A note on generic structures and the finite set property

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1 Introdction

It is known that an algebraically closed field is strongly minimal and has elimination of imaginaries. Three typical examples of strongly minimal structures are infinite sets without relations, infinite vector spaces over a finite field, and algebraically closed fields. In general, any strongly minimal structure is classified as one of set-like case, group-like case and field-like case. Then an algebraically closed structure is a field-like strongly minimal one. Another known example of field-like strongly minimal structures is Hrushovski's strongly minimal structure [5]. Note that his structures belong to field-like case, but they do not interpret an infinite group.

Then a natural question arises whether Hrushovski's strongly minimal structure has elimination of imaginaries or not. In [3], Baldwin and Verbovskiy deal with the question. In this short note, we show that if M is a saturated $(\mathbf{K}, \leq)$-generic structure and $(\mathbf{K}, \leq)$ has the free amalgamation property, then M does not have the finite set property (Proposition 4.4). As a corollary, we prove that Hrushovski's pseudoplanes do not have elimination of imaginaries (Corollary 4.5).

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2 Generic structures

In this section, we review the basics of generic structures. (For details, see papers of Baldwin-Shi [2] and Wagner [9].)

In this paper, a graph means a simple hypergraph, i.e., a structure (A, R) with a universe A and a relation R satisfying that $\models \forall x_1...x_n[R(x_1...x_n) \to \bigwedge_{i\neq j} x_i \neq x_j]$, and $\models \forall \bar{x}[R(\bar{x}) \to R(\sigma\bar{x})]$ for any permutation σ.

Then A denotes a set of vertices in (A, R), and R^A a set of hyper-edges in (A, R). Let $A, B, C, \cdots$ denote hypergraphs. A predimension $\delta(A)$ of finite A is defined by

$$\delta(A) = |A| - \alpha|R^A|,$$

where $0 < \alpha \leq 1$. We write $\delta(A/B) = \delta(A \cup B) - \delta(A)$.

For finite $A \subset B$, A is said to be closed in B, denoted by $A \leq B$, if $\delta(X/A) \geq 0$ for any $X \subset B - A$. It is checked that if $A \leq B$ and $X \subset B$ then $A \cap X \leq B \cap X$. Then, for possibly infinite A, B, $A \leq B$ can be defined by $A \cap X \leq B \cap X$ for any finite $X \subset B$.

For $A \subset B$, there is the smallest C with $A \subset C \leq B$. This C is called the closure of A in B, and denoted by $\mathrm{cl}_B(A)$. A structure M has finite closures, if $\mathrm{cl}_M(A)$ is finite for any finite $A \subset M$.

Let $\mathbf{K}$ be a class of finite hypergraphs A with $\emptyset \leq A$ such that $\mathbf{K}$ is closed under substructures and isomorphism. Then $(\mathbf{K}, \leq)$ has the amalgamation property (AP), if whenever $A \leq B \in \mathbf{K}$ and $A \leq C \in \mathbf{K}$ then there is $D \in \mathbf{K}$ with $B' \leq D$ and $C' \leq D$ for some $B' \cong_A B$ and $C' \cong_A C$.

A countable hypergraph M is said to be $(\mathbf{K}, \leq)$-generic, if it satisfies that, if $A \subset_{\text{fin}} M$ then $A \in \mathbf{K}$; if $A \leq B \in \mathbf{K}$ and $A \leq M$ then there is a $B' \leq M$ with $B' \cong_A B$; M has finite closures.

It is known that if $(\mathbf{K}, \leq)$ has AP then there exists the $(\mathbf{K}, \leq)$-generic structure M. By the back-and-forth argument, it is seen that M is ultra-homogeneous, i.e., for any finite A, B, B', if $A \leq B, B' \leq M$ and $B \cong_A B'$ then $\mathrm{tp}(B/A) = \mathrm{tp}(B'/A)$. $\mathrm{Th}(M)$ is said to be ultra-homogeneous, if any model is ultra-homogeneous. $\mathrm{Th}(M)$ is said to have finite closures, if any model has finite closures. Then we can see the following:

Note 2.1 A generic structure M is saturated if and only if $\mathrm{Th}(M)$ is ultra-homogeneous and has finite closures.

Let M be a generic structure and $\mathcal{M}$ a big model of $\mathrm{Th}(M)$. For finite $A, B \subset \mathcal{M}$, let $d_{\mathcal{M}}(A) = \delta(\mathrm{cl}_{\mathcal{M}}(A))$ and let $d_{\mathcal{M}}(B/A) = d_{\mathcal{M}}(BA) - d_{\mathcal{M}}(A)$.

We sometimes abbreviate $\mathrm{cl}_{\mathcal{M}}(*)$ and $d_{\mathcal{M}}(*)$ to $\mathrm{cl}(*)$ and $d(*)$ respectively. For infinite C, let $d(B/C) = \inf\{d(B/C_0) : C_0 \subset_{\mathrm{fin}} C\}$. For infinite B, by $d(B/A) = d(B/C)$ we mean that $d(B_0/A) = d(B_0/C)$ for any finite $B_0 \subset B$. For A, B, C with $A = B \cap C$, B and C are free over A, denoted by $B \perp_A C$, if $R^{B\cup C} = R^B \cup R^C$.

Example 2.2 1. Hrushovski's strongly minimal structures [5]: Let L be a language consisting a ternary relation R, and $\alpha = 1$. By taking some suitable class $\mathbf{K}$ of finite L-structures A with $\delta(A') \geq 0$ for any $A' \subset A$, $(\mathbf{K}, \leq)$-generic structure M is strongly minimal and saturated. (The choice of $\mathbf{K}$ is complicated.) Moreover, it can be checked that M is not locally modular and has no infinite definable groups.

