Ultradiscrete equations for bifurcations in low-dimensional dynamical systems

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abstract

We discuss ultradiscrete equations in low-dimensional, especially, one-dimensional dynamical systems. The ultradiscrete equations are derived from the normal forms of the local bifurcations i.e., saddle-node, transcretical and pitchfork bifurcations, of one-dimensional continuous dynamical systems. With the aid of the graphical analysis, the dynamical properties of the obtained ultradiscrete equations are revealed. In particular, we show that these ultradiscrete equations exhibit the bifurcations characterized by piecewise linearity, say ultradiscrete bifurcations.

1 Ultradiscretization

Recently, ultradiscretization has been applied to nonlinear differential equations found in non-integrable non-equilibrium dissipative systems such as reaction-diffusion systems[1]-[8]. In these studies, we have successfully studied applications of ultradiscretization to dynamical systems with bifurcation structures[6]-[8]. In the present article, we treat ultradiscrete equations derived from the nonlinear equations well-known as the normal forms of the one-dimensional local bifurcations.

First we focus on the following equation[9]:

$$\frac{du}{dt} = c + u(u-2). \tag{1}$$

c is the bifurcation parameter and the saddle-node bifurcation occurs at $c = 1$. If $c < 1$, eq.(1) has the two fixed points u_- and u_+, where $u_\pm = 1 \pm \sqrt{1-c}$ and $u = u_-$ and u_+ are stable and unstable, respectively. At $c = 1$, $u = 1$ is half-stable. When $c > 1$, there is no fixed point. From eq.(1) we obtain the following discrete equation by tropical discretization[1]:

$$u_{n+1} = \frac{u_n + \Delta t[(u_n)^2 + c]}{1 + 2\Delta t}. \tag{2}$$

Note that eq.(2) shows the saddle-node bifurcation at $c = 1$. By taking the ultradiscrete limit[10]

$$\begin{cases} \lim_{\varepsilon\to+0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) \ = \ \max(A, B, \dots), \\ \lim_{\varepsilon\to+0} \varepsilon \log(e^{A/\varepsilon} \cdot e^{B/\varepsilon} \cdot \dots) = A + B + \dots, \end{cases} \tag{3}$$

after the variable transformations

$$\Delta t = e^{T/\varepsilon}, \quad u_n = e^{U_n/\varepsilon}, \quad c = e^{C/\varepsilon}, \tag{4}$$

we derive the ultradiscrete equation

$$U_{n+1} = \max\{U_n, T + \max(2U_n, C)\} - \max\{0, T\}. \tag{5}$$

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If we set $T \geq \max\{0, -C/2\}$, eq.(5) is converted into the following ultradiscrete equation:

$$U_{n+1} = \max(2U_n, C). \tag{6}$$

Equation (6) is the ultradiscrete equation for the one-dimensional normal form of the saddle-node bifurcation with the parameter C.

Next we focus on

$$\frac{du}{dt} = (u-1)(c-u) \tag{7}$$

which exhibits the transcritical bifurcation at $c = 1$ where c is the bifurcation parameter. By using the tropical discretization, we obtain

$$u_{n+1} = u_n \frac{u_n + \Delta t(1+c)u_n}{u_n + \Delta t[(u_n)^2 + c]}. \tag{8}$$

After the variable transformations

$$\Delta t = e^{T/\varepsilon}, \quad u_n = e^{U_n/\varepsilon}, \quad c = e^{C/\varepsilon}. \tag{9}$$

and ultradiscretization by (3), we have the ultradiscrete equation

$$U_{n+1} = U_n + \max\{U_n, T + U_n + \max(0, C)\} - \max\{U_n, T + \max(2U_n, C)\}. \tag{10}$$

Assuming $T \geq -C/2$, we obtain from eq.(10) the ultradiscrete equation for the normal form of the transcritical bifurcation as

$$U_{n+1} = 2U_n + \max(0, C) - \max(2U_n, C). \tag{11}$$

Finally we focus on the following nonlinear equation:

$$\frac{du}{dt} = 3cu(u-1) - u^3 + 1, \tag{12}$$

where c is positive bifurcation parameter and supercritical pitchfork bifurcation occurs at $c = 1$. The discrete equation of eq.(12) by the tropical discretization is

$$u_{n+1} = \frac{u_n + \Delta t[3c(u_n)^2 + 1]}{1 + \Delta t[(u_n)^2 + 3c]}. \tag{13}$$

Setting the variable transformations

$$\Delta t = e^{T/\varepsilon}, \quad u_n = e^{U_n/\varepsilon}, \quad c = e^{C/\varepsilon}, \tag{14}$$

we obtain the ultradiscrete equation

$$U_{n+1} = \max\{U_n, T + \max(2U_n + C, 0)\} - \max\{0, T + \max(2U_n, C)\}. \tag{15}$$

