Some remarks on locally o-minimal structures

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概要

abstract Locally o-minimal structures are some local adaptation from o-minimal structures. They were treated, e.g. in [1], [2]. We try to characterize types of definably complete locally o-minimal structures. And we argue about the dp-rank of them.

1. Introduction

We recall some definitions and fundamental facts at first.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is $o-minimal$ if every definable subset of M^1 is a finite union of points and intervals.

M is *locally* $o-minimal$ if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is $definably$ *complete* if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Here we consider densely linearly ordered structures only.

Example 2. [1], [2]

$(\mathbf{R}, +, <, \mathbf{Z})$ where $\mathbf{Z}$ is the interpretation of a unary predicate, and $(\mathbf{R}, +, <, \sin)$ are definably complete locally o-minimal structures.

Fact 3. [1] *Definably complete local o-minimality is preserved under elementary equivalence.*

Thus we argue in a sufficiently large saturated model $\mathcal{M}$ as usual.

O-minimal structures are characterized by means of behavior of $1-$types. They consider two kinds of $1-$types by the way to cut linear orders of parameter sets, e.g. in [5]. Here we consider nonisolated types only.

Definition 4. Let M be a densely linearly ordered structure and $A \subset M$.

And let $p(x) \in S_1^{or}(A)$, that is, $p(x)$ is complete over A w.r.t. the order relation.

We say that $p(x)$ is *cut over* A if for any $a \in A$, if $a < x \in p(x)$, then there is $b \in A$ such that $a < b < x \in p(x)$, and similarly if $x < a \in p(x)$, then there is $c \in A$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1^{or}(A)$ is *noncut over* A if $q(x)$ is not a cut type.

And sometimes we call $q(x) \in S_1(A)$ *cut* (*noncut*) over A if $q(x)$ contains a cut (noncut) $p(x) \in S_1^{or}(A)$.

Remark 5. *Let M be a densely linearly ordered structure and $A \subset M$. And let $p(x) \in S_1^{or}(A)$ be noncut.*

There are four kinds of noncut types.

$p(x) = \{b < x < a : b < a \in A\}$ for some fixed a, or $\{a < x < b : a < b \in A\}$ for some fixed a.

Here we call these types bounded noncut types.

And $p(x) = \{b < x : b \in A\}$ or $\{x < b : b \in A\}$.

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2. Characterization of definably complete locally o-minimal structures

In o-minimal structures, types of the order relation are complete. Similar argument hold in definably complete locally o-minimal structures to some extent.

Lemma 6. *Let M be a definably complete locally o-minimal structure and $A \subset M$.*

Then any bounded noncut type $p(x) \in S_1^{or}(dcl(A))$ is complete over $dcl(A)$.

Proof ;

Let $p(x) \in S_1^{or}(dcl(A))$ be bounded noncut, that is, $p(x) = \{c < x < a : c < a \in dcl(A)\}$ for some fixed $a \in dcl(A)$ (Another case is proved similarly). By local o-minimality, for any formula $\varphi(x, \bar{b})$ over $dcl(A)$, there is an interval $I \subset M$ such that $a \in I$ and $I \cap \varphi(M, \bar{b})$ is a union of finite points and intervals. Thus there is $c \in I$ such that either for any $d \in I$ with $c < d < a$, $M \models \varphi(d, \bar{b})$, or for any $d \in I$ with $c < d < a$, $M \models \neg\varphi(d, \bar{b})$. Now we assume that for any $d \in I$ with $c < d < a$, $M \models \varphi(d, \bar{b})$. We consider the formula $\forall y(x < y < a \longrightarrow \varphi(y, \bar{b}))$. By definably completeness, there is the infimum $e \in M$ such that for any f with $e < f < a$, $M \models \varphi(f, \bar{b})$ (If $e = -\infty$, then "$x < a$" implies $\varphi(x, \bar{b})$). And $e \in dcl(A)$. ∎

Notation 7. *In the lemma above, for $p(x) = \{c < x < a : c < a \in dcl(A)\}$ for some* *fixed $a \in dcl(A)$, there is the infimum $e \in M$ such that for any f with $e < f < a$ satisfies the formula $\varphi(y, \bar{b})$. We denote "b_φ" such boundary point in the following.*

Next we characterize about the definability of types. We recall the definition.

Definition 8. Let M be a structure.

