# Triplet of Fibonacci Duals

## — with or without constraint —

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**Abstract**

We consider a dual relation between minimization (primal) problem and maximization (dual) problem from a view point of complementarity. An identity

$$\text{(CI)} \quad \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n \;=\; x_0\mu_1$$

is called *complementary* [20,22]. We present three types of complementary identities, which take a fundamental role in analyzing respective pairs of primal and dual. Moreover, we show that a primal and its dual satisfy *Fibonacci Complementary Duality* [18, 19, 21, 22].

# 1 Introduction

A wide class of quadratic optimization problems has been discussed by Bellman and others [1–12, 23]. Dynamic programming has solved its partial class [2, 17, 18, 26]. Further a dual approach has been treated based upon convex-concavity [14, 16, 25].

Recently some new dual approaches — plus-minus method, extended Lagrangean method, ineqlualty method and others — have been derived in [18–22]. In this paper, we propose a complementary duality based upon an identity.

# 2 Complementary identities

Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$\text{(C}_1\text{)} \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n$$

holds true. This identity is called *complementary* [20,22]. Further we assume that $\mu_n = 0$. Then an identity

$$\text{(C}_2\text{)} \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n$$

holds true. This is a *conditional complementarity*.

On the other hand, we assume that $x_n = 0$. Then an identity

$$(\mathrm{C}_3) \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1}-x_n)\mu_n$$

holds true. This is also a *conditional complementarity*.

# 3 Three pairs

We consider three pairs of minimization (primal) problems and maximization (dual) problems, which are ($\mathrm{P}_1$) vs ($\mathrm{D}_1$), ($\mathrm{P}_2$) vs ($\mathrm{D}_2$) and ($\mathrm{P}_3$) vs ($\mathrm{D}_3$). It is shown that each pair is dual to each other. It turns out that the duality is based upon the complementary identity and an elementary inequality with equality

$$2xy \;\le\; x^2 + y^2 \quad \text{on } R^2 \;;\; x = y. \tag{1}$$

Both the primal ($\mathrm{P}_1$) and the dual ($\mathrm{D}_1$) are *unconditional*. The primal ($\mathrm{P}_2$) is unconditional, while the dual ($\mathrm{D}_2$) is *conditional* on $\mu_n$. The primal ($\mathrm{P}_3$) is conditional on $x_n$, while the dual ($\mathrm{D}_3$) is unconditional.

## 3.1 ($\mathrm{P}_1$) vs ($\mathrm{D}_1$)

Let us consider the first pair:

$$(\mathrm{P}_1) \qquad \begin{array}{l} \text{minimize} \;\; \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + x_n^2 \\ \text{subject to} \quad \text{(i)} \;\; x \in R^n, \quad \text{(ii)} \;\; x_0 = c, \end{array}$$

$$(\mathrm{D}_1) \qquad \begin{array}{l} \text{Maximize} \;\; 2c\mu_1 - \displaystyle\sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 - \mu_n^2 \\ \text{subject to} \;\; \text{(i)} \;\; \mu \in R^n. \end{array}$$

An identity ($\mathrm{C}_1$) with the elementary inequality (1) yields an inequality

$$\begin{aligned} & 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 - \mu_n^2 \\ \le\; & \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + x_n^2 \end{aligned}$$

for any feasible pair $(x,\,\mu)$. Then it turns out that both are dual to each other. An equality condition is

$$(\mathrm{EC}_1)\qquad \begin{aligned} &x_{k-1}-x_k=\mu_k,\quad x_k=\mu_k-\mu_{k+1}\quad 1\le k\le n-1\\ &x_{n-1}-x_n=\mu_n,\quad x_n=\mu_n. \end{aligned}$$

The equality condition $(\mathrm{EC}_1)$ is a linear system of $2n$-equation on $2n$-variable $(x,\mu)$. Let $(x,\mu)$ be a solution. Then both sides become a common value with five expressions.

