APPENDIX TO "DIFFEOMORPHISM CLASSES OF THE DOUBLING CALABI-YAU THREEFOLDS WITH PICARD NUMBER TWO"

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1. Introduction

This is an appendix to the author's paper entitled "Diffeomorphism classes of the doubling Calabi-Yau threefolds with Picard number two [Y21]" where he proved that any two of the doubling Calabi-Yau 3-folds with Picard number 2 are not diffeomorphic to each other when the underlying Fano 3-folds are distinct. We refer the reader to [Y21] for background on the problem and terminology discussed in this note.

As listed in Table 1 below, there are 8 doubling Calabi-Yau 3-folds M with Picard number 2 which have the same Hodge numbers $(h^{1,1}(M), h^{2,1}(M))$. These 8 overlapping Hodge numbers $(h^{1,1}(M), h^{2,1}(M))$ are listed with ✓ on the table. Furthermore, in Table 1, V denote the underlying Fano 3-folds which are the ingredients for the doubling construction of Calabi-Yau 3-folds in [DY14]. See [DY14, Section 6], for more details. This note aims to summarize computational details of

(i) the cubic forms, and
(ii) the λ-invariants

which we will use for the proof of Theorem 1.1 in [Y21].

TABLE 1. The doubling Calabi-Yau 3-folds with Picard number 2 and the underlying Fano 3-folds with Picard number 1

ID in [FG]	$-K_V^3$	$h^{1,2}(V)$	$(h^{1,1}(M), h^{2,1}(M))$
1-1	2	52	$(2, 128)$
1-2	4	30	✓ $(2, 86)$
1-3	6	20	$(2, 68)$
1-4	8	14	✓ $(2, 58)$
1-5	10	10	$(2, 52)$
1-6	12	7	$(2, 48)$
1-7	14	5	$(2, 46)$
1-8	16	3	✓ $(2, 44)$
1-9	18	2	✓ $(2, 44)$
1-10	22	0	✓ $(2, 44)$
1-11	8	21	$(2, 72)$
1-12	16	10	✓ $(2, 58)$
1-13	24	5	$(2, 56)$
1-14	32	2	✓ $(2, 58)$
1-15	40	0	$(2, 62)$
1-16	54	0	$(2, 76)$
1-17	64	0	✓ $(2, 86)$

Date: July 3, 2022.
Key words and phrases. Calabi-Yau manifolds, diffeomorphism, cubic intersection form.

2. $(h^{1,1}(M), h^{2,1}(M)) = (2, 86)$ CASE

These doubling Calabi-Yau 3-folds are listed in Table 1 with the underlying Fano 3-folds, (a) ID 1-2 and (b) ID 1-17. Geometric description of the corresponding Fano 3-folds are

(a) a quartic hypersurface in $\mathbb{C}P^4$; $V(4) \subset \mathbb{C}P^4$, and
(b) the projective space $\mathbb{C}P^3$.

2.1. **ID 1-2:** $V(4) \subset \mathbb{C}P^4$ **case.** Let V be a quartic hypersurface in $\mathbb{C}P^4$. Note that V is the Fano 3-fold with $-K_V^3 = 4$ (see [IsPr99, p.215]). By Lefschetz Hyperplane Theorem, we have more specific description of V such as

$$h^{p,q}(V) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 30 & & 30 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}, \qquad g = g(V) = \frac{H^3}{2} + 1 = \frac{-K_V^3}{2} + 1 = 3$$

where g denotes the genus of Fano variety. In particular, $H^3 = 4$ for the ample generator $H \in H^2(V, \mathbb{Z})$. Let $D \in |-K_V|$ be a smooth anticanonical divisor and let $C \in |\mathcal{O}_D(1)|$ be a smooth curve in D which represents the intersection class of $D \cdot D$. Then the degree of C is $2g - 2$ and this is the reason why $g = \frac{-K_V^3}{2} + 1$ is called the *genus* of a Fano 3-fold [IsPr99, p.32]. Taking Y_i to be the blow-ups $\mathrm{Bl}_C(V)$ of V along C, we again denote the exceptional divisors by E_i for $i = 1, 2$. Then the cohomology rings of Y_i are