A type $p(\overline{x}) \in S_n(M)$ is *definable* if for any $L-$formula $\varphi(\overline{x}, \overline{y})$, there is an $L(M)-$formula $d\varphi(\overline{y})$ such that for all $\overline{a} \in M$, $\varphi(\overline{x}, \overline{a}) \in p(\overline{x})$ if and only if $M \models d\varphi(\overline{a})$.

We can prove the next fact.

Fact 9. *Let M be a definably complete locally o-minimal structure and let $p(x) \in S_1(M)$. Then $p(x)$ is definable if and only if $p(x)$ is noncut.*

We can generalize the fact above for $n-$types to a certain extent.

Definition 10. [6] Let $\mathcal{M}$ be a sufficiently large saturated densely linearly ordered structure and $A \subset \mathcal{M}$.

We say that $p(\overline{x}) \in S_n(A)$ is *noncut over A* if ;

1) $n = 1$, we define it by the same way as above, and

2) $n \geq 2$, let $\overline{x} = (x_1, x_2, \cdots, x_n)$, we define inductively,

$q(\overline{x}') := tp(x_1 \cdots x_{n-1}/A)$ is noncut over A and $tp(x_n/Aa_1 \cdots a_{n-1})$ is noncut over $A \cup \{a_1, \cdots, a_{n-1}\}$ for any realizations $a_1 \cdots a_{n-1}$ of $q(\overline{x}')$.

Fact 11. *Let M be a locally o-minimal structure and let $p(\overline{x}) \in S_n(M)$ be noncut. Then $p(\overline{x})$ is definable.*

Next we characterize definably complete locally o-minimal structures by the notion of forking. We recall some definitions.

Definition 12. A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with $tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$.

A formula $\phi(\bar{x}, \bar{a})$ *forks* over A if $\phi(\bar{x}, \bar{a}) \vdash \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i)$ and each $i < n$, $\psi_i(\bar{x}, \bar{b}_i)$ divides over A.

In some papers, they consider the notion of dimension for (definably complete) locally o-minimal structures, e.g. in [3].

Definition 13. Let M be a densely linearly ordered structure without endpoints and let $X \subset M^n$ be a nonempty definable subset.

The dimension $dim(X)$ of X is the maximal nonnegative integer d such that $\pi(X)$ has a nonempty interior for some coodinate projection $\pi : M^n \longrightarrow M^d$.

According to the argument in [7], we recall some lemmas.

Lemma 14. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure and $A \subset \mathcal{M}$. And let $p(x)$, $q(x) \in S_1(A)$ (with $dim(p) = dim(q) = 1$).*

Then either

(a) (i) all $A-$definable $f : p(\mathcal{M}) \longrightarrow q(\mathcal{M})$ are increasing, or

(ii) all $A-$definable $f : p(\mathcal{M}) \longrightarrow q(\mathcal{M})$ are decreasing.

(b) In case (i), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a > dcl(B) \cap p(\mathcal{M})$, then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$,

In case (ii), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a < dcl(B) \cap p(\mathcal{M})$, then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

In the lemma above, we just say that if there is a function f between $p(\mathcal{M})$ and $q(\mathcal{M})$, then f has these properties. There is no definable function between a cut type and a noncut type. By this lemma, they consider characteristic extensions of complete types in o-minimal structures. Here we adapt the argument for noncut types.

Definition 15. Let $p(x_1, \cdots, x_n) \in S_n(A)$ of dimension n and $A \subset B$.

Fix some sequence $\eta = (\eta(1), \cdots, \eta(n))$ where each $\eta(i)$ is 1 or 0.

For $1 \leq i \leq n$, let $p_i(x_1, \cdots, x_i)$ be the restriction of p to the variables $x_1, \cdots, x_i$.

We define an extension $p^\eta_B \in S_n(B)$ of p. Choose a realization $(b_1, \cdots, b_n)$ of p^η_B inductively as follows ;

$b_1 \in p_1(\mathcal{M})$ and if $\eta(1) = 1$, then $b_1 > dcl(B) \cap p_1(\mathcal{M})$,
while if $\eta(1) = 0$, then $b_1 < dcl(B) \cap p_1(\mathcal{M})$.