$$(5\mathrm{V}_1)\qquad \begin{aligned} &\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2+x_n^2\\ &=\ c(c-x_1)\\ &=\ 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2-\mu_n^2\\ &=\ \sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+\mu_n^2+\mu_n^2\\ &=\ c\mu_1. \end{aligned}$$

Let $(x,\mu)$ be a solution of $(\mathrm{EC}_1)$. Then the primal $(\mathrm{P}_1)$ has a minimum value

$$\begin{aligned} m_1\ &=\ \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2+x_n^2\\ &=\ c(c-x_1) \end{aligned}$$

at $x$, while the dual $(\mathrm{D}_1)$ has a maximum value

$$\begin{aligned} M_1\ &=\ 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2-\mu_n^2\\ &=\ \sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+\mu_n^2+\mu_n^2\\ &=\ c\mu_1 \end{aligned}$$

at $\mu$.

**Lemma 1** $(\mathrm{EC}_1)$ *has indeed a unique solution:*

$$\begin{aligned} x\ &=\ (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n)\\ &=\ \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots,F_{2n-2k+1},\ \ldots,\ F_3,\ F_1), \end{aligned}\tag{2}$$

$$\begin{aligned} \mu\ &=\ (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n)\\ &=\ \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots,F_{2n-2k+2},\ \ldots,\ F_4,\ F_2). \end{aligned}\tag{3}$$

*Proof.* From ($EC_1$), we have a pair of linear systems of $n$-variable on $n$-equation:

$$(EQ_1)\qquad \begin{array}{rcl@{\qquad}rcl} c &=& 3x_1 - x_2 & c &=& 2\mu_1 - \mu_2 \\ x_1 &=& 3x_2 - x_3 & \mu_1 &=& 3\mu_2 - \mu_3 \\ &\vdots& & &\vdots& \\ x_{n-2} &=& 3x_{n-1} - x_n & \mu_{n-2} &=& 3\mu_{n-1} - \mu_n \\ x_{n-1} &=& 2x_n & \mu_{n-1} &=& 3\mu_n. \end{array}$$

The left system has a solution $x$ in (2), while the right has a solution $\mu$ in (3). □

The primal ($P_1$) has a minimum value $m_1 = c(c - \hat{x}_1) = \dfrac{F_{2n}}{F_{2n+1}}c^2$ at a path

$$\begin{aligned} \hat{x} &= (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1). \end{aligned}$$

The dual ($D_1$) has a maximum value $M_1 = c\mu_1^* = \dfrac{F_{2n}}{F_{2n+1}}c^2$ at a path

$$\begin{aligned} \mu^* &= (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n-2k+2},\ \ldots,\ F_4,\ F_2) \end{aligned}$$

where $\{F_n\}$ is the *Fibonacci sequence* [13,15,24,27]. This is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1,\ x_0 = 0. \tag{4}$$

| $n$ | $\cdots$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | $\cdots$ | $-1$ | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |

| $n$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 144 | 233 | 377 | 610 | 987 | 1597 | 2584 | 4181 | 6765 | $\cdots$ |

Table 1 Fibonacci sequence $\{F_n\}$

Hence both optimal values are identical:

$$m_1 = M_1 = \frac{F_{2n}}{F_{2n+1}}c^2.$$

An alternate contexture of both optimal points $\mu^*, \hat{x}$ is Fibonacci backward:

$$\begin{aligned} &(\mu_1^*,\ \hat{x}_1,\ \mu_2^*,\ \hat{x}_2,\ \ldots,\ \mu_k^*,\ \hat{x}_k\ \ldots,\ \mu_{n-1}^*,\ \hat{x}_{n-1},\ \mu_n^*,\ \hat{x}_n) \\ = &\frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ F_{2n-2},\ F_{2n-3},\ \ldots, F_{2n-2k+2},\ F_{2n-2k+1},\ \ldots,\ F_4,\ F_3,\ F_2,\ F_1). \end{aligned}$$

Thus *Fibonacci Complementary Duality* (FCD) [18, 19, 21, 22] holds between ($P_1$) and ($D_1$).

