

A result of the Gauss-Bonnet theorem for coherent tangent bundles over surfaces with boundary

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1 Introduction

In this paper, I present an original generalization of Fact3.3 of [2] to the case of 2-manifolds with boundary. The classical Gauss-Bonnet theorem is discussed on regular surfaces, and it is well known that there are two types, the local and global theorem. The following theorem is a generalization of the classical global Gauss-Bonnet's theorem for coherent tangent bundles.

Theorem 1.1 (Fact3.3 of [2]) *Let $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented 2-dimensional manifold M , and suppose that the singular set $\Sigma(\varphi)$ consists of singular points of the first and admissible second kind. We denote by K the Gaussian curvature of the induced metric $ds^2 = \varphi^*\langle \cdot, \cdot \rangle$. Then it holds that*

$$\frac{1}{2\pi} \int_M K d\hat{A} = \chi(M^+) - \chi(M^-) + \#S^+ - \#S^-, \quad (1)$$

$$2\pi\chi(M) = \int_M K dA + 2 \int_{\Sigma(\varphi)} \kappa_s ds, \quad (2)$$

where $\#S^+$ (resp. $\#S^-$) are the numbers of positive (resp. negative) singular points of the second kind, and κ_s is the singular curvature of $\Sigma(\varphi)$.

The concepts described in the above theorem are defined precisely in §2. I discuss a generalization of (1) and (2) for manifolds with boundary (See Theorem3.3 for details).

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2 Preliminaries

From now on, let M be a compact oriented 2-manifold with boundary.

The set of C^∞ -functions on M is denoted by $C^\infty(M)$, and the set of C^∞ -sections of a vector bundle \mathcal{E} is denoted by $\Gamma(\mathcal{E})$. Let \mathcal{E}^* be a dual vector bundle of \mathcal{E} . $\langle \cdot, \cdot \rangle \in \Gamma(\mathcal{E}^* \otimes \mathcal{E}^*)$ is called an *inner product* on \mathcal{E} , when $\langle \cdot, \cdot \rangle$ defines a positive definite inner product on each fiber \mathcal{E}_p ($p \in M$). A map $D : \Gamma(TM) \times \Gamma(\mathcal{E}) \ni (X, \xi) \mapsto D_X \xi \in \Gamma(\mathcal{E})$ is called a *connection* on \mathcal{E} if, for any $f, g \in C^\infty(M)$, $X, Y \in \Gamma(TM)$, $\xi, \eta \in \Gamma(\mathcal{E})$, the following conditions hold.

- (1) $D_{fX+gY} \xi = fD_X \xi + gD_Y \xi$,
- (2) $D_X(\xi + \eta) = D_X \xi + D_X \eta$,
- (3) $D_X(f\xi) = (Xf)\xi + fD_X \xi$.

A connection D on \mathcal{E} with an inner product is called a *metric connection* if it holds that

$$X\langle \xi, \eta \rangle = \langle D_X \xi, \eta \rangle + \langle \xi, D_X \eta \rangle \quad (X \in \Gamma(TM), \xi, \eta \in \Gamma(\mathcal{E})).$$

Definition 2.1 Let \mathcal{E} a vector bundle of rank 2 with an inner product $\langle \cdot, \cdot \rangle$, $D : \Gamma(TM) \times \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$ a metric connection on \mathcal{E} , and $\varphi : TM \rightarrow \mathcal{E}$ a bundle homomorphism. A 5-tuple $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ is called a *coherent tangent bundle* if, for any $X, Y \in \Gamma(TM)$, it holds that

$$D_X \varphi(Y) - D_Y \varphi(X) - \varphi([X, Y]) = 0.$$

Example 2.2 Let (M, g) be a 2-dimensional Riemannian manifold and $\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$ a Levi-Civita connection on TM . If we take an identity map $\text{id} : TM \rightarrow TM$ as a bundle homomorphism, then a 5-tuple $(M, TM, g, \nabla, \text{id})$ is a coherent tangent bundle. This shows that the concept of coherent tangent bundles is a generalization of Riemannian manifolds.

From now on, let \mathcal{E} be an oriented vector bundle.

