THE AKE PRINCIPLE MEETS CLASSIFICATION THEORY

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Abstract. Using a type-amalgamation style criterion of NATP, we give a proof of the AKE principle for NATP in the case of equicharacteristic 0.

In [1], Ahn and Kim introduced a new dividing line, called the *antichain tree property* (ATP), and they provide a nice dichotomy for SOP_1, that is, a complete theory T is SOP_1 if and only if it is TP_2 or ATP as like unstability is equal to IP or TP, and TP is equal to TP_1 or TP_2. So, NATP is located in the map of the universe in Model Theory as follows:

$$\begin{array}{ccccccccc}
\mathrm{DLO} & \longrightarrow & \mathrm{NIP} & \longrightarrow & \mathrm{NTP}_2 & \longrightarrow & \mathrm{NATP} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathrm{Stable} & \longrightarrow & \mathrm{Simple} & \longrightarrow & \mathrm{NSOP}_1 & \longrightarrow & \mathrm{NSOP}_2 \longrightarrow \cdots
\end{array}$$

On the other hand, one of most fundamental principles in model theory of valued fields is the *Ax-Kochen-Ershov* principle (in short, the AKE principle), roughly saying that elementary properties of unramified henselian valued fields of characteristic 0 is determined by elementary properties of its residue field and value group. In [4, 8], Ax and Kochen, and independently Ershov showed that two henselian valued fields of equicharacteristic 0 are elementary equivalent if and only if their residue fields and value groups are elementary equivalent.

In [2], we studied several basic properties of NATP, for example, witness of NATP in a single variable and the equivalence ATP and k-ATP, and provided several algebraic examples of NATP theories. Most of all, we show that the Ax-Kochen-Ershov principle for NATP in the case of equicharacteristic 0. Namely, a given henselian valued field of

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equicharacteristic 0 is NATP if and only if its residue field is NATP. In this note, we aim to give a new proof of the AKE principle for NATP in the case of equicharacteristic 0, given in [2, Theorem 3.27], via a type-amalgamation criterion of NATP ([2, Theorem 3.27]).

Theorem 0.1. [7, Théorèm 8][5, Theorem 7.6][2, Theorem 3.27] *Let* $\mathcal{K} = (K, \Gamma, k)$ *be a henselian valued field of equicharacteristic* 0 *in the Denef-Pas language, where* K *is a valued field,* Γ *is a value group, and* k *is a residue field.*

1. *(Delon)* $\mathrm{Th}(\mathcal{K})$ *is NIP if and only if* $\mathrm{Th}(k)$ *is NIP.*
2. *(Chernikov)* $\mathrm{Th}(\mathcal{K})$ *is* NTP_2 *if and only if* $\mathrm{Th}(k)$ *is* NTP_2*.*
3. *(Ahn-Kim-L.)* $\mathrm{Th}(\mathcal{K})$ *is NATP if and only if* $\mathrm{Th}(k)$ *is NATP.*

Also, using similar methods, we can prove the AKE principle for NATP in the case of mixed characteristic in [10], and we can find several examples having NATP in [3], for example, a random parametrization of DLO is NATP, which is TP_2 and SOP.

1. Preliminaries

1.1. **Antichain tree property.**
We briefly recall several notions and facts around the antichain tree property from [2].

Notation 1.1. Let κ and λ be cardinals.

(1) By κ^{λ}, we mean the set of all functions from λ to κ.
(2) By $\kappa^{<\lambda}$, we mean $\bigcup_{i<\lambda} \kappa^i$ and call it a tree.

Let $\eta, \nu \in \kappa^{<\lambda}$.

(3) By $\eta \trianglelefteq \nu$, we mean $\eta \subseteq \nu$. If $\eta \trianglelefteq \nu$ or $\nu \trianglelefteq \eta$, then we say η and ν are comparable.
(4) By $\eta \perp \nu$, we mean that $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$. We say η and ν are incomparable if $\eta \perp \nu$.
(5) By $\eta \wedge \nu$, we mean the maximal $\xi \in \kappa^{<\lambda}$ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$.
(6) By $l(\eta)$, we mean the domain of η.
(7) By $\eta <_{lex} \nu$, we mean that either $\eta \trianglelefteq \nu$, or or $\eta \perp \nu$ and $\eta(l(\eta \wedge \nu)) < \nu(l(\eta \wedge \nu))$.
(8) By $\eta^\frown\nu$, we mean $\eta \cup \{(l(\eta) + i, \nu(i)) : i < l(\nu)\}$.
(9) We say a subset X of $\kappa^{<\lambda}$ is an *antichain* if the elements of X are pairwise incomparable.