### 3.2 $(P_2)$ vs $(D_2)$

Let us consider the second:

$$(P_2)\qquad \begin{aligned} &\text{minimize} \quad \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2 \\ &\text{subject to} \quad \text{(i)} \ \ x\in R^n, \quad \text{(ii)} \ \ x_0=c \end{aligned}$$

$$(D_2)\qquad \begin{aligned} &\text{Maximize} \quad 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2 \\ &\text{subject to} \quad \text{(i)} \quad \mu\in R^n, \quad \text{(ii)} \ \ \underline{\mu_n=0}. \end{aligned}$$

An identity $(C_2)$ with the elementary inequality (1) yields an inequality

$$\begin{aligned} &2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2 \\ \le\ &\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2 \end{aligned}$$

for any feasible pair $(x,\,\mu)$. Then both are dual to each other. An equality condition is

$$(EC_2)\qquad \begin{aligned} &x_{k-1}-x_k=\mu_k, \quad x_k=\mu_k-\mu_{k+1} \quad 1\le k\le n-1 \\ &x_{n-1}-x_n=\mu_n. \end{aligned}$$

The equality condition $(EC_2)$ is a linear system of $(2n-1)$-equation on $2n$-variable $(x,\mu)$. Let $(EC_2')$ be an augmentation of the system $(EC_2)$ with the additional constraint (ii) $\underline{\mu_n=0}$ :

$$(EC_2')\qquad \begin{aligned} &x_{k-1}-x_k=\mu_k, \quad x_k=\mu_k-\mu_{k+1} \quad 1\le k\le n-1 \\ &x_{n-1}-x_n=\mu_n, \quad \underline{\mu_n=0}. \end{aligned}$$

Then $(EC_2')$ is of $2n$-equation on $2n$-variable.

Let $(x,\mu)$ be a solution of $(EC_2')$. Then both sides become a common value with five expressions.

$$(5V_2)\qquad \begin{aligned} &\sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2 \\ =\ &c(c-x_1) \\ =\ &2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-\mu_n^2 \\ =\ &\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+\mu_n^2 \\ =\ &c\mu_1. \end{aligned}$$

The primal $(\mathrm{P}_2)$ has a minimum value

$$\begin{aligned} m_2 &= \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2\right] + (x_{n-1} - x_n)^2 \\ &= c(c - x_1) \end{aligned}$$

at $x$, while the dual $(\mathrm{D}_2)$ has a maximum value

$$\begin{aligned} M_2 &= 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\ &= \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 \\ &= c\mu_1 \end{aligned}$$

at $\mu$.

**Lemma 2** *The system* $(\mathrm{EC}_2')$ *has indeed a unique solution:*

$$\begin{aligned} x &= (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{F_{2n-1}}(F_{2n-3},\ F_{2n-5},\ \ldots, F_{2n-2k-1},\ \ldots,\ F_1,\ F_{-1}), \end{aligned} \tag{5}$$

$$\begin{aligned} \mu &= (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0). \end{aligned} \tag{6}$$

*Proof.* From $(\mathrm{EC}_2')$, we have a pair of linear systems of $n$-variable on $n$-equation:

$$(\mathrm{EQ}_2) \quad \begin{array}{rcl@{\qquad}rcl} c &=& 3x_1 - x_2 & c &=& 2\mu_1 - \mu_2 \\ x_1 &=& 3x_2 - x_3 & \mu_1 &=& 3\mu_2 - \mu_3 \\ & \vdots & & & \vdots & \\ x_{n-3} &=& 3x_{n-2} - x_{n-1} & \mu_{n-3} &=& 3\mu_{n-2} - \mu_{n-1} \\ x_{n-2} &=& 3x_{n-1} - x_n & \mu_{n-2} &=& 3\mu_{n-1} - \mu_n \\ x_{n-1} &=& x_n & \underline{\mu_n} &\underline{=}& \underline{0}. \end{array}$$

