DEFINABLE SETS IN DP-MINIMAL ORDERED ABELIAN GROUPS

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Abstract. This article surveys some recent results on ordered abelian groups (possibly with additional definable structure) from the subclass of NIP theories which are *dp-minimal*. To put these results in context, the first part of the article reviews and compares various other generalizations of o-minimality (such as local o-minimality and o-stability) and their consequences. It is useful to make the further assumption that there is a cardinal bound on the number of convex subgroups definable in elementary extensions of the structure. Under this hypothesis, some classic theorems on o-minimal structures, such as the monotonicity theorem for unary definable functions, can be suitably generalized.

1. Introduction

In this article I will survey some recent results concerning definable structures on ordered abelian groups, a model-theoretic tameness notion known as dp-minimality, and various other topological and algebraic tameness notions for ordered structures. In the later sections I will announce some new results obtained jointly with Verbovskiy (to appear soon) on dp-minimal ordered abelian groups with boundedly many definable convex subgroups. Some concrete examples will be given as an aid to the reader's intuition, and relationships with other well-studied hypotheses (such as o-stability and local o-minimality) will be discussed.

The inspiration for all the theorems mentioned here comes from the seminal work by Knight, Pillay, and Steinhorn ([25], [32], and [33]) on *o-minimal structures*, that is, first-order structures $\mathcal{M} = (M; <, \ldots)$ endowed with a distinguished linear ordering $<$ and such that any M-definable subset of M is a finite union of points and open intervals.

Theorem 1.1. *(Knight, Pillay, and Steinhorn,* [25]*) Let $\mathcal{M}$ be a densely ordered o-minimal structure.*

1. *(Monotonicity Theorem) For any M-definable function $f : M \to M$, there is a finite partition of M into points and open intervals such that on each open interval I in the partition, $f \restriction I$ is continuous and either strictly increasing, strictly decreasing, or constant.*
2. *(Cell Decomposition Theorem) For any M-definable subset $X \subseteq M^n$, there is a finite decomposition of X into definable subsets called* cells*, defined inductively as follows: a definable set $C \subseteq M$ is a cell if it is a point or an open interval, and a subset $C \subseteq M^{k+1}$ is a cell if its coordinate projection $\pi(C)$ onto the first k coordinates is a cell, and either (i) C is the graph of a continuous function on $\pi(C)$, or else*

$$(ii) \qquad C = \{(\overline{x}, y) : \overline{x} \in \pi(C) \text{ and } f(\overline{x}) < y < g(\overline{x})\}$$

where f and g are continuous functions on $\pi(C)$ such that $f(\overline{x}) < g(\overline{x})$ for all $\overline{x} \in \pi(C)$, including the cases where f is the constant function $-\infty$ or g is the constant function $+\infty$.

There is much more that can be said about the "tame topological" properties of functions and sets definable in o-minimal structures, especially when there is an ordered field structure definable on $\mathcal{M}$. The book by van den Dries [45] is a good basic reference on this topic. O-minimality has found applications to fields outside of logic such as Diophantine geometry (see [31]) and Hodge theory (see [6]).

Unfortunately not all ordered structures we wish to study are o-minimal, and in what follows I will consider ordered abelian groups belonging to wider classes of models. In any ordered structure $\mathcal{M}$ which is not o-minimal, the conclusion of the Monotonicity Theorem above cannot hold, since if Z is a definable subset of M which is not finite unions of points and intervals, we can define the characteristic function for Z:

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$$\chi_Z(a) = \begin{cases} 1, & \text{if } a \in Z; \\ 0, & \text{if } a \notin Z. \end{cases} \tag{1}$$

Given such definable functions, the conclusion of the Cell Decomposition Theorem above will also no longer hold. However, reasonable generalizations of the Monotonicity Theorem do hold in contexts such as weakly o-minimal abelian groups and dp-minimal ordered abelian groups with boundedly many definable convex subgroups, as I will explain in detail in the following sections.

This is far from the first attempt at generalizing Theorem 1.1. Aside from all the generalizations of o-minimality mentioned in Section 4 below, other notable attempts to develop tame topology in similar contexts include Mathews' study of the Cell Decomposition Property for t-minimal structures [28]; Dolich, Miller, and Steinhorn's study of structures with o-minimal open core [15]; Cubides-Kovacsics, Darnière, and Leenknegt's investigation of topological dimension in p-minimal fields [10]; and Foransiero's results on stratifications and dimension functions in d-minimal ordered fields [16].

This paper is organized as follows: in Section 2, I recall the key model-theoretic notions of dp-rank and dp-minimality, emphasizing how dp-rank behaves somewhat like a dimension function on definable subsets X of M^n in a dp-minimal structure $\mathcal{M}$. Section 3 discusses some general concepts related to ordered abelian groups (OAGs) and what it means for an OAG $(M; <, +)$ in the "pure" language of ordered groups to be dp-minimal. In Section 4, I review some of the previously-studied generalizations of o-minimality (weak o-minimality, local o-minimality, and several others) with an emphasis on generalizations of the Monotonicity and Cell Decomposition Theorems to these various contexts, while also describing how each notion relates to dp-minimality. The following two sections summarize what is known about unary definable sets and definable functions $f : M \to M$ in a dp-minimal OAG. Section 7 is devoted to illustrative examples, and Section 8 lists a few open questions.

I will mostly follow the usual notational conventions in model theory. For instance, the notation $\mathcal{M} = (M; <, +, \ldots)$ means that $\mathcal{M}$ is a model (structure) with universe M and whose language includes a binary predicate $<$, a binary function symbol $+$, and perhaps other relations and functions. All such structures considered here will be ordered abelian groups, often with extra definable structure. When I say that $X \subseteq M^n$ is "definable," I mean that it is definable by a formula with parameters from M (not $\emptyset$-definable). Notation such as $\overline{x}$ and $\overline{a}$ always refers to finite tuples. In most cases, a model-theoretic property (such as dp-minimality) defines an elementary class and transfers freely between a structure $\mathcal{M}$ and its complete theory, but not always: weak o-minimality and local o-minimality are notable exceptions.

Given an ordered structure $(M; <, \ldots)$, all topological concepts applied to subsets of M refer to the order topology on M (basic open sets are open intervals), and a subset X of M^n is said to be open (or closed, dense, *et cetera*) if it is so in the product topology.

2. Dp-rank and dp-minimality

Throughout this section, T is some complete theory. All formulas are in the language of T, and all tuples $\overline{a}$ come from some sufficiently saturated model of T (usually ω-saturated is sufficient).