We use a language $\mathcal{L}_0 := \{\trianglelefteq, <_{lex}, \wedge\}$ for a tree with a natural interpretation in Notation 1.1. We say that an antichain A of $\kappa^{<\lambda}$ is *universal* if any finite antichian of $\kappa^{<\lambda}$ is embedded into A as $\mathcal{L}_0$-structures.

Let T be a complete theory in a language $\mathcal{L}$, and let M be a monster model of T. For a tuple a of elements in M and a parameter A, we write $\mathrm{tp}_{\mathcal{L}}(a/A)$ for the complete type of a over A and write $\mathrm{qftp}_{\mathcal{L}}(a/A)$ for the quantifier free type of a over A. If there is no confusion, we omit the subscription $\mathcal{L}$. Let $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ be a tree-indexed tuple of parameters from M. For $\bar{\eta} = (\eta_0, \dots, \eta_m)$, we denote $(a_{\eta_0}, \dots, a_{\eta_m})$ by $a_{\bar{\eta}}$. We say that $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ is *strongly indiscernbile* if for tuples $\bar{\eta}, \bar{\nu}$ of elements in $\kappa^{<\lambda}$,

$$\mathrm{qftp}_{\mathcal{L}_0}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}_0}(\bar{\nu}) \Rightarrow \mathrm{tp}_{\mathcal{L}}(a_{\bar{\eta}}) = \mathrm{tp}_{\mathcal{L}}(a_{\bar{\nu}}).$$

Definition 1.2. [2, Definition 1.1, Definition 3.19] Let $\varphi(x; y)$ be a $\mathcal{L}$-formula. For a positive integer $k \geq 2$, we say that $\varphi(x; y)$ has the *k-antichain tree property* (k-ATP) if there is a tree of parameters $(a_\eta)_{\eta\in 2^{<\omega}}$ such that

- For any antichain $A \subset 2^{<\omega}$, $\{\varphi(x; a_\eta) : \eta \in A\}$ is consistent.
- For any comparable distinct elements $\eta_0, \eta_1 \in 2^{<\omega}$,
$$\{\varphi(x; a_{\eta_0}), \varphi(x; a_{\eta_1})\}$$
is inconsistent.

The *antichain tree property* (ATP) means the 2-antichain tree property. We say that T has k-ATP if there is a formula having k-ATP. We write NATP for non ATP.

Fact 1.3. [2, Theorem 3.17, Lemma 3.20]

1. *If there exists a witness of ATP, then there exists a witness of ATP in a single free variable.*
2. *T is ATP if and only if T has k-ATP for some $k \geq 2$. Moreover, if a formula $\varphi(x; y)$ has k-ATP for some $k \geq 2$, then a formula of the form*
$$\varphi(x; y_0) \wedge \cdots \wedge \varphi(x; y_m)$$
has ATP for some $m \geq 1$.

We recall a type-amalgamation style criterion for NATP.

Fact 1.4. [2, Theorem 3.27] *Let κ and κ' be cardinals such that $2^{|T|} < \kappa < \kappa'$ with $cf(\kappa) = \kappa$. The following are equivalent.*

1. *T is NATP.*
2. *For any strongly indiscernible tree $(a_\eta)_{\eta\in 2^{<\kappa'}}$ with $|a_\emptyset| \leq |T|$ and b with $|b| = 1$, there are some $\rho \in 2^\kappa$ and b' such that*
   - (a) *$(a_{\rho^\frown 0^i})_{i<\kappa'}$ is indiscernible over b', and*
   - (b) *$b \equiv_{a_\rho} b'$.*

(3) *For any strongly indiscernible tree* $(a_\eta)_{\eta\in 2^{<\kappa'}}$ *with* $|a_\emptyset| \le |T|$ *and* b *with* $|b| = 1$*, there are some* $\rho \in 2^\kappa$ *such that for* $p(x; a_\rho) = \mathrm{tp}(b/a_\rho)$*,*

$$\bigcup_{i<\kappa'} p(x; a_{\rho^\frown 0^i})$$

is consistent.