The left system has a solution $x$ in (5), while the right has a solution $\mu$ in (6). □

The primal $(\mathrm{P}_2)$ has a minimum value $m_2 = c(c - \hat{x}_1) = \dfrac{F_{2n-2}}{F_{2n-1}}c^2$ at a path

$$\begin{aligned} \hat{x} &= (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n-1}}(F_{2n-3},\ F_{2n-5},\ \ldots, F_{2n-2k-1},\ \ldots,\ F_1,\ F_{-1}). \end{aligned}$$

The dual ($D_2$) has a maximum value $M_2 = c\mu_1^* = \dfrac{F_{2n-2}}{F_{2n-1}}c^2$ at a path

$$\begin{aligned}\mu^* &= (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*)\\ &= \frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0).\end{aligned}$$

Hence both optimal values are identical:

$$m_2 = M_2 = \frac{F_{2n-2}}{F_{2n-1}}c^2.$$

An alternate contexture of both optimal points $\mu^*, \hat{x}$ is Fibonacci backward:

$$\begin{aligned}&(\mu_1^*,\ \hat{x}_1,\ \mu_2^*,\ \hat{x}_2,\ \ldots,\ \mu_k^*,\ \hat{x}_k\ \ldots,\ \mu_{n-1}^*,\ \hat{x}_{n-1},\ \mu_n^*,\ \hat{x}_n)\\ = &\frac{c}{F_{2n-1}}(F_{2n-2},\ F_{2n-3},\ F_{2n-4},\ F_{2n-5},\ \ldots, F_{2n-2k},\ F_{2n-2k-1},\ \ldots,\ F_2,\ F_1,\ F_0,\ F_{-1}).\end{aligned}$$

Thus FCD holds between ($P_2$) and ($D_2$).

## 3.3 ($P_3$) vs ($D_3$)

Let us consider the third:

$$(P_3)\qquad \begin{aligned}&\text{minimize}\quad \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2\\ &\text{subject to}\quad \text{(i)}\ \ x \in R^n,\quad \text{(ii)}\ \ x_0 = c,\ \underline{x_n = 0}\end{aligned}$$

$$(D_3)\qquad \begin{aligned}&\text{Maximize}\quad 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2\\ &\text{subject to}\quad \text{(i)}\quad \mu \in R^n.\end{aligned}$$

An identity ($C_3$) with the elementary inequality (1) yields an inequality

$$\begin{aligned}&2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2\\ \le &\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2\end{aligned}$$

for any feasible pair $(x,\ \mu)$. Then both are dual to each other. An equality condition is

$$(EC_3)\qquad \begin{aligned}&x_{k-1} - x_k = \mu_k,\quad x_k = \mu_k - \mu_{k+1}\quad 1 \le k \le n-1\\ &x_{n-1} - x_n = \mu_n.\end{aligned}$$

The equality condition ($\mathrm{EC}_3$) is a linear system of $(2n-1)$-equation on $2n$-variable $(x,\mu)$. Let ($\mathrm{EC}_3'$) be an augmentation of the system ($\mathrm{EC}_3$) with the additional constraint (ii) $\underline{x_n = 0}$ :

$$(\mathrm{EC}_3') \qquad \begin{array}{l} x_{k-1} - x_k = \mu_k, \quad x_k = \mu_k - \mu_{k+1} \quad 1 \le k \le n-1 \\ x_{n-1} - x_n = \mu_n, \quad \underline{x_n = 0}. \end{array}$$

Then ($\mathrm{EC}_3'$) is of $2n$-equation on $2n$-variable.