Definition 2.1. (Adler, [1]) An *ict-pattern of depth* κ is a $\kappa \times \omega$ array of formulas

$$\begin{array}{cccc} \varphi_0(\overline{x};\overline{a}_{0,0}) & \varphi_0(\overline{x};\overline{a}_{0,1}) & \varphi_0(\overline{x};\overline{a}_{0,2}) & \ldots \\ \varphi_1(\overline{x};\overline{a}_{1,0}) & \varphi_1(\overline{x};\overline{a}_{1,1}) & \varphi_1(\overline{x};\overline{a}_{1,2}) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

such that for every function $\eta : \kappa \to \omega$, the partial type

$$\{\varphi_i(\overline{x};\overline{a}_{i,\eta(i)})\} \, : \, i < \kappa\} \cup \{\neg\varphi_i(\overline{x};\overline{a}_{i,j}) \, : \, i < \kappa, j \neq \eta(i)\} \tag{2}$$

is consistent.

If $p(\overline{x})$ is a partial type, an ict-pattern as above is *in* $p(\overline{x})$ just in case for every $\eta : \kappa \to \omega$, the set of formulas in (2) is consistent with $p(\overline{x})$.

Using ict-patterns, the dp-rank of a partial type can be defined as follows:

Definition 2.2. [44] A partial type $p(\overline{x})$ *has dp-rank less than* κ if there is no ict-pattern in $p(\overline{x})$ of depth κ. When the least cardinal κ such that $p(\overline{x})$ **does not** have dp-rank less then κ is a successor cardinal $\kappa = \lambda^+$, then $p(\overline{x})$ *has dp-rank* λ, abbreviated as dp-rk$(p) = \lambda$.

Note that $\text{dp-rk}(p(\overline{x})) = 0$ if and only if $p(\overline{x})$ is *algebraic* (has finitely many realizations), and that if $\overline{x}$ is an n-tuple of variables in a theory with infinite models, then there is always an ict-pattern in $p(\overline{x})$ of depth at least n (so $\text{dp-rk}(p(\overline{x})) \geq n$). As proved by Alder in [1], a theory T is NIP if and only if there is some cardinal κ bounding the depth of any ict-pattern in a type $p(\overline{x})$ in T.

One motivation for the study of dp-rank is that it satisfies some of the general properties one wants from a dimension function. The next theorem compiles some basic properties of dp-rk; these are mostly straightforward to check, except for (6), which was originally proved by Kaplan, Onshuus, and Usvyatsov in [23], or see Simon's book [39].

Theorem 2.3. *Let* $\text{dp-rk}(\overline{a}/B)$ *be the dp-rank of the complete type* $\text{tp}(\overline{a}/B)$. *Then we have:*

(1) $\text{dp-rk}(\overline{a}/B) = 0$ *if and only if* $\overline{a}$ *is in the algebraic closure of* B.
(2) If $\overline{a} \in \text{acl}(\overline{b} \cup B)$, *then* $\text{dp-rk}(\overline{a}/B) \leq \text{dp-rk}(\overline{b}/B)$.
(3) If $B \subseteq C$, *then* $\text{dp-rk}(\overline{a}/B) \geq \text{dp-rk}(\overline{a}/C)$.
(4) If $\overline{a}$ *and* B *are from a model* M *and* $f \in \text{Aut}(M)$, *then* $\text{dp-rk}(f(\overline{a})/f(B)) = \text{dp-rk}(\overline{a}/B)$.
(5) If X *and* Y *are nonempty definable sets, then* $\text{dp-rk}(X \cup Y) \geq \max(\text{dp-rk}(X), \text{dp-rk}(Y))$ *and* $\text{dp-rk}(X \times Y) \geq \text{dp-rk}(X) + \text{dp-rk}(Y)$.
(6) (Sub-additivity of dp-rank) $\text{dp-rk}(\overline{a}\overline{b}/C) \leq \text{dp-rk}(\overline{a}/C \cup \overline{b}) + \text{dp-rk}(\overline{b}/C)$.

In contrast with nonforking relation in simple theories, I am not aware of any attempt to axiomatize the properties of dp-rank in NIP theories.

Definition 2.4. [30] The theory T is *dp-minimal* if the partial type $x = x$ (in a single variable x) has dp-rank 1. A structure $\mathcal{M}$ is said to be dp-minimal if its complete theory is.

As an immediate consequence of Theorem 2.3 (6), note that if $\mathcal{M}$ is a model of a dp-minimal theory T and $X \subseteq M^n$ is definable, then $\text{dp-rk}(X) \leq n$.

The following yields many natural examples of dp-minimal theories, and proofs can be found in [14], [38], and [39].

Fact 2.5. *The theory* T *is dp-minimal if it satisfies any one of the following conditions:*

(1) T *is weakly o-minimal.*
(2) T *is any complete theory of a "colored linear ordering," that is, an expansion of a linear ordering by unary predicates.*
(3) T *is any complete theory of a* tree, *that is, a partial ordering* $(M; \leq)$ *such that for any* $a \in M$, *the set* $\{x \in M : x \leq a\}$ *is linearly ordered by* $\leq$.
(4) T *is strongly minimal.*
(5) T *is the complete theory of the ordered abelian group* $(\mathbb{Z}; <, +)$.
(6) T *is the complete theory of any finite extension of the p-adic field* $\mathbb{Q}_p$ *in the language of rings.*
(7) T *is the complete theory of an algebraically closed, non-trivially valued field in the language of rings plus a binary predicate denoting when* $v(x) \leq v(y)$ *(where* v *is the valuation function).*

A complete algebraic characterization of which fields (in the pure language of rings) are dp-minimal has been obtained by Johnson [21]. Dp-minimal ordered abelian groups in the language $\{<, +\}$ have also been classified by Jahnke, Simon, and Walsberg, as recalled below in the following section.

Although I have loosely talked about dp-rank as a sort of dimension function, it is would be more precise to say that it is an analogue of the notion of *weight* in stable theories as defined by Shelah in [36]. The exact relation was established by Onshuus and Usvyatsov in [30]: the dp-rank of a partial type $p(\overline{x})$ in a stable theory is the supremum of the preweights of **all** extensions of $p(\overline{x})$, including even forking extensions.