If, for each point $p \in M$, $(ds^2)_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is defined by

$$(ds^2)_p(\mathbf{v}_1, \mathbf{v}_2) = \langle \varphi_p(\mathbf{v}_1), \varphi_p(\mathbf{v}_2) \rangle \quad (\mathbf{v}_1, \mathbf{v}_2 \in T_p M),$$

ds^2 is called the *first fundamental form* of φ . A point $p \in M$ is called a *regular point* of φ if $\varphi_p : T_p M \rightarrow \mathcal{E}_p$ is a linear isomorphism. A point $p \in M$ is called a *singular point* of φ if φ_p is not a linear isomorphism. The set of singular points of φ is denoted by $\Sigma(\varphi)$.

Let $(U; u, v)$ be a positive local coordinate system of M , $\{\mathbf{e}_1, \mathbf{e}_2\}$ an positive orthonormal basis field on $\mathcal{E}|_U$, and $\{\omega_1, \omega_2\}$ a dual basis field of $\{\mathbf{e}_1, \mathbf{e}_2\}$. If a section $\mu : U \rightarrow \mathcal{E}^*|_U \wedge \mathcal{E}^*|_U$ is defined by

$$\mu := \omega_1 \wedge \omega_2,$$

then μ is independent of the choice of positive orthonormal basis fields of $\mathcal{E}|_U$. Therefore, μ is defined on M . A differential 2 form $d\hat{A}$ on M is called a *signed area form* of \mathcal{E} if, for each point $p \in M$, $(d\hat{A})_p : T_pM \times T_pM \rightarrow \mathbb{R}$ is defined by

$$(d\hat{A})_p(\mathbf{v}_1, \mathbf{v}_2) = \mu(\varphi_p(\mathbf{v}_1), \varphi_p(\mathbf{v}_2)) \quad (\mathbf{v}_1, \mathbf{v}_2 \in T_pM).$$

If a *signed area density function* $\lambda \in C^\infty(U)$ is defined by

$$\lambda := \mu \left(\varphi \left(\frac{\partial}{\partial u} \right), \varphi \left(\frac{\partial}{\partial v} \right) \right),$$

then it holds that

$$\Sigma(\varphi) \cap U = \{p \in U \mid \lambda(p) = 0\}, \quad d\hat{A} = \lambda du \wedge dv.$$

In particular, $d\hat{A}$ defines a C^∞ -differential 2 form on M . On the other hand, $dA := |\lambda| du \wedge dv$ does not depend on the choice of positive local coordinate systems of M . Thus, dA defines a continuous differential 2 form on M , and dA is called an *area form* of \mathcal{E} . If we set

$$M^+ := \left\{ p \in M \setminus \Sigma(\varphi) \mid dA_p = d\hat{A}_p \right\}, \quad M^- := \left\{ p \in M \setminus \Sigma(\varphi) \mid dA_p = -d\hat{A}_p \right\},$$

then $\Sigma(\varphi)$ is coincide with ∂M^+ and ∂M^- , respectively. Using a signed area density function λ , we have

$$M^+ \cap U = \{p \in U \mid \lambda(p) > 0\}, \quad M^- \cap U = \{p \in M \mid \lambda(p) < 0\}.$$

Since $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the positive orthonormal basis field on $\mathcal{E}|_U$, it holds that

$$2\langle D_X \mathbf{e}_i, \mathbf{e}_i \rangle = \langle D_X \mathbf{e}_i, \mathbf{e}_i \rangle + \langle \mathbf{e}_i, D_X \mathbf{e}_i \rangle = X \langle \mathbf{e}_i, \mathbf{e}_i \rangle = 0 \quad (X \in \Gamma(TU)).$$

Thus, $D_X \mathbf{e}_1$ is orthogonal to \mathbf{e}_1 , and a C^∞ -differential 1 form ω on U is defined by

$$D_X \mathbf{e}_1 = -\omega(X) \mathbf{e}_2, \quad D_X \mathbf{e}_2 = \omega(X) \mathbf{e}_1.$$

The exterior derivative $d\omega$ does not depend on the choice of positive orthonormal basis fields of $\mathcal{E}|_U$. Therefore, $d\omega$ defines a C^∞ -differential 2 form on M . Let K be the Gaussian curvature of ds^2 on the set of regular points of φ . Then it holds that

$$d\omega = K d\hat{A} = \begin{cases} K dA & (\text{on } M^+ \setminus \Sigma(\varphi)), \\ -K dA & (\text{on } M^- \setminus \Sigma(\varphi)). \end{cases}$$