Here, we assume $T \geq \max(-C, 0)$. Thus, the following ultradiscrete equation is obtained from eq.(15):

$$U_{n+1} = \max(2U_n + C, 0) - \max(2U_n, C). \tag{16}$$

2 Graphical analysis

In this section, we discuss the dynamical properties of the following max-plus equations corresponding to the ultradiscrete equations (6), (11), and (16):

$$\text{(saddle-node)} \qquad U_{n+1} = \max(PU_n, C), \tag{17}$$

$$\text{(transcritical)} \qquad U_{n+1} = PU_n + \max(0, C) - \max(PU_n, C), \tag{18}$$

$$\text{(supercritical pitchfork)} \qquad U_{n+1} = \max(PU_n + C, 0) - \max(PU_n, C). \tag{19}$$

where we use the general fixed value $P > 1$ instead of the value 2 in eqs. (6), (11), and (16). Their dynamics can be visualized by using a graphical analysis[11, 12]. In particular, the parameter C becomes the bifurcation parameter and the bifurcation occurs at $C = 0$ in them.

2.1 Saddle-node bifurcation

First we treat eq.(17). Figure 1 shows the graphs of eq.(17) for (a) $C > 0$, (b) $C = 0$, and (c) $C < 0$. For (a) $C > 0$, eq.(17) has no fixed point because the graph of eq.(17) does not touch the diagonal $U_{n+1} = U_n$; when $U_n < C/P$, $U_{n+1} = C$ and U_{n+2} increases along $U_{n+2} = PU_{n+1} > U_{n+1}$. At $C = 0$, eq.(17) touches the diagonal at the origin of the graph and then $U_n = 0$ is the only fixed point and it is half-stable as shown in Fig. 1 (b). For (c) $C < 0$, the graph intersects the diagonal at the two points $U_n = 0$ and $U_n = C$. When $U_n \leq C/P$, $U_{n+1} = C$. When $C/P < U_n < 0$, U_n tends to C/P first along $U_{n+1} = PU_n$, and after that U_n finally arrives at C. When $U_n > 0$, U_n goes to positive infinity along $U_{n+1} = PU_n$. Therefore, $U_n = C$ and $U_n = 0$ are the stable and unstable fixed points, respectively. This bifurcation is the saddle-node bifurcation characterized by piecewise linearity. The bifurcation diagram is given as Fig. 2. In the diagram, the solid arrows represent the transition of U_n to the stable point just at the next step. The dotted arrows show the transition satisfying $U_{n+1} = PU_n$.

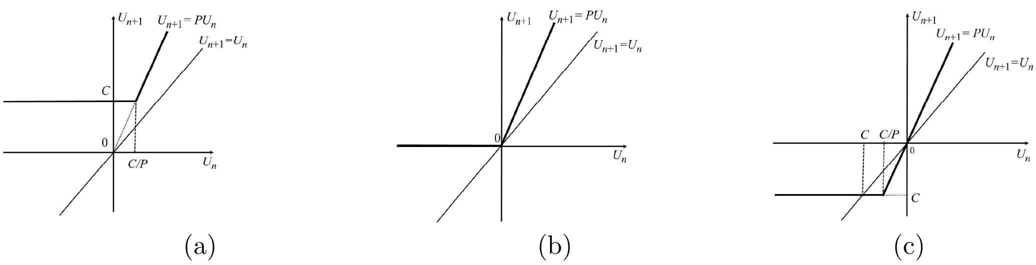


Figure 1: The graphs of eq.(17). (a) $C > 0$, (b) $C = 0$, and (c) $C < 0$.

2.2 Transcritical bifurcation

Next we focus on eq.(18) whose graph is given by Fig. 3. For (a) $C > 0$, the graph intersects the diagonal at the two points $U_n = 0$ and $U_n = C$. When $U_n < 0$, U_n goes to negative infinity along $U_{n+1} = PU_n$. When $0 < U_n \leq C/P$, U_n increases toward C/P along $U_{n+1} = PU_n$, and after that U_n reaches C in the end. When $U_n > C/P$, $U_{n+1} = C$. Therefore, $U_n = 0$ and $U_n = C$ are the unstable and stable fixed points, respectively. For (b) $C = 0$, we have the unique half-stable

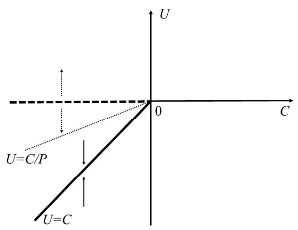
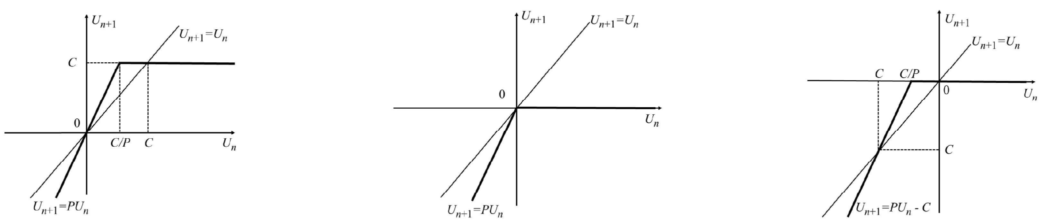


Figure 2: The bifurcation diagram for the ultradiscrete saddle-node bifurcation generated from eq.(17).

fixed point $U_n = 0$; if $U_n > 0$, $U_{n+1} = 0$ and if $U_n < 0$, $U_{n+1} = PU_n(< U_n)$. When $C < 0$, the graph of eq.(18) intersects the diagonal at the two points $U_n = 0$ and $U_n = C$ again; $U_n = 0$ and $U_n = C$ are stable and unstable, respectively as shown in Fig. 3 (c). This is the transcritical bifurcation of eq.(18). Figure 4 shows the bifurcation diagram; the bifurcation occurs at $C = 0$.



