Definable $\mathcal{C}^r$ sheaf on o-minimal spectrum

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概要

Consider an o-minimal expansion of the real field $\widetilde{\mathbb{R}}$ and a definable $\mathcal{C}^r$ submanifold M of $\mathbb{R}^m$, where r is a nonnegative integer. Let $\mathcal{L}$ be the first-order language of $\widetilde{\mathbb{R}}$. The o-minimal spectrum $\widetilde{M}$ of M is the set of all complete m-types of the first-order theory $\mathrm{Th}_{\mathbb{R}}(\widetilde{\mathbb{R}})$ which imply a formula defining M. A stalk of the sheaf of definable $\mathcal{C}^r$ functions on $\widetilde{M}$ at a point $\alpha \in \widetilde{M}$ is a local ring. Its residue field is naturally an $\mathcal{L}$-structure. We show that the residue field is a minimal elementary extension of the o-minimal structure $\widetilde{\mathbb{R}}$ containing $C^r_{\mathrm{df}}(M)/\operatorname{supp}(\alpha)$ and satisfying that, for any $\overline{a} \in (C^r_{\mathrm{df}}(M))^n$ and any formula $\phi(\overline{x})$, the extension satisfies the sentence $\phi(\overline{a})$ if and only if the definable subset of M defined by $\phi(\overline{a})$ is an element of α. Here, the notation $C^r_{\mathrm{df}}(M)$ denotes the ring of all definable $\mathcal{C}^r$ functions on M.

1 Introduction and definitions

We fix an o-minimal expansion of the real field $\widetilde{\mathbb{R}}$ in this paper. We also assume that the interpretation of any function symbol of the language $\mathcal{L}$ in $\widetilde{\mathbb{R}}$ is of class $\mathcal{C}^r$ on its domain of definition throughout the paper. The definition of o-minimal structures and their basic properties are found in [4, 5]. The term 'definable' means 'definable in the o-minimal structure $\widetilde{\mathbb{R}}$' in this paper. A typical example of $\widetilde{\mathbb{R}}$ is the ordered field structure on the real field. A definable set is a semialgebraic set in this case.

Consider a Euclidean space $\mathbb{R}^n$ and the real spectrum of the polynomial ring $X = \mathrm{Sper}(\mathbb{R}[X_1, \ldots, X_n])$. Real spectrum of a commutative ring is defined in [2, Section

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7.1]. A subset $\widetilde{U}$ of X is defined for any semialgebraic subset U of $\mathbb{R}^n$. The sets $\widetilde{U}$ are open bases in the spectral topology of X when U are semialgebraic open subsets of $\mathbb{R}^n$. The definition of $\widetilde{U}$ is found in [2, Proposition 7.2.2]. Sheaves on subsets of X are defined and investigated in semialgebraic geometry. For instance, given an affine Nash submanifold M of $\mathbb{R}^n$, the sheaf $\mathcal{N}_M$ is defined on $\widetilde{M}$ such that, for any semialgebraic open subset U of M, the ring $\mathcal{N}_M(\widetilde{U})$ is the ring of all Nash functions on U. The stalk of the sheaf $\mathcal{N}_M$ at $\alpha \in \widetilde{M}$ is the real closure $k(\alpha)$ of the quotient field of $\mathbb{R}[X_1, \ldots, X_n]/\operatorname{supp}(\alpha)$ with the ordering induced by the prime cone α by [2, Proposition 8.8.1, Proposition 8.8.2, Proposition 8.8.3]. Note that the real closed field containing $\mathbb{R}$ is an elementary extension of the real field $\mathbb{R}$ as $\mathcal{L}_{\text{of}}$-structures, where $\mathcal{L}_{\text{of}}$ is the first order language of ordered fields, because the theory of real closed fields has quantifier elimination by [2, Proposition 5.2.2].

A sheaf $\widetilde{S^0}_{\widetilde{T}}$ on $\widetilde{T}$ is another example, where T is a semialgebraic subset of $\mathbb{R}^n$. The ring $\widetilde{S^0}_{\widetilde{T}}(\widetilde{U})$ coincides with the ring of all semialgebraic continuous functions on a semialgebraic open subset U of T. The residue field of the stalk of this sheaf at $\alpha \in \widetilde{T}$ is also the real closure $k(\alpha)$ by [2, Proposition 7.3.2, Proposition 7.3.3, Proposition 7.3.4].

We want to generalize these results to general o-minimal cases. In this paper, we consider a definable $\mathcal{C}^r$ manifold M and definable $\mathcal{C}^r$ functions on its definable subsets, where r is a nonnegative integer. We can neither use the real spectrum of the polynomial rings nor expect quantifier elimination in our cases. We must find another appropriate space. Candidates for such a space may be the spectrum or the real spectrum of the ring $C^r_{\text{df}}(M)$, where $C^r_{\text{df}}(M)$ denotes the ring of all definable $\mathcal{C}^r$ functions on M. However, they have too much points as demonstrated in Section 2. Another candidate is the o-minimal spectrum defined in [12, 6]. We consider sheaves on the o-minimal spectrum.

We introduce notations necessary so as to describe our results more precisely. Consider an o-minimal expansion of the real field $\widetilde{\mathbb{R}}$ and a definable $\mathcal{C}^r$ manifold M. Note that all definable $\mathcal{C}^r$ manifolds are affine by [10, Theorem 1.1] and [8, Theorem 1.3]. We use this fact without explicitly stated in this paper. Assume that M is a definable $\mathcal{C}^r$ submanifold of $\mathbb{R}^m$. The o-minimal spectrum $\widetilde{M}$ is the set of all complete m-types of the first-order theory $\text{Th}_{\mathbb{R}}(\widetilde{\mathbb{R}})$ which imply a formula defining M. It is equipped with the topology, called spectral topology, generated by the basic open sets of the form

$$\widetilde{U} = \{p \in \widetilde{M} \mid (\text{the formula defining } U) \in p\},$$

where U are definable open subsets of M.

The notation D_M denotes the set of all definable subsets of M. The set $\mathcal{D}_M$ of all D_M-ultrafilters is our main concern. The definitions of filters are found in [1]. We define a topology in $\mathcal{D}_M$ as follows: The open bases of the topology are the subsets of the form

$$\widetilde{U} = \{\alpha \in \mathcal{D}_M \mid U \in \alpha\},$$

where U are definable open subsets of M. The topological space $\mathcal{D}_M$ is homeomorphic to $\widetilde{M}$ by [6, Section 2]. We identify $\widetilde{M}$ with $\mathcal{D}_M$ in the rest of this paper.

We first investigate the relation between real spectrum and o-minimal spectrum. For that purpose, we consider three other topological spaces. Let DC_M be the lattice consisting of all definable closed subset of M. The first topological space $\mathcal{DC}_M$ is the set of all prime DC_M-filters with the following topology. The open bases of the topology of DC_M are the subsets of the form

$$\widetilde{U} = \{\alpha \in \mathcal{DC}_M \mid M \setminus U \notin \alpha\},$$

where U are definable open subsets of M.

The notation $\mathrm{S}_{\widetilde{\mathbb{R}}}$ denotes the set of all definable $\mathcal{C}^r$ functions on $\mathbb{R}$ which are odd, increasing, bijective and r-flat at the origin. A subset $T \subset C^r_{\mathrm{df}}(M)$ is called $\mathrm{S}_{\widetilde{\mathbb{R}}}$-*fixed* if any definable $\mathcal{C}^r$ function g on M with $\phi \circ g \in T$ for some $\phi \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ is contained in T. The second topological space is a topological subspace of the spectrum $\mathrm{Spec}(C^r_{\mathrm{df}}(M))$ of the ring $C^r_{\mathrm{df}}(M)$ with the Zariski topology. Its underlying set consists of all $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime ideals. It is denoted by $\mathrm{Spec}_{\mathrm{fixed}}(C^r_{\mathrm{df}}(M))$. The last topological space $\mathrm{Sper}_{\mathrm{fixed}}(C^r_{\mathrm{df}}(M))$ is a topological subspace of the real spectrum $\mathrm{Sper}(C^r_{\mathrm{df}}(M))$ of the ring $C^r_{\mathrm{df}}(M)$ with the spectral topology. Its underlying set is the set of all $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime cones. See [2, Section 7.1] for the definitions of real spectrum of a commutative ring and its topology.