3. Ordered abelian groups (OAGs) and dp-minimality

By an *ordered abelian group*, I mean a structure $\mathcal{M} = (M; <, +, \ldots)$ such that $<$ is a total (linear) ordering on M, $+$ is the operation of an abelian group structure on M, and for any x, y, and $z \in M$, $x < y$ if and only if $x + z < y + z$ (that is, the order is translation-invariant). For the rest of this article I will sometimes use OAG as an abbreviation for "Ordered Abelian Group." Note that in general the language of an OAG may contain predicates, functions, and constants other than $<$ and $+$.

For OAGs with no definable structure other than that which is definable in the language of ordered groups, dp-minimality has a simple characterization:

Theorem 3.1. *(Jahnke, Simon, and Walsberg* [20]*) If $\mathcal{M} = (M; <, +)$ is an ordered abelian group in the language $\{<, +\}$, then $\mathcal{M}$ is dp-minimal if and only if for every prime p, the group M/pM is finite.*

Note the condition that M/pM is finite makes no reference to the ordering $<$ on the group. Indeed an infinite abelian group can usually be ordered in more than one way, but surprisingly this choice of ordering does not affect whether the structure is dp-minimal or not.

It turns out that dp-minimal groups $(M; \cdot, \ldots)$ endowed with a definable linear ordering $<$ which is left-invariant (or right-invariant) are always abelian. Simon first proved abelianity for dp-minimal groups with a definable ordering which is bi-invariant in [38], and later this was generalized to the case when the ordering is only assumed to be left-invariant by Dobrowolski and myself [12]. Separately, Verbovskiy showed in [47] that any o-stable group (as defined in the next section) with a definable bi-invariant order is abelian.

I will now fix some basic terminology for OAGs which will be useful for the rest of the paper. A subset C of any ordered structure $\mathcal{M} = (M; <, \ldots)$ is *convex* if whenever $a, c \in C$, $b \in M$, and $a < b < c$, then $b \in C$. Convex subgroups of an OAG will be especially important. A *cut* in a linearly ordered structure $(M; <, \ldots)$ is a partition $\langle C, D \rangle$ of the universe M such that every element of C precedes any element of D.

Definition 3.2. Let $\mathcal{M} = (M; <, +, \ldots)$ be an OAG.

1. $\mathcal{M}$ is *Archimedean* if it has no convex subgroups other than the trivial subgroups $\{0\}$ and $\mathcal{M}$ itself.
2. $\mathcal{M}$ is *non-valuational* if it has no nontrivial definable convex subgroups.
3. $\mathcal{M}$ has *boundedly many definable convex subgroups* if there is a fixed cardinal κ such that in every elementary extension of $\mathcal{M}$, there are no more than κ definable convex subgroups.

Note that the property of being non-valuational is preserved under passing to elementary extensions, unlike the property of being Archimedean. A common alternative definition of non-valuational, equivalent to condition (2) above, is that for any cut $\langle C, D \rangle$ which is definable in $\mathcal{M}$ and for every positive $\varepsilon \in M$, there are $c \in C$ and $d \in D$ such that $d - c < \varepsilon$.

4. Generalizations of o-minimality

In this section, I will consider various minimality notions for ordered structures which generalize o-minimality: weak o-minimality, local o-minimality, quasi o-minimality, o-minimalism, o-stability, and viscerality. For each notion, I will say what is known about its connections with Monotonicity and Cell Decomposition and explain its relationship with dp-minimality.

4.1. **Weak o-minimality.** The first generalization of o-minimality I will consider is weak o-minimality.

Definition 4.1. A linearly ordered structure $\mathcal{M} = (M; <, \ldots)$ is *weakly o-minimal* if every definable subset X of M is a finite union of convex sets. (Recall that $C \subseteq M$ is *convex* if for any two elements $x, z \in C$, if $y \in M$ and $x < y < z$, then $y \in C$.)

A complete theory T of linearly ordered structures is called weakly o-minimal in case **all** of its models are.

The concept of weak o-minimality was first defined by Dickmann [11], and soon after Macpherson, Marker, and Steinhorn [27] established fundamental results such as a generalization of the Monotonicity Theorem. Note that in contrast to o-minimality, there are weakly o-minimal structures which have elementary extensions which are not weakly o-minimal – see [27] for an example of this phenomenon.

There are examples of weakly o-minimal ordered abelian groups $\mathcal{M} = (M; <, +, \ldots)$ in which there is a nontrivial convex definable subgroup C of $(M; +)$. Note that such a subgroup C is bounded above but has no maximum element in M, so such an $\mathcal{M}$ is not o-minimal. For details on this and other examples, see [27].

The Monotonicity Theorem for o-minimal structures (Theorem 1.1) has a generalization to weakly o-minimal structures. For the statement, recall that a function $f : C \to M$ defined on an open convex set C is *locally increasing* if for every $x \in C$ there is an open interval I such that $x \in I \subseteq C$ and $f \restriction I$ is strictly increasing. The notions of *locally decreasing* and *locally constant* are similarly defined.

Theorem 4.2. [27] *Let $\mathcal{M} = (M; <, \ldots)$ be a weakly o-minimal, densely ordered structure such that either (i) the complete theory of $\mathcal{M}$ is weakly o-minimal, or (ii) $\mathcal{M}$ has the structure of an ordered* *abelian group. Then for any definable function $f : M \to M$, there is a finite partition of M into definable convex subsets such that for every infinite set C in the partition, (a) C is open, (b) $f \restriction C$ is continuous, and (c) $f \restriction C$ is either locally increasing, locally decreasing, or locally constant.*

Very soon after the theorem above was proved, Arefiev showed in [3] that neither hypothesis (i) nor hypothesis (ii) is actually necessary and that the conclusion holds in any densely ordered weakly o-minimal structure.

As for cell decomposition, Wencel has proved in [50] that a slight variation of the conclusion of Theorem 1.1 (2) holds in non-valuational weakly o-minimal OAGs. To my knowledge, no satisfactory generalization of the Cell Decomposition Theorem has been established for weakly o-minimal expansions of ordered abelian groups in general.

The connection of weak o-minimality with dp-minimality is simple:

Fact 4.3. [14] *Any weakly o-minimal theory is dp-minimal.*

4.2. **Local o-minimality.** Still more general than weak o-minimality is the concept of local o-minimality, originally defined by Toffalori and Vozoris [42].

Definition 4.4. Let $\mathcal{M} = (M; <, \dots)$ be any densely ordered structure.

(1) $\mathcal{M}$ is *locally o-minimal* if for every definable subset X of M and every point $a \in M$, there are points $b < a$ and $c > a$ in M such that
   (a) (b, a) is either a subset of X or disjoint from X; and
   (b) (a, c) is either a subset of X or disjoint from X.