We finis this subsection with a fact on the existence on a 'homogeneous' universal antichain.

Fact 1.5. [2, Corollary 3.23(b)] *Let* $(a_\eta)_{\eta\in 2^{<\kappa'}}$ *be a strongly indiscernible tree and* b *be a tuple of elements in* M *for some cardinals* κ *and* κ' *such that* $2^{|T|+|a_\emptyset|+|b|} < \kappa < \kappa'$ *and* $cf(\kappa) = \kappa$*. Then, there is a universal antichain* $S \subset 2^\kappa$ *such that*

$$a_\eta \equiv_b a_{\eta'}$$

for all $\eta, \eta' \in S$*.*

1.2. **Denef-Pas language.** Let $\mathcal{K} = (K, \Gamma, k, \nu : K \to \Gamma \cup \{\infty\}, \mathrm{ac} : K \to k)$ be a henselian valued field of characteristic $(0, 0)$ in the **Denef-Pas language** $\mathcal{L}_{Pas} = \mathcal{L}_K \cup \mathcal{L}_{\Gamma,\infty} \cup \mathcal{L}_k \cup \{\nu, \mathrm{ac}\}$ where $\nu : K \to \Gamma \cup \{\infty\}$ is a valuation map and $\mathrm{acl} : K \to k$ is an angular component map.

We recall Delon's description of types in a single variable on the valued field sort (c.f. [7, 11]). Let $\mathcal{M} = (M, \ldots)$ be a monster model of $\mathrm{Th}(\mathcal{K})$. Take $x \in M \setminus K$, and define

$$I_K(x) := \{\gamma \in \Gamma : \exists a \in K\ (\gamma = \nu(x - a))\}.$$

Then, there are three possibilities of $I_K(x)$:

- (Immediate) $I_K(x) = \{\nu(x - a) : a \in K\}$ and does not have a maximal element.
- (Residual) $I_K(x) = \{\nu(x - a) : a \in K\}$ and has a maximal element.
- (Valuational) $I_K(x) \neq \{\nu(x - a) : a \in K\}$.

Fact 1.6. [7, Théorèm 5][11, Theorem 2.1.1] *There are three possibilities of* $\mathrm{tp}(x/K)$*:*

(1) *(Immediate) Let* (a_ρ, γ_ρ) *be a sequence indexed by a well-ordered set such that* $a_\rho \in K$*,* $\gamma_\rho = \nu(x - a_\rho)$*, and* (γ_ρ) *is cofinal in* $I_K(x)$*. Then,*

$$\{\nu(x - a_\rho) = \gamma_\rho\} \models \mathrm{tp}(x/\mathcal{K}).$$

(2) *(Residual) There are* $a \in K$ *and* $\gamma \in \Gamma$ *such that* $\nu(x - a) = \gamma$*,* $\mathrm{ac}(x - a) \notin k$*, and*

$$\{\nu(x - a) = \gamma\} \cup \mathrm{tp}(\mathrm{ac}(x - a)/k) \models \mathrm{tp}(x/\mathcal{K}).$$

(3) (Valuational) There is $a \in K$ such that $\nu(x-a) \notin \Gamma$ and

$$\mathrm{tp}(\nu(x-a)/\Gamma) \cup \mathrm{tp}(\mathrm{ac}(x-a)/k) \models \mathrm{tp}(x/\mathcal{K}).$$

2. Proof of Theorem 0.1(3)

In this section, we aim to give a new proof of Theorem 0.1(3) using a type-amalgamation style criterion for NATP (Fact 1.4(3)) and a description of type in a single variable on the valued field sort (Fact 1.6).