Let $(x,\mu)$ be a solution of ($\mathrm{EC}_3'$). Then both sides become a common value with five expressions.

$$\begin{aligned} & \sum_{k=1}^{n} \left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 \\ &= c(c-x_1) \\ (5\mathrm{V}_3) \qquad &= 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\ &= \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 \\ &= c\mu_1. \end{aligned}$$

The primal ($\mathrm{P}_3$) has a minimum value

$$\begin{aligned} m_3 &= \sum_{k=1}^{n} \left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 \\ &= c(c-x_1) \end{aligned}$$

at $x$, while the dual ($\mathrm{D}_3$) has a maximum value

$$\begin{aligned} M_3 &= 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 \\ &= \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 \\ &= c\mu_1 \end{aligned}$$

at $\mu$.

**Lemma 3** *The system* ($\mathrm{EC}_3'$) *has indeed a unique solution:*

$$\begin{aligned} & x = (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{F_{2n}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0), \end{aligned} \tag{7}$$

$$\begin{aligned} & \mu = (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1). \end{aligned} \tag{8}$$

*Proof.* From $(\mathrm{EC}_3')$, we have a pair of linear systems of $n$-variable on $n$-equation:

$$(\mathrm{EQ}_3)\quad \begin{array}{rcl|rcl} c &=& 3x_1 - x_2 & c &=& 2\mu_1 - \mu_2 \\ x_1 &=& 3x_2 - x_3 & \mu_1 &=& 3\mu_2 - \mu_3 \\ & \vdots & & & \vdots & \\ x_{n-3} &=& 3x_{n-2} - x_{n-1} & \mu_{n-3} &=& 3\mu_{n-2} - \mu_{n-1} \\ x_{n-2} &=& 3x_{n-1} - x_n & \mu_{n-2} &=& 3\mu_{n-1} - \mu_n \\ \underline{x_n} &=& \underline{0} & \mu_{n-1} &=& 2\mu_n. \end{array}$$

The left system has a solution $x$ in (7), while the right has a solution $\mu$ in (8). □

The primal $(\mathrm{P}_3)$ has a minimum value $m_3 = c(c - \hat{x}_1) = \dfrac{F_{2n-1}}{F_{2n}}c^2$ at a path

$$\begin{aligned} \hat{x} &= (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n}}(F_{2n-2},\ F_{2n-4},\ \ldots, F_{2n-2k},\ \ldots,\ F_2,\ F_0). \end{aligned}$$

The dual $(\mathrm{D}_3)$ has a maximum value $M_3 = c\mu_1^* = \dfrac{F_{2n-1}}{F_{2n}}c^2$ at a path

$$\begin{aligned} \mu^* &= (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*) \\ &= \frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1). \end{aligned}$$

Hence both optimal values are identical:

$$m_3 = M_3 = \frac{F_{2n-1}}{F_{2n}}c^2.$$

An alternate contexture of both optimal points $\mu^*, \hat{x}$ is Fibonacci backward:

$$\begin{aligned} &(\mu_1^*,\ \hat{x}_1,\ \mu_2^*,\ \hat{x}_2,\ \ldots,\ \mu_k^*,\ \hat{x}_k\ \ldots,\ \mu_{n-1}^*,\ \hat{x}_{n-1},\ \mu_n^*,\ \hat{x}_n) \\ =\ &\frac{c}{F_{2n}}(F_{2n-1},\ F_{2n-2},\ F_{2n-3},\ F_{2n-4},\ \ldots, F_{2n-2k+1},\ F_{2n-2k},\ \ldots,\ F_3,\ F_2,\ F_1,\ F_0). \end{aligned}$$

Thus FCD holds between $(\mathrm{P}_3)$ and $(\mathrm{D}_3)$.

# References


[1] E.F. Beckenbach and R.E. Bellman, Inequalities, Springer-Verlag, Ergebnisse 30, 1961.

[2] R.E. Bellman, Dynamic Programming, Princeton Univ. Press, NJ, 1957.

