Pseudo definable spaces

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1 Introduction

An expansion $\mathcal{M} = (M, <, \dots)$ of dense linear order $<$ without endpoints is *locally* o-*minimal* if for any point $x \in M$ and for any definable subset A of M, there exists an open interval $I \ni x$ such that $A \cap I$ is a finite union of open intervals and points [14]. We say that $\mathcal{M}$ is *definably complete* if any nonempty subset A of M has $\sup A, \inf A \in M \cup \{-\infty, \infty\}$ [11].

Let $\mathcal{M}$ be an expansion of dense linear order without endpoints.

We define *pseudo definable spaces* to generalize definable spaces which are introduced in [3]. This paper considers a definable imbedding theorem of pseudo definable spaces under certain conditions demonstrated in [7], which is a continued paper of [4]. The third author [9], the first author and the second author [6] presented other parts of [7] in 2021 Model Theory Workshop. Another proof of [9] is presented by [15].

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2 History of definable spaces and generalization of them

In 1981, Definition 3 of [2] introduces semialgebraic spaces in a real closed field R. They are defined as ringed spaces. Roughly speaking, they are gluing spaces of semialgebraic sets by semialgebraic homeomorphisms.

Robson proves the following imbedding theorem [13].

Theorem 2.1 ([13]). *Every regular semialgebraic space is semialgebraically imbeddable into some* R^l.

van den Dries introduces definable spaces in an o-minimal expansion of a real closed field [3] which are generalizations of semialgebraic spaces. They are defined as ringed spaces too.

van den Dries [3] proves the following imbedding theorem.

Theorem 2.2 ([3]). *Every regular definable space is definably imbeddable into some* R^l.

We wish to define generalized spaces of definable ones in an expansion of dense linear order $<$ without endpoints and consider a definable imbedding theorem under certain conditions.

We recall some definitions and results.

Definition 2.3 ([8]). *A locally o-minimal structure* $\mathcal{M} = (M, <, \dots)$ *is a unformly locally o-minimal structure of the first kind if for any positive integer* n*, any definable set* $X \subset M^{n+1}$ *and* $a \in M$*, there exists an open interval* $I \ni a$ *such that the definable sets* $X_y \cap I$ *are finite unions of points and intervals for all* $y \in M^n$*. Here* $X_y = \{x \in M | (x, y) \in X\}$.

Definition 2.4 ([5]). *A locally o-minimal structure* $\mathcal{M} = (M, <, \dots)$ *is a unformly locally o-minimal structure of the second kind if for any positive integer* n*, any definable set* $X \subset M^{n+1}$*,* $a \in M$ *and* $b \in M^n$*, there exist an open interval* $I \ni a$ *and an open box* $B \ni b$ *such that the definable sets* $X_y \cap I$ *are finite unions of points and intervals for all* $y \in B$.

Proposition 2.5 ([5]). *A uniformly locally o-minimal expansion of the second kind of an ordered field is o-minimal.*

The full definability of multiplication leads to the structure being too simple. Thus we consider restricted definability of multiplication.

Definition 2.6. *Let $\mathcal{M}$ be an expansion of dense linear order without endpoints. An expansion $\mathcal{M} = (M, <, 0, +, \dots)$ of ordered abelian group has definable bounded multiplication (DBM) compatible with $+$ if there exist a map $\cdot : M \times M \to M$ and $1 \in M$ such that*

(1) $(M, <, 0, 1, +, \cdot)$ is an ordered field.

(2) for any bounded open interval I, the restriction $\cdot|I \times I$ is definable in $\mathcal{M}$.

To cosider somewhat different generalization of definable spaces by van den Dries, we introduce pseudo definable spaces. They are not ringed spaces because our multiplication is not fully definable. We cannot define sheaves of ringned spaces under our expasion of dense linear order without endpoints.

Definition 2.7. *Let $\mathcal{M} = (M, <, \dots)$ be an expansion of dense linear order without endpoints. A pair $(S, \{\phi_i : U_i \to U'_i\}_{i \in I})$ consisting of a topological space S and a finite collection of homeomorphisms is a pseudo $\mathcal{M}$ definable space if*

(1) $\{U_i\}$ is a finite open cover of S.

(2) U'_i is a definable subset of M^{m_i} for any i.