(a) (b) (c)

Figure 3: The graphs of eq.(18) where (a) $C > 0$, (b) $C = 0$, and (c) $C < 0$.

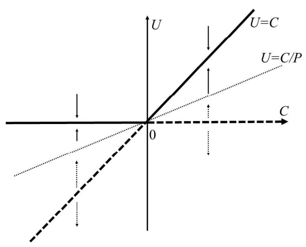


Figure 4: The bifurcation diagram of eq.(18).

2.3 Supercritical pitchfork and flip bifurcations

Finally, we consider eq.(19). Figure 5 shows the graphs of eq.(19) with three different cases of C. Note that U_{n+1} of eq.(19) is an odd function of U_n as shown in Fig. 5. Set $C > 0$. The graph of eq.(19) intersects the diagonal at the three fixed points $U_n = 0, \pm C$. When $U_n \geq C/P$ ($U_n \leq -C/P$), $U_{n+1} = +C$ ($U_{n+1} = -C$). When $-C/P < U_n < 0$, there exists a certain

$m(> n)$ at which $U_m \leq -C/P$ and $U_{m+1} = -C$. Similarly, $0 < U_n < C/P$ finally goes to C. Therefore, $U_n = \pm C$ are stable and $U_n = 0$ is unstable. At $C = 0$, $U_n = 0$ is only the stable fixed point; $U_{n+1} = 0$ for any initial U_n as shown in Fig. 5 (b). In Fig. 5 (c) $C < 0$, $U_n = 0$ retains to be the unique fixed point, however it is no longer stable but unstable. Further we find a cycle $\mathcal{C} = \{+C, -C\}$ with period 2 around $U_n = 0$; it is attracting in the following sense. (i) Whenever $|U_n| > C$, $U_{n+m} \in \mathcal{C}$ for any positive integer m. (ii) If U_n satisfies $0 < |U_n| < C$, U_n leaves from 0 oscillating around 0 and arrives at the point U_m such that $C/P \leq |U_m|$. After that, $U_l \in \mathcal{C}$ for $l > m$. Thus, any U_n starting from $U_0 \neq 0$ is finally absorbed by the cycle $\mathcal{C}$. Equation (19) therefore exhibits the bifurcation as putting the supercritical pitchfork and flip bifurcations together.

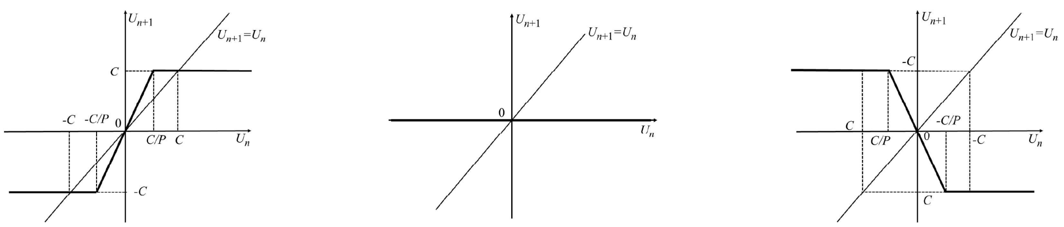


Figure 5: The graphs of eq.(19). (a) $C > 0$, (b) $C = 0$, and $C < 0$.

3 Conclusion

We derive ultradiscrete equations from the one-dimensional normal forms of saddle node, transcritical, and supercritical pitchfork bifurcations and we get dynamical descriptions of them by means of graphical analysis. These derived equations exhibit the ultradiscrete bifurcations which are the similar bifurcation properties to the original normal forms characterized by the piecewise linearity. In the ultradiscrete equation of the supercritical pitchfork bifurcation, another ultradiscrete bifurcation similarly to the flip bifurcation occurs where there is a stable cycle around a unstable fixed point. (More details are shown in [6].)

acknowledgment

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[9] In general, the existence of the negative values makes it difficult to perform the ultradiscretization. To avoid this difficulty, we adopt here the nonlinear differential equations (1), (7), and (12) as the one-dimensional normal forms with saddle-node, transcritical, and pitchfork bifurcations, respectively, instead of the following familiar forms: $\frac{du}{dt} = c + u^2$, $\frac{du}{dt} = cu - u^2$, and $\frac{du}{dt} = cu - u^3$.

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