(2) $\mathcal{M} = (M; <, \dots)$ is *strongly locally o-minimal* if for every point $a \in M$, there are points $b < a$ and $c > a$ in M such that for *every* definable subset X of M,
   (a) $(b, a) \cap X$ is a finite union of points and intervals, and
   (b) $(a, c) \cap X$ is a finite union of points and intervals.

In the literature, it is more common to define local o-minimality as "for every definable $X \subseteq M$ and every $a \in M$, there is an open interval I containing a such that $I \cap X$ is a finite union of points and intervals," making the analogy with classic o-minimality more transparent. But the condition (1) above is easily seen to be equivalent to this and has the advantage of making it clear that any structure elementarily equivalent to a locally o-minimal structure is also locally o-minimal, as observed by Toffalori and Vozoris [42]. They also observe that there are locally o-minimal structures which are not strongly locally o-minimal, and:

Fact 4.5. [42] *If $\mathcal{M}$ is a weakly o-minimal structure, then $\mathcal{M}$ is locally o-minimal.*

Many interesting concrete examples of locally o-minimal structures have been constructed by Toffalori and Vozoris [42], and later by Kawakami, Takeuchi, Tanaka, and Tsuboi [24]. The latter introduced the useful notion of a *simple product* as a general method for building locally o-minimal structures.

Fact 4.6. *(*[24] *and* [42]*) Each of the following structures is locally o-minimal:*

(1) $(\mathbb{R}; <, +, \sin)$*, where* $\sin$ *is a symbol for the sine function;*
(2) $(\mathbb{R}^*; <, +, \mathbb{R})$*, where $\mathbb{R}^*$ is any nonstandard elementary extension of the real field $\mathbb{R}$, and there is a unary predicate for the "standard reals" in $\mathbb{R}$;*
(3) $(\mathbb{R}; <, +, P)$*, for any $P \subseteq \mathbb{Z}$.*

Note that none of the examples above is weakly o-minimal, and that the examples in (3) may not be even NIP (much less dp-minimal).

Generalizations of the Monotonicity Theorem to locally o-minimal structures have been considered. In [24], Kawakami, Takeuchi, Tanaka, and Tsuboi proved a localized version of Monotonicity (and Cell Decomposition) for strongly locally o-minimal structures. In the more recent work [17], Fujita developed a dimension theory for definable sets in certain locally o-minimal structures, especially definably complete structures in which the image of a definable discrete set under a coordinate projection map is discrete. Under these hypotheses, definable functions $f : M \to M$ are shown to have what he calls the *strong local monotonicity property*, which is similar to the local monotonicity for weakly o-minimal structures (Theorem 4.2 above) except with the addition of a closed discrete set on which f may be discontinuous.

4.3. **Quasi o-minimality.** The next generalization of o-minimality, (weak) quasi-o-minimality, was introduced by Belegradek, Stolboushkin, and Taitslin in [8] and studied further by Belegradek, Peterzil, and Wagner [7] and Kudaibergenov [26].

Definition 4.7. (As in [7]) An ordered structure $\mathcal{M} = (M; <, \ldots)$ is *quasi-o-minimal* if for every $\mathcal{N} \equiv \mathcal{M}$, every N-definable subset of N is a finite Boolean combination of intervals and $\emptyset$-definable sets.

An ordered structure $\mathcal{M} = (M; <, \ldots)$ is *weakly quasi-o-minimal* if for every $\mathcal{N} \equiv \mathcal{M}$, every N-definable subset of N is a finite Boolean combination of convex sets and $\emptyset$-definable sets.

Some simple examples of quasi-o-minimal structures from [7] which are not locally o-minimal include $(\mathbb{R}; <, \mathbb{Q})$ (the ordered set of real numbers with a unary predicate for the rationals) and $(\mathbb{Z}; <, +)$ (the integers as an ordered group, i.e. Presburger arithmetic). Note that unlike with weak o-minimality, quasi-o-minimal structures include some discretely ordered groups.

A striking generalization of the Monotonicity Theorem has been recently proved for weakly quasi-o-minimal structures by Moconja and Tanović [29]. To explain their result, first consider the following construction: say E is an $\emptyset$-definable equivalence relation on a densely ordered structure $(M; <, \ldots)$ such that every E-class is convex. Then we can define a new linear ordering $<_E$ by reversing the order $<$ within each E-class but preserving the order between different E-classes; that is,

$$x <_E y \Leftrightarrow [E(x, y) \text{ and } y < x] \text{ or } [\neg E(x, y) \text{ and } x < y].$$

Given a finite sequence of refining equivalence relations $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_n$ with convex classes, the process above can be iterated to obtain $(\ldots((<_{E_1})_{E_2})\ldots)_{E_n}$.

Theorem 4.8. [29] *If $f : D \to M$ is an $\emptyset$-definable function in a weakly quasi-o-minimal structure $(M; <, \ldots)$, then there is an $\emptyset$-definable partition $D = D_1 \cup \ldots \cup D_k$ and $\emptyset$-definable linear orderings $<_1, \ldots, <_k$ constructed by the iterative process described above such that $f : D_i \to M$ is increasing with respect to $<_i$ for each $i \in \{1, \ldots, k\}$.*

As the authors point out, the above theorem yields new information in the special case of weakly o-minimal structures.

The relation between weak quasi o-minimality and dp-minimality is as follows:

Proposition 4.9. *If $\mathcal{M} = (M; <, \ldots)$ is a weakly quasi-o-minimal structure, then it is dp-minimal.*

Proof. Recall a criterion for dp-minimality from [14]: a theory is dp-minimal just in case for any indiscernible sequence $\{\overline{a}_i : i \in I\}$ and any element c from a model M of T, the index set I can be partitioned into finitely many convex pieces $I_1, \ldots, I_k$, at most two of which are infinite, such that for any $\ell \in \{1, \ldots, k\}$ and any $i, j \in I_\ell$, $\text{tp}(\overline{a}_i/c) = \text{tp}(\overline{a}_j/c)$. Let Σ be the set of all formulas $\varphi(x; \overline{y})$ in the language of T such that $|\overline{y}| = |\overline{a}_i|$ and for every tuple $\overline{a}$ from a model of T, the formula $\varphi(x; \overline{a})$ defines a convex set. We may further assume that every $\varphi(x; \overline{y}) \in \Sigma$ is equivalent to a basic relation $R_{\varphi(x;\overline{y})}$ in the language of T (expanding the language of T as necessary). Now let T_0 be the reduct of T to the language L_0 containing just $<$ and the relations $R_{\varphi(x;\overline{y})}$ for each $\varphi(x; \overline{y}) \in \Sigma$, and note that T_0 is weakly o-minimal. So T_0 is dp-minimal by Fact 4.3, and by the criterion for dp-minimality mentioned above, there is a convex partition $I = I_1 \cup \ldots I_k$ such that at most two I_ℓ are infinite and if $i, j \in I_\ell$ then $\text{tp}_{L_0}(\overline{a}_i/c) = \text{tp}_{L_0}(\overline{a}_j/c)$. By weak quasi-o-minimality, $\text{tp}(c/\overline{a}_i)$ (in T) is determined by $\text{tp}(c)$ and $\text{tp}_{L_0}(c/\overline{a}_i)$. Therefore if $i, j \in I_\ell$, then