(3) $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable homeomorphism whenever $U_i \cap U_j \neq \emptyset$.

A subset X of the pseudo definable space S is definable when $\varphi_i(X \cap U_i)$ are definable for all $i \in I$. The dimension of a definable set X is defined by $\dim X = \max_{i \in I} \dim \phi_i(X \cap U_i)$.

Theorem 2.8 (Invariance of domain [5])**.** *Let $U \subset M^n, V \subset M^m$ be definable open sets. If there exists a definable homeomorphism $f : U \to V$, then $n = m$.*

Corollary 2.9. *The dimension of a pseudo definable space is well-defined.*

Definition 2.10 ([12])**.** *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R and $X \subset R^n$ a definable set. We say that X is definably compact if for every $a, b \in R$ with $a < b$ and for every definable map $f : (a, b) \to X$, $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ exist in X.*

Theorem 2.11 ([12])**.** *Suppose that $\mathcal{M}$ is an o-minimal expansion of a real closed field R and $X \subset R^n$ is a definable set. Then X is definably compact if and only if it is closed and bounded.*

Definition 2.12 is an extension of Definition 2.10 to a definable C^r manifold.

Definition 2.12 ([1]). *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R, X a Hausdorff definable C^r manifold and $r > 0$. We call X definably compact if for every $a, b \in R$ with $-\infty \leqq a < b \leqq \infty$ and for every definable map $f : (a, b) \to X$, $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ exist in X.*

Proposition 2.13. *In Definition 2.12, if R is the field of real numbers $\mathbb{R}$, then X is definably compact if and only if it is compact.*

The above two definitions require the space is Hausdorff. Johnson [10] introduces a generalized definition of definable compactness in the case where the space is not Hausdorff.

Definition 2.14. *(1) Let X be a set. A family $\mathcal{F}$ of subsets of X is filtered collection if for any $B_1, B_2 \in F$, there exists $B_3 \in F$ with $B_3 \subset B_1 \cap B_2$.*

(2) Let $\mathcal{M}$ be an expansion of dense linear order without endpoints. Let X and Y be pseudo definable spaces. The parameterized family $\{S_t\}_{t \in Y}$ of definable subsets of X is called definable if the union $\cup_{y \in Y} \{y\} \times S_y$ is definable in $T \times X$. The parameterized family $\{S_y\}_{y \in Y}$ of definable subsets of X is a definable filtered collection if it is definable and a filtered collection.

(3) A definable space X is definably compact if every definable filtered collection of closed nonempty subsets of X has a nonempty intersection.

The itme (1) is found in [10]. The item (3) coinceides with Johnson's definition of definable compactness in [10] when X is a definable subset.

3 Our result

A pseudo definable space X is *Hausdorff* if for any distinct points $x, y \in X$, there exist open sets $U \ni x, V \ni y$ such that $U \cap V = \emptyset$. Moreover X is *regular* if for any $x \in X$ and for any open subset U of X with $x \in U$, there exists an open subset V of X with $x \in V$ and the closure of V in X is contained in U.

The following is our result.

Theorem 3.1 ([7]). *Let $\mathcal{M} = (M, <, 0, +, \dots)$ be a definably complete locally o-minimal expansion of an ordered group having definable bounded multiplication $\cdot$ compatible with $+$. Every regular definably compact pseudo definable space X is definably imbeddable into some M^n.*

Idea of proof of our result.

In an o-minimal expansion of a real closed field case, a unbounded interval is definably homeomorphic to a bounded interval.

In our case, it is not clear the above fact.

In the proof of our result, we need the follwing proprosition.

Proposition 3.2 ([7]). *If X is definably compact, then there exists a definable atlas $\{\phi_i : U_i \to U_i'\}_{i \in I}$ of X such that U_i' are bounded for all $i \in I$.*

If $\mathcal{M}$ is an o-minimal expansion of real closed field, then every U_i' may be assumed that it is bounded.

We assume that X is definably compact. By Proposition 3.2, we can reduce the case where all U_i' are bounded. Thus by our assumption that a definably complete locally o-minimal structure $\mathcal{M}$ has bounded definable multiplication and by a way similar to the proof of o-minimal case, we can construct a definable imbedding into some M^k.

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