Our first main theorem is the following theorem:

Theorem 1.1. *Consider an o-minimal expansion of the real field* $\widetilde{\mathbb{R}}$. *Let* M *be a definable* $\mathcal{C}^r$ *manifold. The five topological spaces* $\widetilde{M}$, $\mathcal{D}_M$, $\mathcal{DC}_M$, $\mathrm{Spec}_{\mathit{fixed}}(C^r_{\mathit{df}}(M))$ *and* $\mathrm{Sper}_{\mathit{fixed}}(C^r_{\mathit{df}}(M))$ *are all homeomorphic to each other. Furthermore, the spaces*

$\mathrm{Spec}_{\mathit{fixed}}(C^r_{df}(M))$ *and* $\mathrm{Sper}_{\mathit{fixed}}(C^r_{df}(M))$ *coincide with the spectrum* $\mathrm{Spec}(C^r_{df}(M))$ *and the real spectrum* $\mathrm{Sper}(C^r_{df}(M))$*, respectively, when the o-minimal structure* $\widetilde{\mathbb{R}}$ *is polynomially bounded.*

There is a sheaf $\mathfrak{D}^r_M$ on $\widetilde{M}$ such that the ring $\mathfrak{D}^r_M(\widetilde{U})$ coincides with the ring $C^r_{\mathrm{df}}(U)$ of all definable $\mathcal{C}^r$ functions on a definable open subset U of M. The stalk $(\mathfrak{D}^r_M)_\alpha$ of the sheaf $\mathfrak{D}^r_M$ at a point $\alpha \in \widetilde{M}$ is a local ring. The residue field of this local ring is denoted by $k(\alpha)$. Let $\mathcal{L}$ be the language of the o-minimal structure $\widetilde{\mathbb{R}}$. We view the field $k(\alpha)$ as an $\mathcal{L}$-structure. We denoted this $\mathcal{L}$-structure by $\widetilde{k(\alpha)}$. Consider an $\mathcal{L}$-formula $\phi(\overline{x})$ with n free variables $\overline{x} = (x_1, \ldots, x_n)$. For any $\overline{a} = (a_1, \ldots, a_n) \in k(\alpha)^n$, we define that $\phi(\overline{a})$ *is satisfied in* $\widetilde{k(\alpha)}$ if the definable set $\{x \in M \mid \widetilde{\mathbb{R}} \models \phi(F_1(x), \ldots, F_n(x))\}$ is contained in the ultrafilter α, where $F_i : U \to \mathbb{R}$ are definable $\mathcal{C}^r$ functions on a definable open subset U of M which are simultaneously representatives of the elements $a_i \in k(\alpha)$ for all $1 \leq i \leq n$. We show that the above definition is well-defined in Section 3. In [12], Pillay gave the same definition only in the case in which M is an Euclidean space and $r = 0$. Our second main theorem is the following theorem. It is a variant of [3, Section 5.2, Section 5.3].

Theorem 1.2. *Consider an o-minimal expansion of the real field* $\widetilde{\mathbb{R}}$ *and its language* $\mathcal{L}$*. Let* M *be a definable* $\mathcal{C}^r$ *manifold. The* $\mathcal{L}$*-structure* $\widetilde{k(\alpha)}$ *is an elementary extension of* $\widetilde{\mathbb{R}}$ *whose underlying set contains the ring* $C^r_{df}(M)/\mathrm{supp}(\alpha)$*. Here, the notation* $\mathrm{supp}(\alpha)$ *is a prime ideal defined by* $\mathrm{supp}(\alpha) = \{F \in C^r_{df}(M) \mid F^{-1}(0) \in \alpha\}$*.*

Let $\mathcal{K}$ *be an elementary extension of* $\widetilde{\mathbb{R}}$ *whose underlying set* K *contains the ring* $C^r_{df}(M)/\mathrm{supp}(\alpha)$*. Assume further that, for any* $\mathcal{L}$*-formula* $\phi(\overline{x})$ *and* $\overline{F} = (F_1, \ldots, F_n) \in (C^r_{df}(M))^n$*, the following two conditions are equivalent:*

- $\mathcal{K} \models \phi(\overline{F})$*, and*
- *the ultrafilter* α *contains the definable set* $\{x \in M \mid \widetilde{\mathbb{R}} \models \phi(F_1(x), \ldots, F_n(x))\}$*.*

Then, there exists a unique elementary embedding $\widetilde{k(\alpha)} \prec \mathcal{K}$*.*

This paper is organized as follows: We first demonstrate Theorem 1.1 in Section 2. Propositions similar to Theorem 1.1 are found in [9], and the results in [9] are often used in this section. We show that the above interpretation in $\widetilde{k(\alpha)}$ is well-defined in Section 3. Section 3 is also devoted to the proof of Theorem 1.2.

2 Correspondence among $\mathcal{D}_M$, $\mathcal{DC}_M$, $\mathrm{Spec}(C^r_{\mathsf{df}}(M))$ and $\mathrm{Sper}(C^r_{\mathsf{df}}(M))$

We first show that the topological space $\widetilde{M}$ is compact.

Proposition 2.1. *Let M be a definable $\mathcal{C}^r$ manifold. The topological space $\widetilde{M}$ is compact.*

Proof. The set D_M is a boolean subalgebra of the boolean algebra of subsets of M. The Stone space of D_M defined in [2, Section 7.1] has the same underlying set as $\mathcal{D}_M$, and its topology is finer than the topology of $\mathcal{D}_M$. Since the Stone space is compact, $\mathcal{D}_M$ is also compact. The topological space $\widetilde{M}$ is also compact because they are homeomorphic. □

The following theorem is a part of Theorem 1.1.

Theorem 2.2. *Let M be a definable $\mathcal{C}^r$ manifold. The map $\tau : \mathcal{D}_M \to \mathcal{DC}_M$ given by*

$$\tau(\alpha) = \{C \in \mathrm{DC}_M \mid C \in \alpha\}$$

is a homeomorphism.

Proof. We may assume that M is a definable subset of a Euclidean space $\mathbb{R}^n$ because M is affine. It is easy to show that $\tau(\alpha)$ is a prime DC_M-filter.

We first demonstrate that τ is injective. Let $\alpha_1, \alpha_2 \in \mathcal{D}_M$ with $\tau(\alpha_1) = \tau(\alpha_2)$. We have only to show that $\alpha_1 \subset \alpha_2$ by symmetry. Consider an arbitrary element $C \in \alpha_1$ and a definable cell decomposition of $\mathbb{R}^n$ partitioning C by [4, Chapter 3, (2.11)]. Since α_1 is an ultrafilter, at least one cell contained in C is an element of α_1. Let D be such a cell of the minimum dimension. We lead a contradiction assuming that $D \notin \alpha_2$. Let E be the closure of D, which is an element of α_1 because $D \in \alpha_1$ and $D \subset E$. It is simultaneously an element of $\tau(\alpha_1)$. We have $E \in \tau(\alpha_2)$ because $\tau(\alpha_1) = \tau(\alpha_2)$. In particular, E is an element of α_2. Since E is a union of the cells, there exists a cell D' which is contained in E and is simultaneously an element of α_2. Note that the dimension of D' is smaller than that of D because $D \notin \alpha_2$. We can show that the closure E' of D' is an element of α_1 in the same way as above. At least one of the cells contained in E' is an element of α_1. This cell is of dimension strictly smaller than the dimension of D. It contradicts the assumption that D has the minimum dimension. We have shown that $\alpha_1 \subset \alpha_2$. We have demonstrated that τ is injective.

Secondly, we demonstrate that τ is surjective. For any $\beta \in \mathcal{DC}_M$, define $d(\beta)$ as the minimum of the dimensions of all the elements in β. We define a subset α of D_M as follows:

$$\alpha = \{C \in \mathrm{D}_M \mid V \cap C \neq \emptyset \text{ and } \dim(V \cap C) \geq d(\beta) \text{ for all } V \in \beta\}.$$

We first show that α is an ultrafilter.

(i) It is obvious that $M \in \alpha$ and $\emptyset \notin \alpha$.

(ii) We show that $C_1 \cap C_2 \in \alpha$ when $C_1 \in \alpha$ and $C_2 \in \alpha$. We have to show that $V \cap C_1 \cap C_2 \neq \emptyset$ and $\dim V \cap C_1 \cap C_2 \geq d(\beta)$ for any $V \in \beta$. There exists a definable closed set $V' \in \beta$ of dimension $d(\beta)$ contained in V for any $V \in \beta$. In fact, let $W \in \beta$ with $\dim W = d(\beta)$, then the intersection $V' = W \cap V$ is an element of β of dimension $d(\beta)$. We have $V \cap C_1 \cap C_2 \neq \emptyset$ and $\dim V \cap C_1 \cap C_2 \geq d(\beta)$ if $V' \cap C_1 \cap C_2 \neq \emptyset$ and $\dim V' \cap C_1 \cap C_2 \geq d(\beta)$. Therefore, we may assume that V is of dimension $d(\beta)$ without loss of generality. Consider a definable cell decomposition of $\mathbb{R}^n$ partitioning V, C_1 and C_2. Let $\{D_i\}_{i=1}^m$ be the collection of cells of dimension $d(\beta)$ contained in V. The closure of D_i is denoted by E_i for each $1 \leq i \leq m$. We have $V = \bigcup_{i=1}^m E_i \cup F$, where F is a definable closed set of dimension smaller than $d(\beta)$. Since β is a prime DC_M-filter, we get $E_i \in \beta$ for some $1 \leq i \leq m$. The equality $\dim(E_i \cap C_1) = d(\beta)$ should be satisfied because $E_i \in \beta$ and $C_1 \in \alpha$. We get $D_i \subset C_1$ because D_i is a cell of the definable cell decomposition partitioning C_1. We also get $D_i \subset C_2$ in the same way. We have demonstrated that D_i is contained in $V \cap C_1 \cap C_2$. We have $V \cap C_1 \cap C_2 \neq \emptyset$ and $\dim V \cap C_1 \cap C_2 \geq d(\beta)$. It means that $C_1 \cap C_2 \in \alpha$.

(iii) It is obvious that any element of D_M containing an element of α is also an element of α.

(iv) We finally show that, for any $C_1, C_2 \in \mathrm{D}_M$ with $C_1 \cup C_2 \in \alpha$, at least one of C_1 and C_2 is an element of α. Assume the contrary. There exist $V_1, V_2 \in \beta$ with $\dim(V_i \cap C_i) < d(\beta)$ for $i = 1, 2$. We have $\dim((C_1 \cup C_2) \cap V_1 \cap V_2) =$

$\max\{\dim(C_1 \cap V_1 \cap V_2), \dim(C_2 \cap V_1 \cap V_2)\} \leq \max\{\dim(C_1 \cap V_1), \dim(C_2 \cap V_2)\} < d(\beta)$. It is a contradiction because $V_1 \cap V_2 \in \beta$ and $C_1 \cup C_2 \in \alpha$.