$$\text{tp}_{L_0}(\overline{a}_i/c) = \text{tp}_{L_0}(\overline{a}_j/c)$$

$$\Rightarrow \text{tp}_{L_0}(c/\overline{a}_i) \equiv \text{tp}_{L_0}(c/\overline{a}_j)^1$$

$$\Rightarrow \text{tp}(c/\overline{a}_i) \equiv \text{tp}(c/\overline{a}_j), \text{ by weak quasi o-minimality}$$

$$\Rightarrow \text{tp}(c, \overline{a}_i) = \text{tp}(c, \overline{a}_j), \text{ by indiscernibility}$$

$$\Rightarrow \text{tp}(\overline{a}_i/c) = \text{tp}(\overline{a}_j/c).$$

[1]That is, these types are in the same orbit under the action of the automorphism group of a sufficiently-saturated model of T.

Applying our criterion for dp-minimality again, it follows that T is dp-minimal.

□

4.4. **O-minimalism.** Schoutens in [35] attempted to give an axiomatic characterization of all the features of o-minimal structures. In particular, given a fixed language $L = \{<, \ldots\}$ with a distinguished binary predicate $<$ for a linear ordering, he defined T^{omin} to the be intersection of all theories of o-minimal structures in the language L in which $<$ is a dense linear ordering without endpoints, and he called the models of T^{omin} *o-minimalistic* structures. In this work, Schoutens isolated the axiom schemes of Definable Completeness ("every definable subset of the universe has an infimum") and Type Completeness ("rational cuts determine 1-types" – compare with o-stability below) which hold in any o-minimalistic theory, and from these two schemes, together denoted DCTC, derived many interesting consequences. In particular, he established a version of Cell Decomposition into "quasi-cells" which are like cells but may be only definable locally, and the following version of Monotonicity:

Theorem 4.10. [35] *If $f : M \to M$ is definable in an o-minimalistic structure, then there is a discrete, closed, and bounded definable set $D \subseteq M$ such that for any interval I between two consecutive points in D (or between $\pm\infty$ and an endpoint of D), the restriction $f \upharpoonright I$ is continuous and strictly monotone (or constant).*

Also note:

Fact 4.11. [35] *Any o-minimalistic structure is locally o-minimal.*

Rennet has shown [34] that in general L^{omin} is not recursively axiomatizable. He also showed that the implication in Fact 4.11 is not reversible, even for definably complete expansions of real closed fields.

4.5. **O-stability.** While any structure in which an infinite linear ordering is definable is unstable, Baizhanov and Verbovskiy [5] have observed that many concepts from Shelahian stability theory can be usefully generalized to ordered structures by working locally within each cut.

Definition 4.12. (Baizhanov and Verbovskiy [5])

(1) If λ is an infinite cardinal and $\mathcal{M} = (M; <, \ldots)$ is an ordered structure, $\mathcal{M}$ is *o-λ-stable* if for every $A \subseteq M$ of cardinality at most λ and every cut $\langle C, D \rangle$ in $\mathcal{M}$, there are at most λ 1-types over A which are consistent with $\langle C, D \rangle$.
(2) A theory T of linearly ordered structures is called *o-λ-stable* if every model of T is λ-o-stable.
(3) A theory T of linearly ordered structures is *o-stable* if there is some infinite cardinal λ such that T is o-λ-stable.

Many other stability theoretic notions can be naturally generalized to this context, so we have, for instance, o-superstability, the strict order property within a cut, Morley o-rank (see [48]), *et cetera.*

Verbovskiy [47] has generalized the Monotonicity Theorem to non-valuational o-stable ordered abelian groups (which are automatically abelian): any definable unary function $f : M \to M$ is piecewise monotone and continuous, where each of the finitely many pieces is definable but not necessarily convex.[2] Note that this is the same conclusion that I obtained in [18] for dp-minimal denseley-ordered abelian groups (some of which are valuational), as explained below, although under a different set of hypotheses. Also note that Verbovskiy in [47] proved some tame topological properties for unary definable sets in non-valuational o-stable groups, such as that any infinite definable subset of M is dense in some interval, a property which also holds in any dp-minimal densely ordered abelian group (see below).

Verbovskiy showed [46] that every dp-minimal ordered structure is o-stable. There are examples of o-stable theories which are not dp-minimal: see Example 7.3 below.

4.6. **Viscerally ordered structures.** In joint work with Dolich [13], we introduced the following notion.

Definition 4.13. An densely ordered structure $(M; <, \ldots)$ is *viscerally ordered* if every infinite definable subset X of R has interior in the order topology.

Viscerally ordered structures evidently generalize weakly o-minimal structures, and for divisible ordered abelian groups, viscerality is implied by dp-minimality (see Theorem 5.3 below). There are many examples of divisible ordered abelian groups which are locally o-minimal but not viscerally ordered because they

[2]Another proof of this fact was recently given by Verbovskiy and Dauletiyarova in [49].

define infinite discrete sets. On the other hand, there are also viscerally ordered divisible OAGs which are not locally o-minimal: see Example 7.1 below.

We can show that at least the continuity clause of the Monotonicity Theorem holds for these structures:

Theorem 4.14. [13] *If $f : M \to M$ is a definable function in a viscerally ordered structure $(M; <, \ldots)$, then the set of discontinuities of f is finite.*

In [13] we also proved a form of Cell Decomposition for viscerally ordered abelian groups, in which a *cell* is taken to be any definable $C \subseteq M^n$ for which there is some coordinate projection $\pi : M^n \to M^m$ which maps C homeomorphically onto its image (in which case C is thought of as a *m-cell*). Note that in particular any open definable subset of M is a 1-cell. The natural topological dimension function in viscerally ordered structures turns out to have many nice properties, some of which are similar to those studied by Fujita in locally o-minimal structures [17].