2. Hrushovski's pseudoplane [6]: Let L be a language consisting a binary relation R. For a fuction $f : \omega \to \mathbb{R}^{\geq 0}$, let $\mathbf{K}_f$ be a class of finite L-structures A with $\delta(A') \geq f(|A'|)$ for any $A' \subset A$. By taking suitable α and f, $(\mathbf{K}_f, \leq)$ has the free amalgamation property, i.e., whenever $A \leq B \in \mathbf{K}_f$ and $A \leq C \in \mathbf{K}_f$ then $B \oplus_A C \in \mathbf{K}_f$. Then there exists the $(\mathbf{K}, \leq)$-generic structure M_f. Moreover, by choosing unbounded f, M_f turns out to be ω-categorical (and hence saturated).

3 Finite set property

The notions and definitions appearing in this section can be found in for instance [7, 8].

Let T be a complete theory, and $\mathcal{M}$ a big model of T.

Definition 3.1 Let $\bar{s}$ be a tuple in $\mathcal{M}$, and F a subset of tuples of $\mathcal{M}$.

1. $\bar{s}$ is said to code F, if, for any $\sigma \in \mathrm{Aut}(\mathcal{M})$, σ fixes $\bar{s}$ pointwise if and only if σ fixes F setwise.

2. T is said to have the finite set property, if for any finite $F \subset \mathcal{M}$ there is a tuple $\bar{s}$ which codes F.

Note 3.2 1. It is known that T has elimination of imaginaries if and only if T has weak elminination of imaginaries and the finite set property.

2. The theory of an algebraically close field has elimination of imaginaries.

3. It is seen that the theory of a saturated generic structure has weakly elimination of imaginaries. (See for instance [4, 8]).

Algebraically closed fields are field-like strongly minimal structures. Another known example of field-like strongly minimal structures is Hrushovski's strongly minimal structure (Example 2.2.1). By Note 3.2.2,, the following question arises.

Question 3.3 (Verbovskiy [3]) Does Hrushovski's strongly minimal structure have elimination of imaginaries?

By Note 3.2.3, Hrushovski's strongly minimal structures have weakly elimination of imaginaries. So Question 3.3 is equivalent to the question of whether Hrushovski's strongly minimal structure has the finite set property.

4 Proposition

Let M be a saturated $(\mathbf{K}, \leq)$-generic structure and $T = \mathrm{Th}(M)$.

Definition 4.1 T is said to satisfy acl $=$ cl, if, for any subset $A \subset M$, $\mathrm{acl}(A) = \mathrm{cl}(A)$. M is said to satisfy acl $=$ cl, if T satisfies acl $=$ cl.

Note 4.2 For any $A \subset M$, $\mathrm{cl}(A) \subset \mathrm{acl}(A)$.

Proof. We can assume that A is finite. Since M is saturated, by Note 2.1, $\mathrm{cl}(A)$ is finite. Since $\mathrm{cl}(A)$ is uniquely determined for A, $\mathrm{cl}(A)$ is algebraic over A. Hence $\mathrm{cl}(A) \subset \mathrm{acl}(A)$.

Note 4.3 If $(\mathbf{K}, \leq)$ has the free amalgamation property, M satisfies acl $=$ cl.

Proof. By Note 4.2, it is enough to show that $\mathrm{acl}(A) \subset \mathrm{cl}(A)$. Suppose that there would be some $b \in \mathrm{acl}(A) - \mathrm{cl}(A)$. We can assume that A is finite. Let B be the closure of $\mathrm{cl}(A)$ and all conjugates of b over $\mathrm{cl}(A)$. Since b is algebraic over $\mathrm{cl}(A)$, B is finite. Take b' with $b' \cong_{\mathrm{cl}(A)} b, b'$. By the free amalgamation property, we can assume that $\mathrm{cl}(A) \leq \mathrm{cl}(A)b' \leq M$ and $b' \perp_{\mathrm{cl}(A)} B$. Then $\mathrm{tp}(b'/\mathrm{cl}(A)) = \mathrm{tp}(b/\mathrm{cl}(A))$ and $b' \notin B$. This contradicts our assumption that B contains all conjugates of b over $\mathrm{cl}(A)$. Hence $\mathrm{acl}(A) \subset \mathrm{cl}(A)$.

Proposition 4.4 Let M be a saturated $(\mathbf{K}, \leq)$-generic structure. If M satisfies acl $=$ cl, then it does not have the finite set property.

Proof. Let $T = \mathrm{Th}(M)$. Suppose that T would have the finite set property. Let $F = \{a, b\}$ be a hypergraph. By genericity of M, we can assume that $F \leq M$. By the finite set property, there is a tuple $\bar{s}$ which codes F. Then we have $s \in \mathrm{acl}(F)$. By $\mathrm{acl} = \mathrm{cl}$, we have $\bar{s} \in \mathrm{acl}(F) = \mathrm{cl}(F) = F$. By ultrahomogeneity of M, there is $\sigma \in \mathrm{Aut}(M)$ with $\sigma(ab) = ba$. Then $\sigma(\bar{s}) \neq \bar{s}$. A contradiction.

Hrushovski's pseduplane is an ω-saturated $(\mathbf{K}, \leq)$-generic structure, and $(\mathbf{K}, \leq)$ has the free amalgamation property (see Example 2.2.2). By Proposition 4.4 and Note 3.2.1, we have the following corollary.

Corollary 4.5 Hrushovski's pseudoplanes do not have elimination of imaginaries.

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