We have shown that the subset α is a D_M-ultrafilter.

We next demonstrate that $\beta = \tau(\alpha)$. The inclusion $\beta \subset \tau(\alpha)$ is obvious. We show the opposite inclusion. The set $\tau(\alpha)$ is described as follows:

$$\tau(\alpha) = \{V \in \mathrm{DC}_M \mid W \cap V \neq \emptyset \text{ and } \dim W \cap V \geq d(\beta) \text{ for all } W \in \beta\}.$$

Take an arbitrary element $V \in \tau(\alpha)$ and an element $W \in \beta$ of dimension $d(\beta)$. Consider a definable cell decomposition of $\mathbb{R}^n$ partitioning V and W. Let $\{D_i\}_{i=1}^m$ be the collection of cells of dimension $d(\beta)$ contained in W. The closure of D_i is denoted by E_i for each $1 \leq i \leq m$. We have $W = \bigcup_{i=1}^m E_i \cup F$ for some definable closed subset F of M of dimension smaller than $d(\beta)$. A definable closed set E_i is an element of β for some $1 \leq i \leq m$ because β is a prime filter. We have $\dim(V \cap E_i) = d(\beta)$ because $V \in \tau(\alpha)$. Hence, the cell D_i is contained in V. The closure E_i is also contained in V because V is closed. We get $V \in \beta$ because $E_i \subset V$ and $E_i \in \beta$. We have shown that $\beta = \tau(\alpha)$. We have demonstrated that τ is surjective.

It remains to show that the bijective map τ is a homeomorphism. Set $\widetilde{U}^{\mathrm{D}} = \{\alpha \in \mathcal{D}_M \mid U \in \alpha\}$ and $\widetilde{U}^{\mathrm{DC}} = \{\beta \in \mathcal{DC}_M \mid M \setminus U \notin \beta\}$ for all definable open subsets U of M. We have only to show that $\tau(\widetilde{U}^{\mathrm{D}}) = \widetilde{U}^{\mathrm{DC}}$. We first show the inclusion $\tau(\widetilde{U}^{\mathrm{D}}) \subset \widetilde{U}^{\mathrm{DC}}$. Let $\alpha \in \widetilde{U}^{\mathrm{D}}$. We have $U \in \alpha$, and $M \setminus U \notin \alpha$; hence, $M \setminus U \notin \tau(\alpha)$. We have shown that $\tau(\alpha) \in \widetilde{U}^{\mathrm{DC}}$. The next task is to illustrate the opposite inclusion. We assume that $\beta \in \widetilde{U}^{\mathrm{DC}}$. We have $M \setminus U \notin \beta$. Since τ is onto, there is $\alpha \in \mathcal{D}_M$ with $\beta = \tau(\alpha)$. We get $M \setminus U \notin \alpha$. Since α is an ultrafilter, we have $U \in \alpha$. We have shown the opposite inclusion. □

Consider the ring $C^r_{\mathrm{df}}(M)$ of all definable $\mathcal{C}^r$ functions on a definable $\mathcal{C}^r$ manifold M. The author showed that three topological spaces $\mathcal{DC}_M$, $\mathrm{Spec}(C^r_{\mathrm{df}}(M))$ and $\mathrm{Sper}(C^r_{\mathrm{df}}(M))$ are all homeomorphic to each other when the o-minimal structure $\widetilde{\mathbb{R}}$ is polynomially bounded in [9, Theorem 2.11, Corollary 2.12].

An open basis of $\mathcal{DC}_M$ is defined as a set of the form $\{\beta \in \mathcal{DC}_M \mid V \notin \beta\}$ in [9], where $V = \bigcup_{i=1}^k \{x \in M \mid f_i(x) \leq 0\}$ for some $f_1, \ldots, f_k \in C^r_{\mathrm{df}}(M)$. It seems slightly different from the definition in this paper, but they are identical. In fact, an open basis in [9] is an open basis in this paper because $U = M \setminus V$ is a definable open set.

On the contrary, for any definable open subset U in M, there exists a definable $\mathcal{C}^r$ function on M with $f^{-1}(0) = M \setminus U$ by [9, Lemma 2.1]. Set $V = \{x \in M \mid f^2(x) \leq 0\}$, then we get $\widetilde{U} = \{\beta \in \mathcal{DC}_M \mid V \notin \beta\}$. An open basis in this paper is an open basis in [9].

The example in [9, Example 3.1] shows that $\mathcal{DC}_M$ is not homeomorphic to the spectrum $\mathrm{Spec}(C^r_{\mathrm{df}}(M))$ when the o-minimal structure $\widetilde{\mathbb{R}}$ is not polynomially bounded. We consider appropriate subsets $\mathrm{Spec}_{\mathrm{fixed}}(C^r_{\mathrm{df}}(M))$ and $\mathrm{Sper}_{\mathrm{fixed}}(C^r_{\mathrm{df}}(M))$ of $\mathrm{Spec}(C^r_{\mathrm{df}}(M))$ and $\mathrm{Sper}(C^r_{\mathrm{df}}(M))$, and show that they are homeomorphic to $\mathcal{DC}_M$.

We review the maps defined in [9]. The map $\mathcal{I} : \mathcal{DC}_M \to \mathrm{Spec}(C^r_{\mathrm{df}}(M))$ is given by

$$\mathcal{I}(\beta) = \{f \in C^r_{\mathrm{df}}(M) \mid f^{-1}(0) \in \beta\},$$

and it is continuous by [9, Proposition 2.4]. The map $\alpha : \mathcal{DC}_M \to \mathrm{Sper}(C^r_{\mathrm{df}}(M))$ is given by

$$\alpha(\beta) = \{f \in C^r_{\mathrm{df}}(M) \mid f^{-1}([0, \infty)) \in \beta\},$$

and it is also continuous by [9, Lemma 2.6]. We call this map Λ instead of α because we use the symbol α to represent an element of $\mathcal{D}_M$ in this section. Finally, the continuous map $\Phi_r : \mathrm{Sper}(C^r_{\mathrm{df}}(M)) \to \mathrm{Spec}(C^r_{\mathrm{df}}(M))$ is given by $\Phi_r(P) = \mathrm{supp}(P) = \{f \in C^r_{\mathrm{df}}(M) \mid f \in P \text{ and } -f \in P\}$.

Lemma 2.3. *The maps $\mathcal{I}$ and Λ send a prime DC_M-filter to an $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime ideal and an $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime cone, respectively.*

Proof. The maps $\mathcal{I}$ and Λ send a prime DC_M-filter to a prime ideal and a prime cone by [9, Proposition 2.4, Lemma 2.6]. It is obvious that they are $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed. □

Lemma 2.4. *The map $\mathcal{Z} : \mathrm{Spec}_{\mathit{fixed}}(C^r_{\mathit{df}}(M)) \to \mathcal{DC}_M$ defined by*

$$\mathcal{Z}(\mathfrak{p}) = \{f^{-1}(0) \mid f \in \mathfrak{p}\}$$

is a continuous map, and the equality $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$ holds true for any $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime ideal $\mathfrak{p}$ of $C^r_{\mathit{df}}(M)$.

Proof. The set $\mathcal{Z}(\mathfrak{p})$ is a DC_M-filter by [9, Proposition 2.4]. We show that it is a prime DC_M-filter. Let A and B be definable closed subsets of M with $A \cup B \in \mathcal{Z}(\mathfrak{p})$. There are definable $\mathcal{C}^r$ functions $f, g \in C^r_{\mathrm{df}}(M)$ with $f^{-1}(0) = A$ and $g^{-1}(0) = B$ by [9, Lemma 2.2]. Since $A \cup B \in \mathcal{Z}(\mathfrak{p})$, there is a definable $\mathcal{C}^r$ function $h \in \mathfrak{p}$ with

$A \cup B = h^{-1}(0)$. There exist $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ and $u \in C^r_{\mathrm{df}}(M)$ with $\sigma \circ (fg) = uh \in \mathfrak{p}$ by [9, Lemma 2.1]. We have $fg \in \mathfrak{p}$ because $\mathfrak{p}$ is $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed; and, we get $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal. We have shown that $A \in \mathcal{Z}(\mathfrak{p})$ or $B \in \mathcal{Z}(\mathfrak{p})$. The set $\mathcal{Z}(\mathfrak{p})$ is a prime DC_M-filter.

We next show the $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$ for any $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime ideal $\mathfrak{p}$ of $C^r_{\mathrm{df}}(M)$. The inclusion $\mathfrak{p} \subset \mathcal{I}(\mathcal{Z}(\mathfrak{p}))$ is obvious. We show the opposite inclusion. Let $f \in \mathcal{I}(\mathcal{Z}(\mathfrak{p}))$, there exists a definable $\mathcal{C}^r$ function $g \in \mathfrak{p}$ with $f^{-1}(0) = g^{-1}(0)$. There exist $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ and $h \in C^r_{\mathrm{df}}(M)$ with $\sigma \circ f = gh \in \mathfrak{p}$ by [9, Lemma 2.1]. Since $\mathfrak{p}$ is $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed, we have $f \in \mathfrak{p}$.