In fact our results in [13] were proved in the more general context of *visceral definable uniform structures* in which there is a uniform topology (more or less, an M-valued psuedometric) with a definable base. Any P-minimal field is an example of a dp-minimal visceral definable uniform structure without a definable ordering. It happens that around the same time as we were working on Theorem 4.14 and Cell Decomposition for visceral theories, Simon and Walsberg independently obtained similar results on tame topology and dimension theory for dp-minimal visceral structures in [41] (note that what they called "**(Inf)**" there is equivalent to viscerality). Our results are more general since we did not assume dp-minimality.

4.7. **Summary.** Some of the relations between the concepts in this section can be summed up in the diagram of implications below. Note that these implications apply on the level of individual structures, and none of them is reversible.

o-minimal	$\Rightarrow$	weakly o-minimal	$\Rightarrow$	weakly quasi-o-minimal	$\Rightarrow$	dp-minimal	$\Rightarrow$	o-stable
$\Downarrow$		$\Downarrow$						
o-minimalistic	$\Rightarrow$	locally o-minimal						

There are simple examples of non-valuational dp-minimal OAGs which are neither locally o-minimal nor weakly quasi-o-minimal: let p be any prime, and let

$$\mathbb{Q}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \,:\, a, b \in \mathbb{Z} \text{ and } p \text{ does not divide } b\}$$

considered as an ordered subgroup of $\mathbb{Q}$. Then for any prime q, $|\,\mathbb{Q}_{(p)} \,/q\, \mathbb{Q}_{(p)}\,|$ is either 1 (if $q \neq p$) or p (if $q = p$), and in any case it is finite, so $(\mathbb{Q}_{(p)}; <, +)$ is dp-minimal by Theorem 3.1. The definable subgroup $p\mathbb{Q}_{(p)}$ is dense and codense in the universe, so it is not locally o-minimal, and nontrivial cosets of $p\mathbb{Q}_{(p)}$ are not equivalent to Boolean combinations of convex sets and $\emptyset$-definable sets.

5. Unary definable sets in dp-minimal ordered abelian groups

In this section I will explain some of the tameness properties (mostly of a topological nature) that can be deduced for definable subsets of M in a dp-minimal OAG $\mathcal{M}$. It is challenging to find interesting properties which hold generally for definable subsets of any dp-minimal OAG, so the results below focus on specific subclasses, such as when the group is densely ordered, or divisible, or has finitely many definable convex subgroups.

One of the first results obtained along these lines was the following.

Theorem 5.1. *(*[18]*, Lemma 3.3) Let $(M; <, +, \ldots)$ be a dp-minimal*[3] *densely ordered abelian group.*

1. *If $X \subseteq M$ is definable and X is nowhere dense, then X is finite.*
2. *More generally, if $X \subseteq \overline{M}$ is a definable subset of a* definable sort *in the Dedekind completion $\overline{M}$ of M (as defined in* [27]*), X is nowhere dense, and every cut in X is non-valuational, then X is finite.*

[3]In fact, this was stated and proved in [18] with the more general hypothesis that the structure is *inp-minimal*,. A few of the other results from the literature cited here also hold with "dp-minimal" replaced by "inp-minimal." Note that within the class of NIP theories, inp-minimality is equivalent to dp-minimality.

Theorem 5.1 rules out many topological pathologies in definable sets in dp-minimal OAGs, such as the Cantor middle-third set in $\mathbb{R}$, and even precludes the definability of unary sets with infinitely many isolated points. It has been the starting point for much of the further analysis given below and led to the development of the notion of viscerality (see Subsection 4.6 above).

Before discussing further properties of definable unary sets, I will prove a simple corollary to Theorem 5.1 which connects dp-minimality with weak and local o-minimality. Recall that not every densely ordered locally o-minimal OAG is weakly o-minimal, however:

Proposition 5.2. *Suppose that $\mathcal{M} = (M; <, +, \ldots)$ is a densely ordered OAG which is locally o-minimal, dp-minimal, and non-valuational. Then $\mathcal{M}$ is weakly o-minimal.*

Proof. Suppose that $X \subseteq M$ is definable and suppose that C is an infinite convex component of X. Then the cut in $\overline{M}$ just below C, and the cut in $\overline{M}$ just above C, are both definable and hence non-valuational. The collection of all such cuts for all possible infinite convex components C of X is definable in $\overline{M}$, so by Theorem 5.1 (2), it is finite. On the other hand, the set Z of all elements of X which do not belong to any infinite convex component of X is also definable, and so if Z were infinite, then by dp-minimality it would be dense on some open interval I. But then both X and $M \setminus X$ would be dense on such an interval I, contradicting local o-minimality. Putting this together, we have shown that X consists of only finitely many convex components, as desired. □

Simon further developed the study of definable sets in dp-minimal OAGs further in [38]. First, he noted that stronger conclusions hold when we assume that the OAG is not just densely ordered, but *divisible* (that is, for every element $g \in M$ and every positive integer n, there is $h \in M$ such that $nh = g$).

Theorem 5.3. *(*[38]*, Theorem 3.6) If $(M; <, +, \ldots)$ is a dp-minimal divisible ordered abelian group, $X \subseteq M$ is definable, and X is infinite, then X has nonempty interior.*

A notable consequence of the Theorem above is:

Theorem 5.4. *(*[38]*, Corollary 3.7) If $\mathcal{M}$ is a dp-minimal divisible ordered abelian group and $\mathcal{M}$ is definably complete, then it is o-minimal.*

In particular, an expansion of the ordered group $(\mathbb{R}; <, +)$ is dp-minimal if and only if it is o-minimal.

The challenge is now to say something interesting about unary definable sets in dp-minimal densely ordered OAGs without the hypothesis of definable completeness or divisibility. Recall that weakly o-minimal OAGs are dp-minimal, but how far can dp-minimal OAGs be from being weakly o-minimal? A partial answer is given in recent work of Simon and Walsberg [40]:

Theorem 5.5. *(*[40]*, Theorem 5.1) Suppose that $\mathcal{M} = (M; <, +, \ldots)$ is a dp-minimal ordered abelian group with finitely many definable convex subgroups. Then $\mathcal{M}$ is dp-minimal if and only if in every elementary extension $\mathcal{N}$ of $\mathcal{M}$, every definable subset X of N is a finite union of sets of the form $C \cap (nN + g)$ where C is convex, $n \in \mathbb{N}$, and $g \in N$.*

Note that in Theorem 5.5, there is no assumption that $\mathcal{M}$ is densely ordered, so it applies equally to discretely ordered groups. A direct corollary of Theorem 5.5 is that if $\mathcal{M}$ is a dp-minimal divisible ordered abelian group with finitely many definable convex subgroups, then $\mathcal{M}$ is weakly o-minimal.