A singular point $p \in \Sigma(\varphi) \cap U$ is called *non-degenerate* if it holds that

$$(\lambda_u(p), \lambda_v(p)) \neq (0, 0).$$

If $p \in \Sigma(\varphi) \cap U$ is non-degenerate, by the implicit function theorem, there exists a regular curve $\gamma(t)$ on U through the point p such that it holds $\lambda(\gamma(t)) = 0$. This curve γ is called a *singular curve*, the tangent vector $\dot{\gamma}(t)$ is called a *singular vector*, and a 1-dimensional vector space generated by $\dot{\gamma}(t)$ is called a *singular*

direction. A tangent vector $\mathbf{v} \in T_p U \setminus \{\mathbf{0}\}$ at $p \in \Sigma(\varphi) \cap U$ is called a *null vector* if it holds that $\varphi_p(\mathbf{v}) = \mathbf{0}$, and a vector space generated by the null vector is called a *null direction*. We remark that a null direction at a non-degenerate singular point is 1-dimensional.

From now on, we assume that $\Sigma(\varphi)$ consists of non-degenerate singular points of φ . A singular point of $\Sigma(\varphi)$ is called a *singular point of the first kind* (resp. *singular point of the second kind*) if the singular and null direction at the point are different (resp. same). If there are only singular points of the first kind around a singular point of the second kind of φ , the point is called an *admissible singular point of the second kind*.

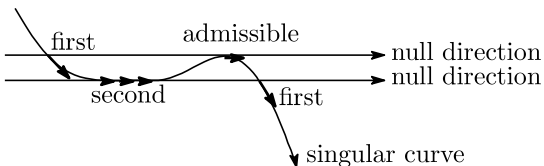


Figure 1:

From now on, we assume that $\Sigma(\varphi)$ consists of singular points of the first kind and admissible second kind.

If a singular point $p \in \Sigma(\varphi) \cap U$ is a first kind, there is a singular curve $\gamma(t)$ through the point p . Since the point p is the first kind, the singular and null directions at the point p are different, and the singular and null directions along γ change continuously, so we retake $(U; u, v)$ such that $\Sigma(\varphi) \cap U$ consists of singular points of the first kind. Then we set

$$\kappa_s(t) := \text{sgn}(\lambda_\eta(t)) \frac{\mu(\varphi_{\gamma(t)}(\dot{\gamma}(t)), D_{\dot{\gamma}(t)}(\varphi \circ \dot{\gamma}))}{|\varphi_{\gamma(t)}(\dot{\gamma}(t))|^3}, \quad (3)$$

where $\eta(t)$ is a null vector at each point $\gamma(t)$ such that $\{\dot{\gamma}(t), \eta(t)\}$ is positively oriented, and we set

$$\lambda_\eta(t) := (d\lambda)_{\gamma(t)}(\eta(t)), \quad |\varphi_{\gamma(t)}(\dot{\gamma}(t))| := \sqrt{\langle \varphi_{\gamma(t)}(\dot{\gamma}(t)), \varphi_{\gamma(t)}(\dot{\gamma}(t)) \rangle}.$$

The equality (3) is called a *singular curvature*. We remark that κ_s is independent of the choice of parameters of γ , the orientation of γ , the orientation of M , and the orientation of \mathcal{E} . If a point $p \in M$ be an admissible singular point of the second kind and $\gamma(t)$ is a singular curve through p , $\kappa_s(t)ds$ ($ds := |\varphi_{\gamma(t)}(\dot{\gamma}(t))|dt$) defines a bounded differential 1 form on $\gamma(t)$.

Consider triangulating M . We triangulate $M^\pm \cup \Sigma(\varphi)$ such that vertexes are singular points of the second kind. Then we triangulate M by subdividing such that their triangles on M^\pm are properly congruent on $\Sigma(\varphi)$. As a result, such a triangulation has the following properties.

- Singular points appear on edges of a triangle, and the interior of the triangle consists of regular points. A singular point appears on an edge

other than a vertex only if the edge consists of singular points (Such an edge is called a *singular edge*).

- A triangle has at most one singular edge.
- All edges other than a singular edge are gathered at singular vertexes from directions other than a null direction.
- There is no possibility that points at both ends of a singular edge are singular points of the second kind at the same time.