For some realization $b_1, \cdots, b_i$ of $p_i(x_1, \cdots, x_i)$, let b_{i+1} be a realization of $p_{i+1}(b_1, \cdots, b_i, x_{i+1})$ such that :

if $\eta(i+1) = 1$, then $b_{i+1} > dcl(B, b_1, \cdots, b_i) \cap p_{i+1}(b_1, \cdots b_i, \mathcal{M})$ and
if $\eta(i+1) = 0$, then $b_{i+1} < dcl(B, b_1, \cdots, b_i) \cap p_{i+1}(b_1, \cdots b_i, \mathcal{M})$.

Lemma 16. [7] *Let $p(\bar{x}) \in S_n(A)$ of dimension n and let $q(y) \in S_1(A)$ of dimension 1.*

Then there is $\eta \in {}^n2$ as in the definition above such that ;

for any $B \supset A$ and any realization $\bar{a}$ of $p^\eta_B(\bar{x})$, $dcl(\bar{a}A) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

By the lemmas above, we can prove the next fact.

Proposition 17. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure. And let $c \in \mathcal{M}$ and $A \subset \mathcal{M}$ with $dim(\mathrm{tp}(c/A)) = 1$, and $A \cup \{c\} \subset B$.*

Moreover let $r(x) \in S_1(B)$ be a bounded noncut type of c over $dcl(B)$ satisfying $r \restriction Ac \nvdash r$.

Then $r(x)$ divides over A.

Sketch of proof ;

Let $r(x) := \{ c < x < d \,:\, d \in dcl(B) \}$ *(Another case is proved similarly). As* $r \restriction Ac \nvdash r$*, there is an* $L(B)$*−formula* $\varphi(x, \bar{b}) \in r(x)$ *such that for any* $L(Ac)$*−formula* $\psi(x) \in r(x)$*,* $b_\varphi < b_\psi$*.*

Case 1. $c \equiv_A b_\varphi$*.*

We consider an automorphism $\sigma \in Aut_A(\mathcal{M})$ *such that* $\sigma(c) = b_\varphi$*. And let* ${b_\varphi}' = \sigma(b_\varphi)$*. Thus* $c < b_\varphi < {b_\varphi}'$*.*

Case 2. $c \not\equiv_A b_\varphi$*.*

Now there is an $L(A)$*−formula* $\psi(x)$ *such that* $\neg\psi(x) \in \mathrm{tp}(c/A)$ *and* $\psi(x) \in \mathrm{tp}(b_\varphi/A)$*. As for any* $L(A)$*−formula* $\psi(x) \in r(x)$*,* $b_\varphi < b_\psi$*, so for any* d *with* $c < d < b_\varphi$*,* $\models \psi(d)$*.*

In the lemmas above, let $p(\bar{x}) = \mathrm{tp}(cb_\varphi/A)$*,* $q(x) = r(x) \restriction A$*, and the noncut extension* $p'(\bar{x}) \in S^\eta(\bar{b}cA)$ *of* p *such that for any* $c'{b_\varphi}' \models p'(\bar{x})$*,* $dcl(c'{b_\varphi}'A) \cap q(\mathcal{M}) > dcl(\bar{b}cA) \cap q(\mathcal{M})$*. If* $c < c' \leq b_\varphi$*, then* $\models \psi(c')$*, a contradiction. Thus* $c < b_\varphi < c' < {b_\varphi}'$*.*

We iterate this construction infinitely many times and prove that the formula $c < x < b_\varphi$ *divides over* A*.* ∎

Corollary 18. *Let* $\mathcal{M}$ *be a sufficiently large saturated definably complete locally o-minimal structure. And let* $A \subset B \subset C \subset \mathcal{M}$ *and* $p(x) \in S_1(C)$ *be a cut type over* C*.*

Suppose that there is $d \in B$ *with* $dim(\mathrm{tp}(d/A)) = 1$ *and a bounded noncut type* $q(x) \in S_1(dcl(B))$ *of* d *such that* $q \restriction Ad \nvdash q$ *and* $p \vdash q$*.*

Then $p(x)$ *divides over* A*.*

3. Dp-rank of locally o-minimal structures

We recall some definitions.