The conclusion of Theorem 5.5 does not hold in arbitrary dp-minimal OAGs: see the discussion in Example 7.1 below. However, it may hold in any dp-minimal ordered abelian group with boundedly many definable convex subgroups.

In forthcoming work by Verbovskiy and myself [19], we will show the following:

Theorem 5.6. [19] *If $\mathcal{M}$ is a dp-minimal ordered abelian group with boundedly many definable convex subgroups and $X \subseteq M$ is definable and unbounded in M, then there is some $n \in \mathbb{N}$ such that X is eventually equal to a finite union of cosets of nM. (That is, there is some $a \in M$ such that for every $b \in M$, if $b > a$, then $b \in X$ if and only if b belongs to one of these cosets of nM.)*

For general dp-minimal densely ordered OAGs, Simon and Walsberg have proved the following:

Theorem 5.7. [40] *If $\mathcal{M} = (M; <, +, \ldots)$ is a dp-minimal densely ordered group and $X \subseteq M$ is definable in $\mathcal{M}$, then X is a finite union of sets of the form $U \cap (nM + g)$ where U is open and $\mathcal{M}$-definable, $n \in \mathbb{N}$, and $g \in M$.*

The above-cited paper includes much more information on particular dp-minimal structures such as a criterion for dp-minimality of Archimedean cyclically ordered abelian groups and a description of dp-minimal expansions of divisible Archimedean groups.

6. Unary functions in dp-minimal ordered abelian groups

In attempting to generalize the local monotonicity theorem for definable functions in weakly o-minimal groups (Theorem 4.2 above) to dp-minimal structures, I obtained the following:

Theorem 6.1. [18] *If $\mathcal{M} = (M, <, +, \ldots)$ is a densely ordered dp-minimal OAG and $f : M \to M$ is definable in $\mathcal{M}$, then there is a finite definable partition $X_1 \cup \ldots \cup X_k$ of M such that the restriction of f to each set X_i is continuous.*

In my paper [18], I claimed to have proved an even stronger result: that the X_i in the Theorem above can be chosen so that furthermore each $f \restriction X_i$ is locally increasing, locally decreasing, or locally constant (as in the weakly o-minimal case). However, as was first pointed out to me by Pierre Simon, the argument contains a flaw, specifically in the proof of Lemma 3.30 of [18]. (Given an interval (a, ∞) partitioned into definable sets Y_0, Y_1, and Y_2, I wanted to apply Theorem 5.1 above to conclude that one of the Y_i is dense in an interval of the form (a, a_1), but this Theorem only allows one to say that some Y_i is dense in some subinterval of (a, ∞).) Fortunately this mistake does not affect my published proof of Theorem 6.1, which was established in [18] by a series of lemmas preceding the error in question. Whether or not the stronger conclusion I had hoped for holds in any densely ordered dp-minimal OAG remains open, although now I suspect that there are counterexamples (see Example 7.4 below).

In a forthcoming article with Verbovskiy, we amend the gap in the proof mentioned above by adding an extra hypothesis, obtaining the following result:

Theorem 6.2. [19] *Let $\mathcal{M} = (M; <, +, \ldots)$ be a densely ordered dp-minimal OAG with boundedly many definable convex components. If $f : M \to M$ is definable in $\mathcal{M}$, then there is a finite definable partition $X_1 \cup \ldots \cup X_k$ of M such that for each $i \in \{1, \ldots, k\}$, (a) $f \restriction X_i$ is continuous, and (b) $f \restriction X_i$ is either locally increasing, locally decreasing, or locally constant.*

7. Some examples

Example 7.1. There are dp-minimal divisible ordered abelian groups with unboundedly many definable convex subgroups. I will present a concrete example of such a structure following the explanation of Simon and Walsberg in [40]. Let $M = \mathbb{R}((t))$, the set of all formal Laurent series over $\mathbb{R}$ in the variable t. Then M is a field, and can be endowed with an ordering $<$ in which t is a positive element between 0 and any element of $\mathbb{R}$ (so that $1 > t > t^2 > \ldots$). The ordered field $(M; <, +, \cdot)$ is dp-minimal by a result of Johnson from [22]. If v is the usual valuation map on the field M (under which $v(t) = 1$ and every element of $\mathbb{R}$ has valuation 0), then the binary relation $\leq_v$ defined by $x \leq_v y \Leftrightarrow v(x) \leq v(y)$ is definable in the field $(M; +, \cdot)$. If we view $\mathcal{M} = (M; <, +, \cdot)$ as an expansion of the additive group, it is divisible and dp-minimal, and in elementary extensions $\mathcal{N}$ of $\mathcal{M}$, there are unboundedly many definable convex subgroups of the form

$$H_x(N) = \{y \in N : x \leq_v y\}.$$

If $P = \{x \in M : \exists y \in M \left[x = y^2\right]\}$, then P is definable in $\mathcal{M}$ and is equal to all the elements whose valuation is even. The set P is open and has infinitely many convex components, showing that $\mathcal{M}$ is not weakly o-minimal.

Finally, note that the example above is viscerally ordered (by dp-minimality and the fact of being divisible, and applying Theorem 5.3 above), but it is not locally o-minimal: every open interval around the point 0 contains infinitely many convex components of the definable set P described above.

Example 7.2. There are dp-minimal divisible ordered abelian groups with infinitely many definable convex subgroups, but not unboundedly many. Such structures can easily be constructed using the observation that if $(M; <, \ldots)$ is any dp-minimal ordered structure and $\mathcal{M}'$ is an expansion of $\mathcal{M}$ by any family of unary convex predicates, then $\mathcal{M}'$ is also dp-minimal. (This is a corollary of a well-known result by Shelah from [37] stating that the so-called "Shelah expansion" of a structure by all externally definable sets preserves dp-rank, and observing that any convex subset of M is externally definable as the intersection of M with an interval in an elementary extension.) Now if $(M; <, +)$ is an ω-saturated elementary extension of $(\mathbb{R}; <, +)$, then there is an infinite chain of nontrivial convex subgroups $(H_i : i \in \omega)$ of M, so if we let $\mathcal{M}$ be the structure $(M; <, +, H_i : i \in \omega)$, then $\mathcal{M}$ is dp-minimal. Using quantifier elimination, it can be checked that the H_i are the only nontrivial convex subgroups definable in any elementary extension of $\mathcal{M}$, and hence it has boundedly many definable convex subgroups.