We finally illustrate that $\mathcal{Z}$ is continuous. Let U be a definable open subset of M. There exists a definable $\mathcal{C}^r$ function $f \in C^r_{\mathrm{df}}(M)$ with $M \setminus U = f^{-1}(0)$ by [9, Lemma 2.2]. We have only to show that

$$\mathcal{Z}^{-1}(\widetilde{U}) = \{\mathfrak{p} \in \mathrm{Spec}_{\mathrm{fixed}}(C^r_{\mathrm{df}}(M)) \mid f \not\in \mathfrak{p}\}.$$

Assume that $f \in \mathfrak{p}$, then $M \setminus U \in \mathcal{Z}(\mathfrak{p})$, and $\mathcal{Z}(\mathfrak{p}) \not\in \widetilde{U}$. On the other hand, if $\mathcal{Z}(\mathfrak{p}) \not\in \widetilde{U}$, we have $M \setminus U \in \mathcal{Z}(\mathfrak{p})$, and $f \in \mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$. □

Lemma 2.5. *If a prime cone $P \in \mathrm{Sper}(C^r_{df}(M))$ is $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed, the support $\mathrm{supp}(P)$ is an $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime ideal.*

Proof. The set $\mathrm{supp}(P)$ is a prime ideal by [2, Proposition 4.3.2]. We have only to show that, if $g \in C^r_{\mathrm{df}}(M)$ and $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ with $\sigma \circ g \in \mathrm{supp}(P)$, the element g is contained in $\mathrm{supp}(P)$. We have $g \in P$ because $\sigma \circ g \in P$ and P is $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed. Remember that $\sigma : \mathbb{R} \to \mathbb{R}$ is an odd function. We also have $-g \in P$ because $\sigma \circ (-g) = -\sigma \circ g \in P$. It means that $g \in \mathrm{supp}(P)$. □

Theorem 2.6. *The restriction*

$$\Phi_r|_{\mathrm{Sper}_{\mathit{fixed}}(C^r_{df}(M))} : \mathrm{Sper}_{\mathit{fixed}}(C^r_{df}(M)) \to \mathrm{Spec}_{\mathit{fixed}}(C^r_{df}(M))$$

is a homeomorphism, and its inverse map is $\Lambda \circ \mathcal{Z}$.

Proof. The continuous map Φ_r is well-defined by Lemma 2.5, The map $\Lambda \circ \mathcal{Z}$ is also well-defined and continuous by Lemma 2.3 and Lemma 2.4. The remaining task is to show that the composition of two maps are the identity maps.

We first show that $P = \Lambda(\mathcal{Z}(\Phi_r(P)))$ for any $P \in \mathrm{Sper}_{\mathrm{fixed}}(C^r_{\mathrm{df}}(M))$. Set $P' =$

$\Lambda(\mathcal{Z}(\Phi_r(P))$, then we have $\operatorname{supp}(P') = \mathcal{I}(\mathcal{Z}(\operatorname{supp}(P)))$ by [9, Lemma 2.6]. Apply Lemma 2.4, then we get $\operatorname{supp}(P') = \operatorname{supp}(P)$. The prime cones P and P' coincide by [9, Proposition 2.8].

The equality $\Phi_r(\Lambda(\mathcal{Z}(\mathfrak{p}))) = \mathfrak{p}$ is easy to prove, where $\mathfrak{p} \in \operatorname{Spec}_{\text{fixed}}(C^r_{\text{df}}(M))$. In fact, we have $\Phi_r(\Lambda(\mathcal{Z}(\mathfrak{p}))) = \mathcal{I}(\mathcal{Z}(\mathfrak{p}))$ by [9, Lemma 2.6]. The right hand side of the equality coincides with $\mathfrak{p}$ by Lemma 2.4. ☐

Theorem 2.7. *The map $\mathcal{I} : \mathcal{DC}_M \to \operatorname{Spec}_{fixed}(C^r_{df}(M))$ is a homeomorphism, and its inverse map is $\mathcal{Z}$.*

Proof. The maps $\mathcal{I}$ and $\mathcal{Z}$ are continuous by [9, Proposition 2.4] and Lemma 2.4. We also have $\mathcal{I}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$ for any $\mathrm{S}_{\widetilde{\mathbb{R}}}$-fixed prime ideal $\mathfrak{p}$ of $C^r_{\text{df}}(M)$. It is obvious that $\mathcal{Z}(\mathcal{I}(\beta)) = \beta$ for any prime DC_M-filter β. ☐

The author promised that Theorem 1.1 is proved in this section. In fact, Theorem 1.1 follows from Theorem 2.2, Theorem 2.6, Theorem 2.7, [6, Section 2] and [9, Theorem 2.11, Corollary 2.12].

3 Sheaf of definable $\mathcal{C}^r$ functions on o-minimal spectrum and its stalk

We introduce several lemmas and propositions used in the proof of Theorem 1.2.

Lemma 3.1. *Let M be a definable $\mathcal{C}^r$ manifold with $0 \leq r < \infty$. Let X and Y be definable closed subsets of M with $X \cap Y = \emptyset$. Then, there exists a definable $\mathcal{C}^r$ function $f : M \to [0,1]$ with $f^{-1}(0) = X$ and $f^{-1}(1) = Y$.*

Proof. There exist definable $\mathcal{C}^r$ functions $g, h : M \to \mathbb{R}$ with $g^{-1}(0) = X$ and $h^{-1}(0) = Y$ by [9, Proposition 2.2]. The function $f : M \to [0,1]$ defined by $f(x) = \frac{g(x)^2}{g(x)^2+h(x)^2}$ satisfies the requirement. ☐

Lemma 3.2. *Let M be a definable $\mathcal{C}^r$ manifold with $0 \leq r < \infty$. Let C and U be definable closed and open subsets of M, respectively. Assume that C is contained in U. Then, there exists a definable open subset V of M with $C \subset V \subset \overline{V} \subset U$.*

Proof. There is a definable continuous function $h : M \to [0,1]$ with $h^{-1}(0) = C$ and

$h^{-1}(1) = M \setminus U$ by Lemma 3.1. The set $V = \{x \in M\,;\, h(x) < \frac{1}{2}\}$ satisfies the requirement. $\square$

Lemma 3.3 (Partition of unity)**.** *Let $M \subset \mathbb{R}^m$ be an a definable $\mathcal{C}^r$ manifold. Given a finite definable open covering $\{U_i\}_{i=1}^q$ of M, there exist nonnegative definable $\mathcal{C}^r$ functions λ_i on M for all $1 \leq i \leq q$ such that $\sum_{i \in I} \lambda_i = 1$ and the closure of the set $\{x \in M \mid \lambda_i(x) > 0\}$ is contained in U_i.*

Proof. Let $h_i(x) = \operatorname{dist}(x, M \setminus U_i)$ be the distance between a point $x \in M$ and the closed set $M \setminus U_i$ for any $1 \leq i \leq q$. Set $V_i = \{x \in M \mid h_i(x) > \max_{1 \leq j \leq q} h_j(x)/2\}$. The closure of V_i in M is contained in U_i. In fact, let x be a point in the closure of V_i. We have $h_j(x) > 0$ for some $1 \leq j \leq q$ because $\{U_i\}_{i=1}^q$ is an open covering. Since $h_i(x) \geq \max_{1 \leq j \leq q} h_j(x)/2 > 0$, we get $x \in U_i$. We next show that $\{V_i\}_{i=1}^q$ is a finite definable open covering of M. Fix an arbitrary point $x \in M$. There exists an integer $1 \leq i \leq q$ with $x \in U_i$, and $h_i(x) > 0$. Let k be the positive integer with $1 \leq k \leq q$ and $h_k(x) = \max_{1 \leq j \leq q} h_j(x) > 0$. It is obvious that the point x belongs to V_k.

There exists a definable $\mathcal{C}^r$ function f_i on M with $f_i^{-1}(0) = M \setminus V_i$ by [9, Lemma 2.2]. Set $\lambda_i = f_i^2 / \sum_{j=1}^q f_j^2$. The definable $\mathcal{C}^r$ functions λ_i on M satisfy the requirements. $\square$

Lemma 3.4. *Let $M \subset \mathbb{R}^n$ be a definable $\mathcal{C}^r$ submanifold of $\mathbb{R}^n$, which is closed in $\mathbb{R}^n$. For any definable $\mathcal{C}^r$ function f on M, there exists a definable $\mathcal{C}^r$ extension F to $\mathbb{R}^n$.*

Proof. There exists a definable open neighborhood U of M and definable $\mathcal{C}^r$ map $\rho : U \to M$ such that the restriction of ρ to M is the identity map by [7, Theorem 1.9]. Let V be a definable open neighborhood of M with $M \subset V \subset \overline{V} \subset U$ given in Lemma 3.2. There exists a definable $\mathcal{C}^r$ function h on $\mathbb{R}^n$ with $h^{-1}(0) = \mathbb{R}^n \setminus V$ and $h^{-1}(1) = M$ by Lemma 3.1. A definable $\mathcal{C}^r$ extension $F : \mathbb{R}^n \to \mathbb{R}$ of f is given by

$$F(x) = \begin{cases} h(x) f(\rho(x)) & \text{if } x \in V, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Lemma 3.5. *Let $M \subset \mathbb{R}^n$ be a definable $\mathcal{C}^r$ submanifold of $\mathbb{R}^n$, which is closed in $\mathbb{R}^n$. Consider a definable continuous function f on M which is of class $\mathcal{C}^r$ on $M \setminus f^{-1}(0)$.*

There exists a definable continuous extension F of f to $\mathbb{R}^n$ which is of class $\mathcal{C}^r$ on $\mathbb{R}^n \setminus F^{-1}(0)$.