Triangles obtained by triangulation are classified into the following four types.

- (1) A triangle consists of regular points.
- (2) A triangle with one singular vertex consists of regular points except for that vertex.
- (3) A triangle with one singular edge consists of regular points except for that edge, and does not have singular points of the second kind.
- (4) A triangle with one singular edge consists of regular points except for that edge, and have a singular point of the second kind.

We remark that, at a non-degenerate singular point p , there exists a positive local coordinate system (u, v) with the origin at point p such that the null direction is parallel to the u -axis along the singular curve passing through point p . If the vertex A of $\triangle ABC$ is a singular point and we take the above local coordinate system (u, v) around A , the angle $\angle A$ is defined as follows:

$$\angle A := \begin{cases} \pi & \text{if the } u\text{-axis passes through the interior of } \triangle ABC, \\ 0 & \text{otherwise.} \end{cases}$$

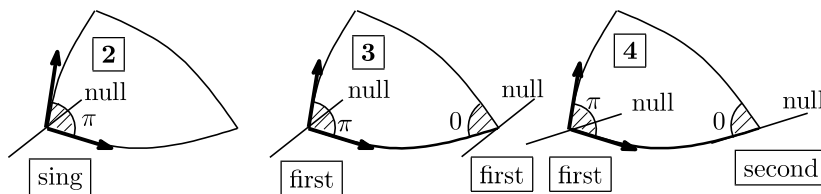


Figure 2:

3 The Gauss-Bonnet Theorem

Let $\gamma(t)$ be the parameterization of one of the sides of the triangles obtained by triangulating M . We define the *geometric curvature* $\tilde{\kappa}_g$ of γ as follows:

$$\tilde{\kappa}_g(t) = \begin{cases} \kappa_g(t) & \text{if } \gamma(t) \in M^+, \\ -\kappa_g(t) & \text{if } \gamma(t) \in M^-, \\ \kappa_s(t) & \text{if } \gamma(t) \in \Sigma(\varphi), \end{cases} \quad \kappa_g(t) := \frac{\mu(\varphi_{\gamma(t)}(\dot{\gamma}(t)), D_{\dot{\gamma}(t)}(\varphi \circ \dot{\gamma}))}{|\varphi_{\gamma(t)}(\dot{\gamma}(t))|^3}.$$

Theorem 3.1 (Local Gauss-Bonnet Theorem) Let $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented 2-manifold. If $\triangle ABC$ is a triangle obtained by triangulating M , then it holds that

$$\int_{\partial\triangle ABC} \tilde{\kappa}_g ds + \int_{\triangle ABC} K dA = \angle A + \angle B + \angle C - \pi.$$

Let $p \in \Sigma(\varphi)$ be an admissible singular point of the second kind. The sum of interior angles on side M^+ (resp. M^-) at p is denoted by $\alpha_+(p)$ (resp. $\alpha_-(p)$). Then it holds that

$$\alpha_+(p) + \alpha_-(p) = 2\pi, \quad \alpha_+(p) - \alpha_-(p) \in \{-2\pi, 0, 2\pi\}.$$

A point p is called *positive* (resp. *null*, *negative*), if $\alpha_+(p) - \alpha_-(p) > 0$ (resp. $\alpha_+(p) - \alpha_-(p) = 0$, $\alpha_+(p) - \alpha_-(p) < 0$).

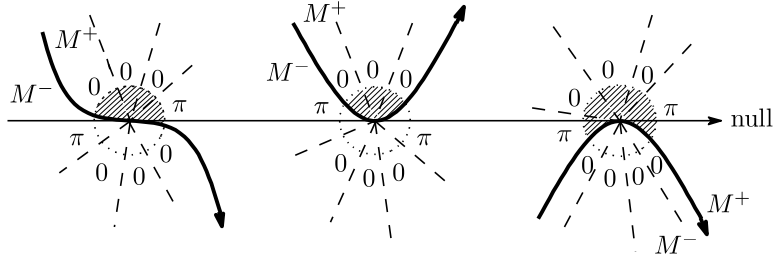


Figure 3:

Suppose that $\Sigma(\varphi)$ and ∂M satisfy the following conditions:

- (1) $\Sigma(\varphi)$ and ∂M are transversal.
- (2) All singular points on $\Sigma(\varphi) \cap \partial M$ are the first kind.
- (3) The null direction of a singular point on $\Sigma(\varphi) \cap \partial M$ is not tangent to ∂M .

