n=1}^{\infty} \int_{M^2\times\Omega} f(x)f(X_n(\omega)y)H_2\, d(m^2 \times P) = 0.$$

(9)
$$\lim_{n\to\infty} \frac{1}{n} \int_{M^2\times\Omega} S_n f(x,\omega) S_n f(y,\omega) H_2\, d(m^2 \times P) = 0.$$

From Theorem 6.3, one recognize that although at the first glance the deterministic condition (DC) seems natural, it is not appropriate in the quenched situation. So we need to consider sample-dependent centering or random centering.

In the rest of this this section, we impose the uniformly continuity condition (UI) in addition to the conditions (AS_2) and (Σ_2) on our RDS in order to utilize the natural extension of the noise system $(\sigma.P)$. The invariant densities H_1 and H_2 for T_1 and T_2 with respect to $(m \times P)$ and $m^2 \times P$ are extended to the invariant densities $\bar{H}_1$ and $\bar{H}_2$ for $\bar{T}_1$ and $\bar{T}_2$ with respect to $m \times \bar{P}$ and $m^2 \times \bar{P}$, respectively. We extend the distribution of $S_n(T_1)f$ with respect to $m \times P$ to that of $S_n(\bar{T}_1)f$ with respect to $m \times \bar{P}$,

For a observable $f \in L^\infty(m)$ we consider the *random centering* (cf. [7], [15])

$$\bar{f}(x, \bar{\omega}) = f(x) - \int_M f(y)\bar{H}_1(y, \bar{\omega})\, m(dy)$$

and the sample-wise asymptotic behavior of the distribution of

$$\frac{1}{\sqrt{n}} S_n(\bar{T}_1)\bar{f}(x,\bar{\omega}) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(f(X_n(\bar{\omega})x) - \int_M f(y)\bar{H}_1(y,\bar{\sigma}^k\bar{\omega})\, m(dy) \right)$$

with respect to m-absolutely continuous probability measures.

With the notation above, we obtain the following.

THEOREM 6.4. *In addition to the assumptions in Theorem* 6.3 *we assume that the condition* (UI). *Then the following condition* (10) *is equivalent to each of the conditions* (1)∼(9) *in Theorem* 6.3.

(10) *There exists a* $\bar{\varphi} \in L^2(\bar{P})$ *such that* $\displaystyle\int_M f(x)\bar{H}_1(x,\bar{\omega})\, dm = \bar{\varphi}(\bar{\sigma}\bar{\omega}) - \bar{\varphi}(\bar{\omega})$ $\bar{P}$*-a.e.* $\bar{\omega}$.

Put

$$\Xi(\bar{\omega}) = \int_M f(y)\bar{H}_1(y,\bar{\omega})\, dm.$$

The conditions (AS_2) and (Σ_2) guarantees that the series given by the autocorrelation coefficients $C(\bar{\sigma},\Xi,n)$ of the strictly stationary random sequence $\{\Xi\circ\bar{\sigma}^n\}_{n\geq 0}$ on $(\bar{\Omega},\bar{\mathscr{F}},\bar{P})$ is absolutely convergent and the condition (8) in Theorem 6.3 yields

$$C(\bar{\sigma},\Xi,0) + 2\sum_{n=1}^{\infty} C(\bar{\sigma},\Xi,n) = 0.$$

It can be shown that this is equivalent to the fact that there exists a function $\bar{\varphi} \in L^2(\bar{P})$ such that

$$\Xi(\bar{\omega}) = \bar{\varphi}(\bar{\sigma}\bar{\omega}) - \bar{\varphi}(\bar{\omega}) \quad P\text{-a.e.}\,\bar{\omega}.$$

Finally, we state a sort of quenched central limit theorem for the extended RDS given by the natural extension $(\bar{\sigma},\bar{P})$ of (σ,P).

THEOREM 6.5. *Under the same notation, we assume that* (AS_2), (Σ_2), *and* (UI). *Then for any* $t \in \mathbb{R}$, *we have*

$$E_{\bar{P}}\left| \int_M \exp\left(\frac{\sqrt{-1}t(S_n(\bar{T}_1)(f-\Xi)}{\sqrt{n}} \right) \bar{H}_1(x,\bar{\omega})\, dm - e^{-vt^2/2} \right| \to 0 \quad (n\to\infty),$$

where $v = v(T_2,F_f)/2$.

We do not have enough space to give the proofs of our results here. The details will be published elsewhere.

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