Proof. We can construct an extension F in the same way as Lemma 3.4. □

Proposition 3.6. *Let M be a definable $\mathcal{C}^r$ manifold. Consider a definable subset A of M and a definable $\mathcal{C}^r$ function on A. Assume that, for any $x_0 \in \overline{A} \setminus A$, the limit of the function f at x_0 exists and it is zero. Then, there exists an element $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ such that the composition $\sigma \circ f$ has a definable $\mathcal{C}^r$ extension to M.*

Proof. Since M is affine, there is a definable $\mathcal{C}^r$ embedding $\iota : M \hookrightarrow \mathbb{R}^n$. Since $\overline{M} \setminus M$ is a definable closed set, there exists a definable $\mathcal{C}^r$ function H on $\mathbb{R}^n$ vanishing only on $\overline{M} \setminus M$ by [5, Theorem C.11]. The image of the definable $\mathcal{C}^r$ embedding $\iota' : M \to \mathbb{R}^{n+1}$ given by $\iota'(x) = (\iota(x), 1/H(x))$ is a closed subset. Hence, we may assume that M is a definable $\mathcal{C}^r$ submanifold of a Euclidean space $\mathbb{R}^n$, which is simultaneously closed in $\mathbb{R}^n$.

Consider a definable continuous function $F : M \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

It is of class $\mathcal{C}^r$ on $M \setminus F^{-1}(0)$. There is a definable continuous extension $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}$ of F such that it is of class $\mathcal{C}^r$ on $\mathbb{R}^n \setminus (\widetilde{F})^{-1}(0)$ by Lemma 3.5. The composition $\sigma \circ \widetilde{F}$ is a definable $\mathcal{C}^r$ function for some $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ by [5, Corollary C.10]. Hence, the composition $\sigma \circ f$ has a definable $\mathcal{C}^r$ extension to M. □

Lemma 3.7. *For any definable continuous function $f : \mathbb{R} \to \mathbb{R}$, there exists a positive definable $\mathcal{C}^r$ function $\rho : \mathbb{R} \to \mathbb{R}$ such that $|f(x)| < \rho(x)$ for any $x \in \mathbb{R}$.*

Proof. We may assume that f is not negative by considering $|f|$ instead of f. There exists a finite subset $\{t_1, \dots, t_m\}$ of $\mathbb{R}$ such that f is of class $\mathcal{C}^r$ on $V_0 = \mathbb{R} \setminus \{t_1, \dots, t_m\}$ by [4, Theorem 3.2 and Exercise 3.3 of Chapter 7]. Set $y_i = f(t_i) + 1$ and $V_i = \{t \in \mathbb{R} \mid f(t) < y_i\}$ for all $1 \leq i \leq m$. The family $\{V_0, V_1, \dots, V_m\}$ is a definable open covering of $\mathbb{R}$. Let $\{\lambda_i\}_{i=0}^m$ be a definable $\mathcal{C}^r$ partition of unity subordinate to $\{V_0, V_1, \dots, V_m\}$ given in Lemma 3.3. Set $\rho(x) = \sum_{i=1}^m y_i \lambda_i(x) + \lambda_0(x)(f(x) + 1)$, then it is a definable $\mathcal{C}^r$ function with $f(x) < \rho(x)$ for any $x \in \mathbb{R}$. □

Lemma 3.8. *For any definable $\mathcal{C}^r$ function $f : \mathbb{R}^n \to \mathbb{R}$, there exists a positive definable $\mathcal{C}^r$ function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{\|x\|\to\infty} g(x) = 0$ and $\lim_{\|x\|\to\infty} f(x)g(x) = 0$.*

Proof. Consider a definable continuous function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(t) = \begin{cases} \max\limits_{\|x\|^2=t} |f(x)| & \text{if } t \geq 0, \\ |f(O)| & \text{otherwise,} \end{cases}$$

where O is the origin of $\mathbb{R}^n$. There exists a positive definable $\mathcal{C}^r$ function $\rho : \mathbb{R} \to \mathbb{R}$ with $\phi(t) < \rho(t)$ for any $t \in \mathbb{R}$ by Lemma 3.7. Set $\kappa(t) = \frac{1}{\rho^2(t)+t^2}$, then we have $\lim_{t\to\infty} \kappa(t) = 0$ and $\lim_{t\to\infty} \phi(t)\kappa(t) = 0$. Set $g(x) = \kappa(\|x\|^2)$, then we have $\lim_{\|x\|\to\infty} g(x) = 0$ and $\lim_{\|x\|\to\infty} f(x)g(x) = 0$. □

Lemma 3.9. *Consider a definable $\mathcal{C}^r$ manifold M. Let $f : U \to \mathbb{R}$ be a definable $\mathcal{C}^r$ function on a definable open subset U of M. Then, there exists a definable $\mathcal{C}^r$ function g on M such that g is positive on U, zero on the boundary of U and $\lim\limits_{U\ni x\to x_0} f(x)g(x) = 0$ for any point x_0 in the boundary of U.*

Proof. We may assume that M is a definable $\mathcal{C}^r$ submanifold of $\mathbb{R}^n$ and closed in $\mathbb{R}^n$ in the same way as the proof of Proposition 3.6. There exists a definable $\mathcal{C}^r$ function H on $\mathbb{R}^n$ such that $\partial U = \overline{U} \setminus U = H^{-1}(0)$ by [5, Theorem C.11]. The definable $\mathcal{C}^r$ map $\iota : \mathbb{R}^n \setminus \partial U \to \mathbb{R}^{n+1}$ is given by $\iota(x) = \left(x, \frac{1}{H(x)}\right)$. Consider the function $f \circ \iota^{-1}$ defined on $\iota(U)$. Since $\iota(U)$ is closed in $\mathbb{R}^{n+1}$, we have its definable $\mathcal{C}^r$ extension F to $\mathbb{R}^{n+1}$ by Lemma 3.4. We can take a positive definable $\mathcal{C}^r$ function G on $\mathbb{R}^{n+1}$ such that $\lim_{\|x\|\to\infty} G(x) = 0$ and $\lim_{\|x\|\to\infty} F(x)G(x) = 0$ by Lemma 3.8. Since the restriction of $G \circ \iota$ to U satisfies the assumption of Proposition 3.6, there exists $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ such that $\sigma \circ G \circ \iota$ has a definable $\mathcal{C}^r$ extension g to M. It is obvious that g is positive on U and zero on the boundary of U. Let x_0 be a point of the boundary of U. The limit $\lim\limits_{U\ni x\to x_0} \dfrac{g(x)}{G \circ \iota(x)} = \lim\limits_{U\ni x\to x_0} \dfrac{\sigma \circ G \circ \iota(x)}{G \circ \iota(x)}$ exists because σ is an element of $\mathrm{S}_{\widetilde{\mathbb{R}}}$ and $\lim\limits_{U\ni x\to x_0} G \circ \iota(x) = 0$. We have $\lim\limits_{U\ni x\to x_0} f(x)g(x) = \left(\lim\limits_{U\ni x\to x_0} F(\iota(x))G(\iota(x))\right) \cdot \left(\lim\limits_{U\ni x\to x_0} \dfrac{g(x)}{G \circ \iota(x)}\right) = 0$. □

Lemma 3.10. *Let $\{C_i\}_{i=1}^m$ be a definable $\mathcal{C}^r$ cell decomposition of $\mathbb{R}^n$ given in [4, Theorem 3.2 and Exercise 3.3 of Chapter 7], where r is a nonnegative integer. For* *any $1 \leq i \leq m$, there exist a definable open neighborhood W_i of C_i in $\mathbb{R}^n$ and a definable $\mathcal{C}^r$ map $\rho_i : W_i \to C_i$ such that the restriction of ρ_i to C_i is the identity map.*

Proof. We fix an integer $1 \leq i \leq m$. The maps $\pi_l : \mathbb{R}^n \to \mathbb{R}^l$ are the projections onto the first l coordinates for all $1 \leq l \leq n$. We inductively define a definable open neighborhood $W_{i,l} \subset \mathbb{R}^l$ of $\pi_l(C_i)$ and a definable $\mathcal{C}^r$ map $\rho_{i,l} : W_{i,l} \to \pi_l(C_i)$ such that the restriction of $\rho_{i,l}$ to $\pi_l(C_i)$ is the identity map.

When $l = 1$, $\pi_1(C_i)$ consists of a single point a or is a connected open interval $I \subset \mathbb{R}$. Set $W_{i,1} = \mathbb{R}$ and $\rho_{i,1}(x) = a$ in the former case. Set $W_{i,1} = I$ and $\rho_{i,1}(x) = x$ in the latter case.