(1) implies that tangent directions of $\Sigma(\varphi)$ and ∂M at a singular point on ∂M are different. (2) implies that a null direction at the singular point on ∂M is not tangent to $\Sigma(\varphi)$.

Definition 3.2 We take a local coordinate system (u, v) around a singular point p on ∂M such that the null direction along the singular curve is parallel to the u -axis. The point p is called *positive* (resp. *negative*) if the u -axis passes through M^+ (resp. M^-)

The sum of interior angles on side M^+ (resp. M^-) at a singular point p on ∂M is denoted by $\beta_+(p)$ (resp. $\beta_-(p)$). Then it holds that

$$\beta_+(p) + \beta_-(p) = \pi, \quad \beta_+(p) - \beta_-(p) \in \{-\pi, \pi\}.$$

By Definition 3.2, a point p is positive (resp. negative) if and only if it holds that $\beta_+(p) - \beta_-(p) = \pi$ (resp. $\beta_+(p) - \beta_-(p) = -\pi$).

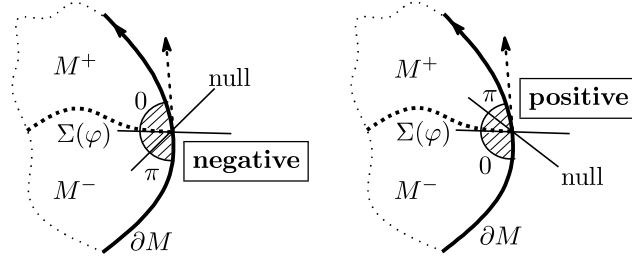


Figure 4:

Theorem 3.3 Let $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented 2-manifold with boundary M , and suppose that the singular set $\Sigma(\varphi)$ consists of singular points of the first kind and admissible second kind. We set

$$\int_M KdA := \int_{M^+} KdA + \int_{M^-} KdA.$$

Then it holds that

$$\begin{aligned} \int_M KdA + 2 \int_{\Sigma(\varphi)} \kappa_s ds + \int_{\partial M} \tilde{\kappa}_g ds &= 2\pi\chi(M) - \pi(\#\Sigma(\varphi) \cap \partial M), \\ \int_{\partial M} \kappa_g ds + \int_M Kd\hat{A} &= 2\pi \{ \chi(M^+) - \chi(M^-) \} + 2\pi(\#S^+ - \#S^-) \\ &\quad + \pi(\#\Sigma(\varphi) \cap \partial M)^+ - \#\Sigma(\varphi) \cap \partial M)^-, \end{aligned}$$

where $\#S^+$ (resp. $\#S^-$) are the numbers of positive (resp. negative) singular points of the second kind, $\#\Sigma(\varphi) \cap \partial M$ is the numbers of singular points on ∂M , and $\#\Sigma(\varphi) \cap \partial M)^+$ (resp. $\#\Sigma(\varphi) \cap \partial M)^-$) are the numbers of positive (resp. negative) singular points on ∂M .

Remark 3.4 The result presented here is closely related to Theorem 2.20 of [4]. Their theorem is a generalization of Theorem B of [1] to the case of manifolds with boundary, and it is generalized by considering peaks instead of admissible singular points of the second kinds. Several applications are presented in [4].

References

- [1] K. Saji, M. Umeraha, K. Yamada, *Behavior of corank-one singular points on wave fronts*, Kyushu Journal of Mathematics, Vol. 62 (2008), 259–280.
- [2] K. Saji, M. Umehara, K. Yamada, *Coherent tangent bundles and Gauss-Bonnet formulas for wave fronts*, J. Geom. Anal. 22 (2012), no. 2, 383–409.
- [3] K. Saji, M. Umeraha, K. Yamada, *The geometry of fronts*, Annals of Mathematics, 169 (2009), 491–529.

- [4] W. Domitrz and M. Zwierzyński, *The Gauss-Bonnet Theorem for Coherent Tangent Bundles over Surfaces with Boundary and Its Applications*, The Journal of Geometric Analysis (2020) 30:3243-3274.