[3] R.E. Bellman, Introduction to the Mathematical Theory of Control Processes, Vol.I, Linear Equations and Quadratic Criteria, Academic Press, NY, 1967.

[4] R.E. Bellman, Introduction to the Mathematical Theory of Control Processes, Vol.II, Nonlinear Processes, Academic Press, NY, 1971.

[5] R.E. Bellman, Methods of Nonlinear Analysis, Vol.I, Nonlinear Processes, Academic Press, NY, 1972.

[6] R.E. Bellman, Methods of Nonlinear Analysis, Vol.II, Nonlinear Processes, Academic Press, NY, 1972.

[7] R.E. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, NY, 1970 (Second Edition is a SIAM edition 1997).

[8] R.E. Bellman and Wm. Karush, On a new functional transform in analysis : the maximum transform, Bull. Amer. Math. Soc., **67**(1961), 501-503.

[9] R.E. Bellman and Wm. Karush, Mathematical programming and the maximum transform, J. SIAM Appl. Math., **10**(1962), 550-567.

[10] R.E. Bellman and Wm. Karush, On the maximum transform and semigroups of transformations, Bull. Amer. Math. Soc., **68**(1962), 516-518.

[11] R.E. Bellman and Wm. Karush, Functional equations in the theory of dynamic programming-XII : an application of the maximum transform, J. Math. Anal. Appl., **6**(1963), 155-157.

[12] R.E. Bellman and Wm. Karush, On the maximum transform, J. Math. Anal. Appl., **6**(1963), 57-74.

[13] A. Beutelspacher and B. Petri, Der Goldene Schnitt 2, überarbeitete und erweiterte Auflange, Elsevier GmbH, Spectrum Akademischer Verlag, Heidelberg, 1996.

[14] J.M. Borwein and A.S. Lewis, Convex Analysis and Nonlinear Optimization Theory and Examples, Springer-Verlag, New York, 2000.

[15] R.A. Dunlap, The Golden Ratio and Fibonacci Numbers, World Scientific Publishing Co.Pte.Ltd., 1977.

[16] W. Fenchel, Convex Cones, Sets and Functions, Princeton Univ. Dept. of Math, NJ, 1953.

[17] S. Iwamoto, Theory of Dynamic Program, Kyushu Univ. Press, Fukuoka, 1987 (*in Japanese*).

[18] S. Iwamoto, Mathematics for Optimization II Bellman Equation, Chisen Shokan, Tokyo, 2013 (*in Japanese*).

[19] S. Iwamoto and Y. Kimura, Semi-Fibonacci programming – odd-variable – , RIMS Kokyuroku, Vol.2158, pp.30–37, 2020.

[20] S. Iwamoto and Y. Kimura, Identical Duals – Gap Function – , RIMS Kokyuroku, Vol.2194, pp.56–67, 2021.

[21] S. Iwamoto, Y. Kimura and T. Fujita, Complementary versus shift dualities, J. Nonlinear Convex Anal., **17**(2016), 1547–1555.

[22] S. Iwamoto, Y. Kimura and T. Fujita, On complementary duals – both fixed points –, Bull. Kyushu Inst. Tech. Pure Appl. Math., **67**(2020), 1–28.

[23] E.S. Lee, Quasilinearization and Invariant Imbedding, Academic Press, New York, 1968.

[24] S. Nakamura, A Microcosmos of Fibonacci Numbers — Fibonacci Numbers, Lucas Numbers, and Golden Section — (Revised), Nippon Hyoronsha, 2008 (*in Japanese*).

[25] R.T. Rockafeller, Conjugate Duality and Optimization, SIAM, Philadelphia, 1974.

[26] M. Sniedovich, Dynamic Programming: foundations and principles, 2nd ed., CRC Press 2010.

[27] H. Walser, DER GOLDENE SCHNITT, B.G. Teubner, Leibzig, 1996.