When $l > 1$, the definable set $\pi_l(C_i)$ is one of the following forms:

$$\pi_l(C_i) = \{(x,t) \in \pi_{l-1}(C_i) \times \mathbb{R} \mid t = f(x)\} \text{ and}$$
$$\pi_l(C_i) = \{(x,t) \in \pi_{l-1}(C_i) \times \mathbb{R} \mid f_1(x) < t < f_2(x)\},$$

where f, f_1 and f_2 are definable $\mathcal{C}^r$ functions on $\pi_{l-1}(C_i)$. Set $W_{i,l} = W_{i,l-1} \times \mathbb{R}$ in the former case. The definable $\mathcal{C}^r$ map $\rho_{i,l} : W_{i,l} = W_{i,l-1} \times \mathbb{R} \to \pi_l(C_i)$ is given by $\rho_{i,l}(x,t) = (\rho_{i,l-1}(x), f(\rho_{i,l-1}(x)))$. Set $W_{i,l} = \{(x,t) \in W_{i,l-1} \times \mathbb{R} \mid f_1(\rho_{i,l-1}(x)) < t < f_2(\rho_{i,l-1}(x))\}$ in the latter case. The definable $\mathcal{C}^r$ map $\rho_{i,l} : W_{i,l} \to \pi_l(C_i)$ is given by $\rho_{i,l}(x,t) = (\rho_{i,l-1}(x), t)$.

The definable open set $W_i = W_{i,n}$ and the definable $\mathcal{C}^r$ map $\rho_i = \rho_{i,n}$ satisfy the conditions required in this lemma. □

We have finished introducing the preliminary results. We begin to define a sheaf on the o-minimal spectrum.

Proposition 3.11. *Let M be a definable $\mathcal{C}^r$ manifold. There exists a sheaf $\mathfrak{D}^r_M$ on $\widetilde{M}$ such that, for any definable open subset U of M, the equality $\mathfrak{D}^r_M(\widetilde{U}) = C^r_{df}(U)$ is satisfied.*

Proof. The proof is the same as the proof of [2, Proposition 7.3.2]. We omit the proof. □

Proposition 3.12. *Let M be a definable $\mathcal{C}^r$ manifold. The stalk $(\mathfrak{D}^r_M)_\alpha$ of the sheaf*

$\mathfrak{D}^r_M$ *at a point* $\alpha \in \widetilde{M}$ *is a local ring, and its maximal ideal is given by*

$$\mathfrak{m}_\alpha = \{f \in (\mathfrak{D}^r_M)_\alpha \mid F^{-1}(0) \in \alpha\},$$

where $F \in C^r_{df}(U)$ *is a representative of the element* $f \in (\mathfrak{D}^r_M)_\alpha$ *and* U *is a definable open subset of* M *with* $U \in \alpha$.

Proof. We first show that $\mathfrak{m}_\alpha$ is an ideal. Let $f \in \mathfrak{m}_\alpha$ and $g \in (\mathfrak{D}^r_M)_\alpha$. The definable $\mathcal{C}^r$ functions $F \in C^r_{\mathrm{df}}(U)$ and $G \in C^r_{\mathrm{df}}(U')$ are their representatives. We may assume that $U' = U$ considering the intersection $U \cap U'$. We have $(GF)^{-1}(0) \supset F^{-1}(0) \in \alpha$; hence $(GF)^{-1}(0) \in \alpha$ and $gf \in \mathfrak{m}_\alpha$. When $f_1, f_2 \in \mathfrak{m}_\alpha$, we can take their representatives $F_1, F_2 \in C^r_{\mathrm{df}}(U)$ for some common definable open subset U of M in the same way as the previous case. We get $(F_1 + F_2)^{-1}(0) \supset F_1^{-1}(0) \cap F_2^{-1}(0) \in \alpha$; hence, $(F_1 + F_2)^{-1}(0) \in \alpha$ and $f_1 + f_2 \in \mathfrak{m}_\alpha$. We have shown that $\mathfrak{m}_\alpha$ is an ideal.

We next show that all the elements in $(\mathfrak{D}^r_M)_\alpha \setminus \mathfrak{m}_\alpha$ are units. Let $f \in (\mathfrak{D}^r_M)_\alpha \setminus \mathfrak{m}_\alpha$ and $F \in C^r_{\mathrm{df}}(U)$ be a representative of f. Set $V = U \setminus F^{-1}(0)$. It is an element of α because $f \notin \mathfrak{m}_\alpha$. The restriction $F|_V$ of F to V is also a representative of f and the function $1/F|_V \in C^r_{\mathrm{df}}(V)$ is a representative of the multiplicative inverse of f. The element f is a unit in $(\mathfrak{D}^r_M)_\alpha$. □

Lemma 3.13. *Let* r *be a nonnegative integer. Let* M *be a definable* $\mathcal{C}^r$ *manifold, and* $\alpha \in \widetilde{M}$. *Given any* $f \in (\mathfrak{D}^r_M)_\alpha$, *there exist* $g, h \in C^r_{df}(M)$ *and* $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ *such that* $g \notin \mathfrak{m}_\alpha$ *and* $\sigma \circ (gf) = h$ *in* $(\mathfrak{D}^r_M)_\alpha$.

Proof. Let $F \in C^r_{\mathrm{df}}(U)$ be a representative of f, where U is a definable open subset of M with $\alpha \in \widetilde{U}$. There exists a definable $\mathcal{C}^r$ function g on M such that g is positive on U and $\lim_{U \ni x \to x_0} g(x)F(x) = 0$ for all $x_0 \in \overline{U} \setminus U$ by Lemma 3.9. We have $g \notin \mathfrak{m}_\alpha$ because g is positive on U and $U \in \alpha$. Using Proposition 3.6, we can find $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ such that $\sigma \circ (gf)$ is extendable to M as a definable $\mathcal{C}^r$ function. Let h be the extension. We have $\sigma \circ (gf) = h$ in $(\mathfrak{D}^r_M)_\alpha$. □

Let α be an arbitrary element of $\widetilde{M}$. We want to define an interpretation of $\mathcal{L}$-formulae in the residue field $k(\alpha)$. For that purpose, we first determine an interpretation in the stalk $(\mathfrak{D}^r_M)_\alpha$. For any constant symbol c, the interpretation of c in $(\mathfrak{D}^r_M)_\alpha$ is given by $c^{(\mathfrak{D}^r_M)_\alpha} = c^{\widetilde{\mathbb{R}}}$. The notation $c^{\widetilde{\mathbb{R}}}$ denotes the interpretation of the constant symbol c in $\widetilde{\mathbb{R}}$. Let g be a function symbol in n variables in $\mathcal{L}$. For any $f_1, \ldots, f_n \in (\mathfrak{D}^r_M)_\alpha$, the interpretation of g in $(\mathfrak{D}^r_M)_\alpha$ is given by $g^{(\mathfrak{D}^r_M)_\alpha}(f_1, \ldots, f_n) = g^{\widetilde{\mathbb{R}}}(F_1, \ldots, F_n) \in (\mathfrak{D}^r_M)_\alpha$, where $F_i : U \to \mathbb{R}$ are definable $\mathcal{C}^r$ functions which are representatives of f_i for all $1 \leq i \leq n$. We finally consider a relation symbol R in n variables. The interpretation of R in $(\mathfrak{D}^r_M)_\alpha$ is given by

$$R^{(\mathfrak{D}^r_M)_\alpha} = \{(f_1, \ldots, f_n) \in ((\mathfrak{D}^r_M)_\alpha)^n \mid \{x \in U \mid (F_1(x), \ldots, F_n(x)) \in R^{\widetilde{\mathbb{R}}}\} \in \alpha\}.$$

It is easy to check that the above definitions are independent of the choice of the representatives $F_1, \ldots, F_n$. Under the above interpretation, the local ring $(\mathfrak{D}^r_M)_\alpha$ is an $\mathcal{L}$-structure. We denote this $\mathcal{L}$-structure by $\widetilde{(\mathfrak{D}^r_M)}_\alpha$.

Proposition 3.14. *Consider a definable $\mathcal{C}^r$ manifold M, where r is a nonnegative integer. Let $\alpha \in \widetilde{M}$ be a D_M-ultrafilter, $\phi(\overline{x})$ be an $\mathcal{L}$-formula with n free variables and $\overline{f} = (f_1, \ldots, f_n) \in ((\mathfrak{D}^r_M)_\alpha)^n$. The $\mathcal{L}$-structure $\widetilde{(\mathfrak{D}^r_M)}_\alpha$ satisfies $\phi(\overline{f})$ if and only if the set*

$$\{x \in U \mid \widetilde{\mathbb{R}} \models \phi(F_1(x), \ldots, F_n(x))\}$$

belongs to the D_M-ultrafilter α, where $F_i : U \to \mathbb{R}$ are definable $\mathcal{C}^r$ functions which are representatives of f_i for all $1 \leq i \leq n$.

Proof. We prove the proposition by induction on the complexity of the formula $\phi(\overline{x})$. The proposition is obviously true when $\phi(\overline{x})$ is an atomic formula. It is easy to show the proposition when $\phi = \phi_1 \wedge \phi_2$ or $\phi = \neg\, \psi$ for some $\mathcal{L}$ formulae ϕ_1, ϕ_2 and ψ. The remaining case is the case in which $\phi(\overline{x}) = \exists y\ \psi(\overline{x}, y)$. We may assume that the formula $\psi(\overline{x}, y)$ satisfies the statement of the proposition by the induction hypothesis.

We first consider the case in which the definable set

$$X = \{x \in U \mid \widetilde{\mathbb{R}} \models \phi(F_1(x), \ldots, F_n(x))\}$$

is an element of α. Consider the definable set Y given by

$$Y = \{(x, y) \in X \times \mathbb{R} \mid \widetilde{\mathbb{R}} \models \psi(F_1(x), \ldots, F_n(x), y)\}.$$

Let $\pi : Y \to X$ be the projection, then the definable map π is onto by the definition of X.