Example 7.3. There are examples of o-stable divisible ordered abelian groups which are not dp-minimal. To give concrete examples, I will use a general construction given by Verbovskiy in [47], which I now briefly recall for convenience. Let K be a subgroup of the additive group of real numbers such that $K \cap \mathbb{Q} = \{0\}$ and let $\mathcal{K} = (K; +, -, \ldots)$ be any definable structure on K in a language with quantifier elimination and in which each additional symbol (other than the binary operation $+$ and the unary operation $-$ for negation) is relational. Let $G = \mathbb{Q} + K$, and let $\mathcal{G} = (G; <, +, -, Q, \ldots)$ as a structure in the language of OAGs expanded by a unary predicate Q naming the subgroup $\mathbb{Q}$ and an n-ary relational predicate $\widehat{R}$ for each basic n-ary relation R in the language of $\mathcal{K}$, interpreted as

$$\mathcal{G} \models \widehat{R}(q_1 + k_1, \ldots, q_n + k_n) \Leftrightarrow \mathcal{K} \models R(k_1, \ldots, k_n)$$

in which $q_i + k_i$ denotes the unique decomposition of an element of G as a sum of elementos from $\mathbb{Q}$ and from K respectively. As proved in [47], the complete theory $T_{\mathcal{G}}$ of $\mathcal{G}$ has quantifier elimination, and if $\mathcal{K}$ is ω-stable, then $T_{\mathcal{G}}$ is o-ω-stable.

To construct a specific subgroup K, pick any infinite subset I of $\mathbb{R}$ which is linearly independent over $\mathbb{Q}$ and let K consist of all finite sums of elements of I with coefficients from

$$\mathbb{Q}_{(2)} = \{a/b \in \mathbb{Q} : a, b \in \mathbb{Z} \text{ and } b \text{ is odd}\}.$$

Then $K \cap \mathbb{Q} = \{0\}$, and as an abelian group, K is ω-stable. Expand this to a structure $\mathcal{K}$ on K with quantifier elimination. Now apply the results of Verbovskiy above to conclude that $T_{\mathcal{G}}$ is o-ω-stable. However, as an ordered abelian group in which $G/2G$ is infinite, it is not dp-minimal by Theorem 3.1 above.

Example 7.4. Finally, we present an example of a divisible ordered abelian group $\mathcal{M} = (M; <, +, f)$ in which $f : M \to M$ is a function which is not locally monotonic on any interval. I suspect that this structure is dp-minimal, which would show that the hypothesis of having boundedly many definable convex subgroups in Theorem 6.2 cannot be removed.

The universe M of the example is again the set $\mathbb{R}((t))$ of all formal Laurent series over $\mathbb{R}$ in the variable t considered as an ordered abelian group with $1 > t > t^2 > \ldots > 0$, and where $f : M \to M$ is the function

$$f\left(\sum_{i=k}^{\infty} a_i t^i\right) = \sum_{i=k}^{\infty} (-1)^i a_i t^{i+1}.$$

It is simple to check that f is a biyection from M onto itself and that f is everywhere continuous in the order topology.

Let $\mathcal{M} = (M; <, +, f)$. I conjecture that this structure is dp-minimal. If so, then it gives a drastic counterexample to the local monotonicity property (as for weakly o-minimal theories): there is no nonempty open interval I in M such that $f \upharpoonright I$ is strictly increasing, strictly decreasing, or constant. But note that this would not contradict Theorem 6.2 above because there are unboundedly many definable convex subgroups in elementary extensions of $\mathcal{M}$. Indeed, if we let X be the set of all $a \in M$ such that a and $f(a)$ have the same sign (positive or negative), then X is a definable open set with infinitely many convex components, and if C is a convex component of X then the least convex subgroup $G_C \leq M$ containing C is definable. This yieldis an infinite uniformly definable family of convex subgroups.

8. Future directions of research and open questions

Question 8.1. *Suppose that $\mathcal{M}$ is a densely ordered dp-minimal OAG with boundedly many definable convex subgroups. Can the local monotonicity theorem (Theorem 6.2 above) be used to develop a good version of Cell Decomposition and dimension of definable subsets of M^n?*

It should be possible to develop the notion of Cell Decomposition for dp-minimal OAGs with boundedly many definable convex subgroups even without assuming that the order is dense. Recall that Cluckers proved a version of Cell Decomposition for the complete theory of $(\mathbb{Z}; <, +)$ in [9], which may give an idea of how this could work for general OAGs.

Question 8.2. *Other than Theorem 5.7 above, what more can be said in general about the properties of unary definable sets in dp-minimal ordered abelian groups?*

Recall that Simon and Walsberg have developed a nice theory of dimension and topology for definable sets in densely ordered dp-minimal structures under the hypothesis of viscerality in [41], so it is natural to ask:

Question 8.3. *Can a theory of tame topology be developed for general dp-minimal ordered abelian groups along the same line as Fujita's work on locally o-minimal structures in* [17]*?*

Question 8.4. *Is the structure $\mathcal{M}$ in Example 7.4 dp-minimal?*

A different sort of question is to take a specific dp-minimal structure $\mathcal{M}$ and ask if we can classify all expansions of $\mathcal{M}$ which are dp-minimal. It was shown in [4] that no proper expansion of the dp-minimal group $(\mathbb{Z}; <, +)$ which is dp-minimal. On the other hand there are many interesting dp-minimal expansions of the group $(\mathbb{Z}, +)$. For example, for any prime p, there is the expansion by a binary relation $\leq_p$ given by $x \leq_p y \Leftrightarrow v_p(x) \leq v_p(y)$ where v_p is the p-adic valuation, and $(\mathbb{Z}; +, \leq_p)$ was shown to be dp-minimal by Alouf and d'Elbée in [2]. In fact this structure has many more (in the sense of cardinality) dp-minimal expansions even than these: in [43], Tran and Waslberg showed that there is a size continuum family of different cyclically-ordered expansions of $(\mathbb{Z}; +)$, each of which is dp-minimal.

Question 8.5. *Is there any reasonable way to classify (up to interdefinability) all dp-minimal expansions of the group $(\mathbb{Z}; <)$? Or all dp-minimal expansions of $(\mathbb{Q}; +)$?*

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