Definition 19. An *independent partition pattern* of a partial type $p(\bar{x})$ is a sequence of formulas $(\, \varphi^\alpha(\bar{x}, \bar{y}^\alpha)\,)_{\alpha<\kappa}$ and tuples $\bar{b}^\alpha_i$ for $a < \kappa$ and $i < \omega$ satisfying that ;

for any $\alpha < \kappa$, $\{\, \varphi^\alpha(\bar{x}, \bar{b}^\alpha_i) \mid i < \omega \,\}$ is k^α−inconsistent for some $k^\alpha < \omega$, and for any $\eta \in \omega^\kappa$, $\{\, \varphi^\alpha(\bar{x}, \bar{b}^\alpha_{\eta(\alpha)}) \mid \alpha < \kappa \,\}$ is consistent with $p(\bar{x})$.

For a theory T, the invariant $\kappa^n_{inp}(T)$ is the smallest infinite cardinal κ such that no n−type has an *inp*−pattern of cardinality κ.

A formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* if there are sequences $(\, \bar{a}_i \,:\, i < \omega\,)$ and $(\, \bar{b}_I \,:\, I \subset \omega\,)$ such that $\models \varphi(\bar{a}_i, \bar{b}_I)$ if and only if $i \in I$.

A formula $\varphi(\bar{x}, \bar{y})$ has the *tree property of the second kind* (TP_2) if there are tuples

$(\bar{b}_i^\alpha)_{\alpha, i<\omega}$ such that $\{\,\varphi(\bar{x}, \bar{b}_i^\alpha) \mid i < \omega\,\}$ is 2−inconsistent for any $\alpha < \omega$, and for any $\eta \in \omega^\omega$, $\{\,\varphi(\bar{x}, \bar{b}_{\eta(\alpha)}^\alpha) \mid \alpha < \omega\,\}$ is consistent.

We call the *burden* of $p(\bar{x})$ the supremum of the cardinalities κ of all *inp*−patterns for p. It is known that the burden of p is equal to the (classical) weight of p in simple theories.

Theorem 20. *e.g.* [15]
For any theory T,
any formula $\varphi(\bar{x}, \bar{y})$ *with* $|\,\bar{x}\,| = n$ *is* NTP_2 *if and only if* $\kappa_{inp}^n(T) \leq |\,T\,|^+ \; (< \infty)$.

Definition 21. An *independent contradictory types pattern* is a sequence of formulas $(\,\varphi^\alpha(\bar{x}, \bar{y}^\alpha)\,)_{\alpha<\kappa}$ and tuples $\bar{b}_i^\alpha$ for $\alpha < \kappa$ and $i < \omega$ satisfying that ;
for any $\eta \in \omega^\kappa$, the following set of formulas is consistent,
$\Gamma_\eta(\bar{x}) := \{\,\varphi^\alpha(\bar{x}, \bar{b}_i^\alpha) \mid \alpha < \kappa,\, i < \omega,\, \eta(\alpha) = i\,\} \cup \{\,\neg\varphi^\alpha(\bar{x}, \bar{b}_i^\alpha) \mid \alpha < \kappa,\, i < \omega,\, \eta(\alpha) \neq i\,\}$.

Theorem 22. *e.g.* [11]
If T *is NIP, then* $\kappa_{ict}(T) = \kappa_{inp}(T)$. *Otherwise,* $\kappa_{ict}(T) = \infty$.

And we recall the definition of dp-rank from [16].

Definition 23. Let $p(\bar{x})$ be a partial type over a set $A \subset \mathcal{M}$. We define the $dp-rank$ of $p(\bar{x})$ as follows.
The dp-rank of $p(\bar{x})$ is always greater than or equal to 0. Let μ be a cardinal.
We say that $p(\bar{x})$ has dp-rank $\leq \mu$ if given any realization a of p and any $1 + \mu$ mutually A−indiscernible sequences, at least one of them is indiscernible over Aa.
And we say that p has dp-rank $= \mu$ if it has dp-rank $\leq \mu$, but it is not the case that it has dp-rank $\leq \lambda$ for any $\lambda < \mu$.
We call p $dp-minimal$ if it has dp-rank 1, we denote rk-dp$(p) = 1$.
We call p *dependent* if it has an ordinal dp-rank, that is, rk-dp$(p) < \infty$.
And we call p *strongly dependent* if rk-dp$(p) \leq \omega$.

There exists many examples whose theories are dp-minimal. For example, structures of superstable with U-rank = 1, C-minimal, p-adics, ordered set with finite width, tree, and so on. Here we recall the next fact. It is proved by the argument about the notion of Vapnik-Chervonenkis density (or VC-minimality).

Theorem 24. [14], [17]
Weakly o-minimal theories are dp-minimal.