We may assume that M is a definable $\mathcal{C}^r$ submanifold of a Euclidean space $\mathbb{R}^m$. Apply the definable $\mathcal{C}^r$ cell decomposition theorem [4, Theorem 3.2 and Exercise 3.3

of Chapter 7]. We get a definable $\mathcal{C}^r$ cell decomposition of $\mathbb{R}^{m+1}$ partitioning Y. One of cells in $\mathbb{R}^m$, say C, is contained in X and belongs to α. There exists a definable $\mathcal{C}^r$ function $h : C \to \mathbb{R}$ such that the definable set $\{(x, h(x)) \mid x \in C\}$ is contained in Y. In fact, a cell D with $\pi(D) = C$ is contained in Y because π is onto. Set $h = u$ if the cell D is of the from $\{(x, y) \in C \times \mathbb{R} \mid y = u(x)\}$ for some definable $\mathcal{C}^r$ function u on C. Set $h = \frac{u_1+u_2}{2}$ if the cell D is of the from $\{(x, y) \in C \times \mathbb{R} \mid u_1(x) < y < u_2(x)\}$ for some definable $\mathcal{C}^r$ functions u_1 and u_2 on C. There exists a definable open subset W of M and a definable $\mathcal{C}^r$ map $\rho : W \to C$ with $C \subset W$ and $\rho|_C = \mathrm{id}$ by Lemma 3.10. We have $W \in \alpha$ because $C \subset W$ and $C \in \alpha$. Set $G = h \circ \rho$, and let g be the image of G in $(\mathfrak{D}^r_M)_\alpha$. The definable set

$$Z = \{x \in U \cap W \mid \widetilde{\mathbb{R}} \models \psi(F_1(x), \ldots, F_n(x), G(x))\}$$

contains C, hence; we have $Z \in \alpha$. We get $\widetilde{(\mathfrak{D}^r_M)_\alpha} \models \psi(\overline{f}, g)$ by the induction hypothesis. We obtain $\widetilde{(\mathfrak{D}^r_M)_\alpha} \models \phi(\overline{f})$.

We next consider the case in which the relation $\widetilde{(\mathfrak{D}^r_M)_\alpha} \models \phi(\overline{f})$ is satisfied. There exists $g \in (\mathfrak{D}^r_M)_\alpha$ with $\widetilde{(\mathfrak{D}^r_M)_\alpha} \models \psi(\overline{f}, g)$. Let $G : U \to \mathbb{R}$ be a representative of g. We may assume that $F_1, \ldots, F_n$ and G have the common domain U by shrinking U if necessary. Set $A = \{x \in U \mid \widetilde{\mathbb{R}} \models \psi(F_1(x), \ldots, F_n(x), G(x))\}$. It belongs to α by the induction hypothesis. For any $x \in A$, the formula $\exists y\ \psi(F_1(x), \ldots, F_n(x), y)$ holds true by taking $y = G(x)$. It means that the set

$$X = \{x \in U \mid \widetilde{\mathbb{R}} \models \phi(F_1(x), \ldots, F_n(x))\}$$

contains A; therefore, the set X belongs to α. □

Proposition 3.15. *Consider a definable $\mathcal{C}^r$ manifold M, where r is a nonnegative integer. Let $\alpha \in \widetilde{M}$ be a D_M-ultrafilter. Let $\phi(\overline{x})$ be an $\mathcal{L}$-formula with n free variables. Let $\overline{f} = (f_1, \ldots, f_n)$, $\overline{g} = (g_1 \ldots, g_n) \in ((\mathfrak{D}^r_M)_\alpha)^n$ with $f_i - g_i \in \mathfrak{m}_\alpha$ for all $1 \leq i \leq n$. Here, $\mathfrak{m}_\alpha$ is the maximal ideal of the local ring $(\mathfrak{D}^r_M)_\alpha$. The $\mathcal{L}$-structure $\widetilde{(\mathfrak{D}^r_M)_\alpha}$ satisfies $\phi(\overline{f})$ if and only if $\phi(\overline{g})$ is true in $\widetilde{(\mathfrak{D}^r_M)_\alpha}$.*

Proof. By symmetry, we have only to show that $\widetilde{(\mathfrak{D}^r_M)_\alpha} \models \phi(\overline{g})$ if $\widetilde{(\mathfrak{D}^r_M)_\alpha} \models \phi(\overline{f})$. Let F_i and G_i be representatives of f_i and g_i for all $1 \leq i \leq n$, respectively. We may assume that the domains of F_i and G_i are common without loss of generality. Let U be the common domain. It is an element of α. Set $Z_i = \{x \in U \mid F_i(x) = G_i(x)\}$ for all $1 \leq i \leq n$, then it belongs to α by the definition of the maximal ideal $\mathfrak{m}_\alpha$. The intersection $Z = \bigcap_{i=1}^n Z_i$ is also an element of α.

Set $X = \{x \in U \mid \widetilde{\mathbb{R}} \models \phi(F_1(x), \dots, F_n(x))\}$ and $Y = \{x \in U \mid \widetilde{\mathbb{R}} \models \phi(G_1(x), \dots, G_n(x))\}$. We have $X \in \alpha$ by the assumption and Proposition 3.14. We get $Y \cap Z \in \alpha$ because $Y \cap Z = X \cap Z$ and $X, Z \in \alpha$. We obtain $Y \in \alpha$ because $Y \cap Z \subset Y$. We finally have $\widetilde{(\mathfrak{D}_M^r)_\alpha} \models \phi(\overline{g})$ by Proposition 3.14. □

Let M be a definable $\mathcal{C}^r$ manifold. The residue field $k(\alpha)$ of the stalk of the sheaf $\mathfrak{D}_M^r$ at a point $\alpha \in \widetilde{M}$ can be considered an $\mathcal{L}$-structure under the following interpretation: For any $\mathcal{L}$-formula $\phi(\overline{x})$ with n free variables and $\overline{a} = (a_1, \dots, a_n) \in (k(\alpha))^n$, the sentence $\phi(\overline{a})$ is true if $\widetilde{(\mathfrak{D}_M^r)_\alpha} \models \phi(f_1, \dots, f_n)$, where $f_i \in (\mathfrak{D}_M^r)_\alpha$ is a representative of a_i for each $1 \leq i \leq n$. The above definition is independent of the choice of the representatives $f_1, \dots, f_n$ by Proposition 3.15. This $\mathcal{L}$-structure is denoted by $\widetilde{k(\alpha)}$. We are finally ready to demonstrate Theorem 1.2.

Theorem 3.16. *The $\mathcal{L}$-structure $\widetilde{k(\alpha)}$ is an elementary extension of $\widetilde{\mathbb{R}}$.*

Let $\mathcal{K}$ be an elementary extension of $\widetilde{\mathbb{R}}$ whose underlying set K contains the ring $C_{df}^r(M)/\operatorname{supp}(\alpha)$. Assume further that, for any $\mathcal{L}$-formula $\phi(\overline{x})$ and $\overline{F} = (F_1, \dots, F_n) \in (C_{df}^r(M))^n$, the following two conditions are equivalent:

- *$\mathcal{K} \models \phi(\overline{F})$, and*
- *the ultrafilter α contains the definable set $\{x \in M \mid \widetilde{\mathbb{R}} \models \phi(F_1(x), \dots, F_n(x))\}$.*

Then, there exists a unique elementary embedding $\widetilde{k(\alpha)} \prec \mathcal{K}$.

Proof. We first demonstrate that $\widetilde{k(\alpha)}$ is an elementary extension of $\widetilde{\mathbb{R}}$. Consider an $\mathcal{L}$-formula $\phi(\overline{x}, y)$. Let $\overline{a} = (a_1, \dots, a_n)$ be a sequence of real numbers and $f \in k(\alpha)$ with $\widetilde{k(\alpha)} \models \phi(\overline{a}, f)$. We have only to show that $\widetilde{\mathbb{R}} \models \phi(\overline{a}, b)$ for some $b \in \mathbb{R}$ by [11, Proposition 2.3.5]. The set $C = \{x \in U \mid \widetilde{\mathbb{R}} \models \phi(a_1, \dots, a_n, F(x))\}$ is contained in α by Proposition 3.14, where $F \in C_{\mathrm{df}}^r(U)$ is a representative of f. In particular, C is not an empty set. Take $c \in C$ and set $b = F(c)$. It is obvious that $\widetilde{\mathbb{R}} \models \phi(\overline{a}, b)$. We have shown that $\widetilde{k(\alpha)}$ is an elementary extension of $\widetilde{\mathbb{R}}$.

Let $\mathcal{K}$ be an elementary extension of $\widetilde{\mathbb{R}}$ satisfying the conditions in the theorem. We construct a map $\iota : k(\alpha) \to K$. Consider an arbitrary element $a \in k(\alpha)$. Let $f \in (\mathfrak{D}_M^r)_\alpha$ be a representative of a. There exist $g, h \in C_{\mathrm{df}}^r(M)$ and $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ such that

$g \notin \operatorname{supp}(\alpha)$ and $\sigma \circ (gf) = h$ in $(\mathfrak{D}^r_M)_\alpha$ by Lemma 3.13. Since $\mathcal{K}$ is an elementary extension of $\widetilde{\mathbb{R}}$, there exists a unique definable $\mathcal{C}^r$ bijective extension $\sigma_K : K \to K$ of σ to K. We define

$$\iota(a) = \sigma_K^{-1}(h)/g. \tag{1}$$

We demonstrate that the map ι is an elementary embedding. Assume that M is a definable $\mathcal{C}^r$ submanifold of $\mathbb{R}^m$. The notation X_i denotes the restriction of the i-th coordinate function on $\mathbb{R}^m$ to M or the its image in $k(\alpha)$ for each $1 \leq i \leq m$. Let $\overline{a} = (a_1, \ldots, a_n) \in (k(\alpha))^n$. Let $F_i : U \to \mathbb{R}$ be a definable $\mathcal{C}^r$ function which is a representative of a_i. We have $U \in \alpha$. The notation $\Phi(x_1, \ldots, x_m)$ denotes the formula representing the definable set U, that is, $U = \{\overline{x} \in \mathbb{R}^m \mid \widetilde{\mathbb{R}} \models \Phi(\overline{x})\}$. We have

$$\mathcal{K} \models \Phi(X_1, \ldots, X_m) \tag{2}$$

by the assumption on $\mathcal{K}$ because $X_1, \ldots X_m$ are definable $\mathcal{C}^r$ functions on M.