Let $\mathcal{K} = (K, \Gamma, k, \nu : K \to \Gamma, \mathrm{ac} : K \to k)$ be a henselian valued field of characteristic $(0,0)$, which is saturated enough, in the Denef-Pas language. We use $x, y, z, \ldots$ for tuples of variables on K, $x^\Gamma, y^\Gamma, z^\Gamma, \ldots$ for tuples of variables on Γ, and $x^k, y^k, z^k, \ldots$ for tuples of variables on k. We first recall the following fact.

Fact 2.1. [2, Lemma 4.27] *Suppose* $\mathrm{Th}(k)$ *is NATP. Let*

$$\varphi(x; yy^\Gamma y^k) \equiv \chi(\nu(x-y); y^\Gamma) \wedge \rho(\mathrm{ac}(x-y); y^k)$$

where $|x| = |y| = 1$, $\chi \in \mathcal{L}_{\Gamma,\infty}$ *and* $\rho \in \mathcal{L}_k$. *Then,* $\varphi(x; yy^\Gamma y^k)$ *does not wintess ATP. Moreover, by Fact 1.3(2), it does not witness* k*-ATP for all* $k \geq 2$.

From now on, we assume that $\mathrm{Th}(k)$ is NATP. By [9], $\mathrm{Th}(\Gamma)$ is automatically NATP. Note that k and Γ are stably embedded (c.f. [6, Lemma 2.3]).

Let κ and κ' be cardinals such that $2^{\aleph_0} < \kappa < \kappa'$ with $cf(\kappa) = \kappa$. Take a strongly indiscernible tree $(a_\eta)_{\eta \in 2^{<\kappa'}}$ with $|a_\eta|$ countable and let b a tuple of element in $\mathcal{K}$ of length 1. Without loss of generality, we may assume that each $a_\eta = (K_\eta, \Gamma_\eta, k_\eta)$ is a elementary substructure of $\mathcal{K}$, where K_η is a valued field, Γ_ρ is a value group, and k_ρ is a residue field. Suppose $b \in \Gamma$. Since Γ is stably embedded, $\mathrm{tp}(b/a_\eta) = \mathrm{tp}(b/\Gamma_\eta)$ for each η. Since $\mathrm{Th}(\Gamma)$ is NATP, by Fact 1.4(3), there is $\rho \in 2^{<\kappa'}$ such that for $p(x; a_\rho) = \mathrm{tp}(b/a_\rho)$,

$$\bigcup_{i<\kappa'} p(x; a_{\rho^\frown 0^i})$$

is consistent. By similar way, the same thing holds for the case that $b \in k$. So, we may assume that $b \in K$.

By Fact 1.5, there is a universal antichain $S \subset 2^\kappa$ such that $a_\eta \equiv_b a_{\eta'}$ for all $\eta, \eta' \in S$. Take $\rho \in S$ arbitrary. Put $p(x; a_\rho) = \mathrm{tp}(b/a_\rho)$ and put $q(x) := \bigcup_{i<\kappa'} p(x; a_{\rho^\frown 0^i})$.

Claim 2.2. *$q(x)$ is consistent.*

Proof. . If $b \in K_\rho$, then by strong indiscernibility, $b \in K_\eta$ for all $\eta \in S$ and so $b \models q$. Now, we assume that $b \notin K_\rho$. Suppose q is not consistent. Then, by compactness, strong indiscernibility, and Fact 1.6, there are

- a formula of the form:
$$\varphi(x; yy^\Gamma y^k) \equiv \chi(\nu(x-y); y^\Gamma) \wedge \rho(\mathrm{ac}(x-y); y^k),$$
- for a positive integer n and for each $i < n$,
$$c_i \in a_{\rho^\frown 0^i},\ c_i^\Gamma \subset \Gamma_{\rho^\frown 0^i},\ c_i^k \subset k_{\rho^\frown 0^i}$$

such that
$$\bigwedge_{i<n} \varphi(x; c_i c_i^\Gamma c_i^k)$$
is inconsistent. Since $b \models \bigcup_{\eta \in S} p(x; a_\eta)$ and S is a universal antichain, the formula $\varphi(x; yy^\Gamma y^k)$ witness n-ATP, a contradiction to Fact 2.1. Thus, $q(x)$ is consistent. □

Now, by Fact 1.4, $\mathrm{Th}(\mathcal{K})$ is NATP.

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