But there are examples of locally o-minimal structures whose theories have the independence property.

Many results are proved under the strong assumption that structures $M \prec \mathcal{M}$ are dp-minimal.

Proposition 25. [13]

Let M be an inp-minimal linearly ordered structure without endpoints and $A \subset M$. And let M be $|A|^+-$saturated and $p(x) \in S_1(M)$.

Then the following are equivalent ;

1. $p(x)$ divides over A.

2. There exist $a, b \in M$ such that $p \vdash a < x < b$ and $a \equiv_A b$.

We can prove the next fact under the same assumption.

Fact 26. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure and $A \subset \mathcal{M}$. And let $Th(\mathcal{M})$ be inp-minimal.*

Moreover let $p(x) \in S_1(dcl(A))$ be an unbounded noncut type.

Then $p(x)$ does not fork over $\emptyset$.

4. Further problems

For definably complete locally o-minimal structures, we can prove the next fact easily.

Lemma 27. *Let M be a definably complete locally o-minimal structure and $A \subset M$ with $dcl(A) \neq \emptyset$.*

Then the isolated $1-$types of $Th(M, a)_{a \in A}$ are dense.

Thus I will try to characterize definably complete locally o-minimal structures by means of prime models.

And I will try to characterize locally o-minimal structures satisfying some additional conditions of their theories. The additional conditions are ; definably complete, dp-minimal, strongly dependent, NIP or NTP_2.

References

[1] C.Toffalori and K.Vozoris, *Note on local o − minimality*, Math.Log.Quart., 55, pp 617−632, 2009.

[2] T.Kawakami, K.Takeuchi, H.Tanaka and A.Tsuboi, *Locally o − minimal structures*, J. Math. Soc. Japan, vol.64, no.3, pp 783−797, 2012.

[3] M.Fujita, *Locally o − minimal structures with tame topological properties*, J. Symb. Logic, to appear.

[4] A.Pillay and C.Steinhorn, *Definable sets in ordered structures. I*, Trans of A.M.S., vol.295, no.2, pp 565−592, 1986.

[5] D.Marker and C.Steinhorn, *Definable types in o − minimal theories*, J. Symb. Logic, vol.59, no.1, pp 185−198, 1994.

[6] A.Dolich, *Forking and independence in o − minimal theories*, J. Symb. Logic, vol.69, pp 215−240, 2004.

[7] Y.Peterzil and A.Pillay, *Generic sets in definably compact groups*, Fund. Math, 193, pp 153−170, 2007.

[8] S.Shelah, *Dependent first order theories, continued*, Israel J. Math., vol.173, pp 1−60, 2009.

[9] S.Shelah, *Strongly dependent theories*, Israel J. Math., vol.204, pp 1−83, 2014.

[10] E.Hrushovski and A.Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc., vol.13, pp 1005−1061, 2011.

[11] H.Adler, *Strong theories, burden, and weight*, Preprint, 2007.

[12] A.Onshuus and A.Usvyatsov, *On DP − minimality, strong dependence and weight*, J. Symb. Logic, vol.76, no.3, pp 737−758, 2011.

[13] P.Simon, *On dp − minimal ordered structures*, J. Symb. Logic, vol.76, no.2, pp 448−460, 2011.

[14] A.Dolich, J.Goodrick and D.Lippel, *Dp − minimality : basic facts and examples*, Notre Dame J. Form. Logic, vol.52, no.3, pp 267−288, 2011.

[15] A.Chernikov, *Theories without the tree property of the second kind*, Ann. Pure. and Appl. Logic, vol.165, pp 695−723, 2014.

[16] I.Kaplan, A.Onshuus and A.Usvyatsov, *Additivity of the dp − rank*, Trans of A.M.S., vol.365, pp 5783−5804, 2013.

[17] M.Aschenbrenner, A.Dolich, D.Haskell, D.MacPherson and S.Starchenko, *Vapnik − Chervonenkis density in some theories without the independence property, I*, Trans of A.M.S., vol.368, no.8, pp 5889−5949, 2016.

[18] L.van den Dries, *Tame topology and o − minimal structures*, London Math. Soc. Lecture Note Ser, 248, Cambridge University Press, 1998.

[19] F.O.Wagner, *Simple Theories*, Mathematics and its applications, vol.503, Kluwer Academic Publishers, Dordrecht, 2000.