The formula $\Psi_i(x_1, \ldots, x_m, y)$ represents the relation $y = F_i(x_1, \ldots, x_m)$. It means that $y = F_i(x_1, \ldots, x_m)$ if and only if $\widetilde{\mathbb{R}} \models \Psi_i(x_1, \ldots, x_m, y)$ for any $(x_1, \ldots, x_m) \in \mathbb{R}^m$ and $y \in \mathbb{R}$. We first show the following claim:

Claim. For any $1 \leq i \leq n$, the unique element $y \in K$ satisfying the formula $\Psi_i(X_1, \ldots, X_m, y)$ in $\mathcal{K}$ is $\iota(a_i)$.

We begin to prove the claim. We get

$$\widetilde{\mathbb{R}} \models \forall x_1 \cdots \forall x_m \exists! y \ (\Phi(x_1, \ldots, x_m) \to \Psi_i(x_1, \ldots, x_m, y)) \tag{3}$$

for all $1 \leq i \leq n$. Since $\widetilde{\mathbb{R}} \prec \mathcal{K}$, the same sentence holds true in $\mathcal{K}$, that is;

$$\mathcal{K} \models \forall x_1 \cdots \forall x_m \exists! y \ (\Phi(x_1, \ldots, x_m) \to \Psi_i(x_1, \ldots, x_m, y)).$$

Using the relation (2), we get

$$\mathcal{K} \models \exists! y \ \Psi_i(X_1, \ldots, X_m, y).$$

It means that only one element $y \in K$ can satisfy the formula $\Psi_i(X_1, \ldots, X_m, y)$ in $\mathcal{K}$. The remaining task to complete the proof of the claim is to demonstrate that $\mathcal{K} \models \Psi_i(X_1, \ldots, X_m, \iota(a_i))$.

There exist definable $\mathcal{C}^r$ functions g_i, h_i on M and $\sigma_i \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ with $g_i \notin \mathrm{supp}(\alpha)$ and $\sigma_i \circ (g_i F_i) = h_i$ in $(\mathfrak{D}^r_M)_\alpha$ by Lemma 3.13. It implies that the definable set

$$\{x \in M \mid \widetilde{\mathbb{R}} \models \forall y\ (\sigma_i(g_i(x)y) = h_i(x) \to \neg\Phi(x) \vee \Psi_i(x, y))\}$$

belongs to α by shrinking U if necessary. We obtain

$$\mathcal{K} \models \forall y\ (\sigma_i(g_i y) = h_i \to \neg\Phi(X_1, \dots, X_m) \vee \Psi_i(X_1, \dots, X_m, y))$$

by the assumption on $\mathcal{K}$. By the definition of $\iota(a_i)$ given in the equality (1), the equality $\sigma_i(g_i\iota(a_i)) = h_i$ is satisfied in $\mathcal{K}$. Hence, we have $\mathcal{K} \models \Psi_i(X_1, \dots, X_m, \iota(a_i))$. We have demonstrated the claim.

The map ι is well-defined because the solution of the relation $\mathcal{K} \models \Psi_i(X_1, \dots, X_m, y)$ is unique and we can show that, if we take another σ_i, g_i and h_i, the element $y = \sigma_i^{-1}(h_i)/g_i$ satisfies the relation $\mathcal{K} \models \Psi_i(X_1, \dots, X_m, y)$ in the same way as above.

We begin to prove that the map ι is an elementary extension. Consider an $\mathcal{L}$-formula $\phi(\overline{x})$ with n free variables. We first show that the condition that $\widetilde{k(\alpha)} \models \phi(\overline{a})$ implies the condition that $\mathcal{K} \models \phi(\iota(\overline{a}))$, where $\iota(\overline{a}) := (\iota(a_1), \dots, \iota(a_n)) \in K^n$. Let $\overline{y} = (y_1, \dots, y_n)$ be free variables. Set

$$\psi(\overline{x}, \overline{y}) = \bigwedge_{i=1}^{n} (\Phi(x_1, \dots, x_m) \to \Psi_i(x_1, \dots, x_m, y_i)) \wedge \phi(\overline{y}).$$

We have

$$\widetilde{k(\alpha)} \models \psi(X_1, \dots, X_m, \overline{a})$$

because we assume that $\widetilde{k(\alpha)} \models \phi(\overline{a})$. The definable set $V = \{x \in M \mid \widetilde{\mathbb{R}} \models \psi(x, F_1(x), \dots, F_n(x))\}$ is contained in α by Proposition 3.14. Set $\psi'(\overline{x}) = \exists \overline{y}\ \psi(\overline{x}, \overline{y})$ and $W = \{x \in M \mid \widetilde{\mathbb{R}} \models \psi'(x)\}$. The definable set W contains the definable set V; and we get $W \in \alpha$. Since $X_1, \dots, X_m$ are definable $\mathcal{C}^r$ functions on M, we get $\mathcal{K} \models \psi'(X_1, \dots, X_m)$ by the assumption on $\mathcal{K}$. It means the following:

$$\mathcal{K} \models \exists \overline{y}\ \psi(X_1, \dots, X_m, \overline{y}).$$

However, by the relation (2) and the above claim, the only $\iota(\overline{a}) \in K^n$ satisfies the first condition $\bigwedge_{i=1}^n (\Phi(X_1, \dots, X_m) \to \Psi_i(X_1, \dots, X_m, y_i))$ of $\psi(X_1, \dots, X_m, \overline{y})$. Hence, we have $\mathcal{K} \models \psi(X_1, \dots, X_m, \iota(\overline{a}))$; therefore, $\mathcal{K} \models \phi(\iota(\overline{a}))$.

We show the opposite implication, that is; we demonstrate that the condition that $\mathcal{K} \models \phi(\iota(\overline{a}))$ implies the condition that $\widetilde{k(\alpha)} \models \phi(\overline{a})$. We have $\mathcal{K} \models \bigwedge_{i=1}^n \Psi_i(X_1, \ldots, X_m, \iota(a_i)) \wedge \phi(\iota(\overline{a}))$ by the above claim and the assumption. We get $\mathcal{K} \models \exists \overline{y} \ \bigwedge_{i=1}^n \Psi_i(X_1, \ldots, X_m, y_i) \wedge \phi(\overline{y})$. Using the assumption on $\mathcal{K}$, the definable set $\{x \in M \mid \exists \overline{y} \ \bigwedge_{i=1}^n \Psi_i(x, y_i) \wedge \phi(\overline{y})\}$ is an element of α. We get

$$\widetilde{k(\alpha)} \models \exists \overline{y} \ \bigwedge_{i=1}^n \Psi_i(X_1, \ldots, X_m, y_i) \wedge \phi(\overline{y}) \tag{4}$$

by Proposition 3.14. On the other hand, the relation (3) implies the relation that

$$\widetilde{k(\alpha)} \models \forall x_1 \cdots \forall x_m \exists ! y \ (\Phi(x_1, \ldots, x_m) \to \Psi_i(x_1, \ldots, x_m, y))$$

because $\widetilde{\mathbb{R}} \prec \widetilde{k(\alpha)}$ as we have demonstrated. The relation $\widetilde{k(\alpha)} \models \Phi(X_1, \ldots, X_m)$ is obviously satisfied by the definition of U and Proposition 3.14. We get

$$\widetilde{k(\alpha)} \models \exists ! y \ \Psi_i(X_1, \ldots, X_m, y) \tag{5}$$

from the above relations. The relation

$$\widetilde{k(\alpha)} \models \Psi_i(X_1, \ldots, X_m, a_i) \tag{6}$$

is obvious by the definition of F_i and Proposition 3.14. Using the relations (4), (5) and (6), we get $\widetilde{k(\alpha)} \models \phi(\overline{a})$. We have demonstrated that the map ι is an elementary embedding.

The remaining task is to show that the map ι is the unique elementary embedding. Let $\iota' : \widetilde{k(\alpha)} \prec \mathcal{K}$ be an elementary embedding. Let v be an arbitrary element of $k(\alpha)$. We have only to show that $\iota(v) = \iota'(v)$. There exist $g, h \in C^r_{\text{df}}(M)$ and $\sigma \in \mathrm{S}_{\widetilde{\mathbb{R}}}$ such that $g \neq 0$ in $k(\alpha)$ and $\sigma \circ (gv) = h$ in $k(\alpha)$ in the same way as above. We have $\sigma_K(g \cdot \iota'(v)) = h$ in $\mathcal{K}$ because ι' is an elementary embedding. Since σ_K is a bijection, we get $\iota(v) = \iota'(v)$ by the equality (1). □

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