$$H^2(Y_i) = \mathbb{C}\langle \pi_i^*(H), E_i \rangle = \mathbb{C}\langle H_i, E_i \rangle$$

and the proper transforms D_i of D in Y_i are $H_i - E_i$. Let $\delta = \langle -D_1, D_2 \rangle = \langle E_1 - H_1, H_2 - E_2 \rangle$. Then we see that any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ is written as

$$(aH_1 + bE_1, cE_2 + (a + b - c)H_2) = (a + b)(H_1, H_2) - (b + c)(H_1 - E_1, 0) - c\delta.$$

Thus we conclude that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0) \rangle$$

up to torsion. Hence in this case, we take $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$ as generators of $H^2(M, \mathbb{Z})$.

Now we compute the cubic products of e_i in $H^6(M, \mathbb{Z})$. Let us denote by $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ two copies of the blow-ups of V along C for $i = 1, 2$. Let L be a fiber over a point on C under the blow-up π_i. Since the intersection number is preserved by the total transform, we see that $H_i^3 = (\pi_i^* H)^3 = H^3 = 4$. Moreover, $H_i L = 0$ and $E_i L = -1$. Let d be the degree of C. Since a hyperplane in V will intersect C in d points, its inverse image H_i in Y_i will meet the exceptional divisor E_i in d fibers. Thus

$$H_i E_i = dL = (2g - 2)L = 4L \qquad \text{and} \qquad E_i^2 = -4H_i^2 + 8L.$$

Then we see that

$$H_i^2 E_i = 4H_i L = 0, \qquad H_i E_i^2 = 4E_i L = -4 \qquad \text{and}$$
$$E_i^3 = -4H_i^2 E_i + 8LE_i = -8.$$

In sum, we find the following table of the multiplication of the intersection forms on $H^{2*}(Y_i, \mathbb{Z})$:

	H_i^2	L	$H^4(Y_i, \mathbb{Z})$
H_i	4	0	
E_i	0	-1	
$H^2(Y_i, \mathbb{Z})$			

	H_i	E_i	$H^2(Y_i, \mathbb{Z})$
H_i	H_i^2	$4L$	
E_i	$4L$	$-4H_i + 8L$	
$H^2(Y_i, \mathbb{Z})$			

Plugging these values into the products, we find that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 8, \\ e_1^2 e_2 &= (H_1, H_2)^2 (H_1 - E_1, 0) = H_1^3 - H_1^2 E_1 = 4, \\ e_1 e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2 E_1 + H_1 E_1^2 = 4 - 4 = 0, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2 E_1 + 3H_1 E_1^2 - E_1^3 = 4 - 0 + 3 \cdot (-4) - (-8) = 0. \end{aligned}$$

Next we calculate the λ-invariant of the resulting doubling Calabi-Yau 3-fold M. Since V is a degree 4 smooth hypersurface in $\mathbb{C}P^4$, the total Chern classes of V are given by the formula

$$\frac{(1+H)^5}{(1+4H)} = (1 + 5H + 10H^2)(1 - 4H + 16H^2) + O(H^3) = 1 + H + 6H^2 + O(H^3).$$

Hence we find that the second Chern classes of Y_i are given by

$$c_2(Y_i) = \pi_i^*(c_2(V) + \eta_C) - \pi_i^*(c_1(V)) \cdot E_i = 7H_i^2 - H_i E_i \tag{2.1}$$

by [GH, p.610], where η_C denotes the class of the blow-up center $C \in |\mathcal{O}_D(1)|$. Then the products of $c_2(M)$ and e_i $(i = 1, 2)$ are

$$\begin{aligned} e_1 \cdot c_2(M) &= 7H_1^3 - H_1^2 E_1 + 7H_2^3 - H_2^2 E_2 = 56 = 8 \cdot 7, \\ e_2 \cdot c_2(M) &= (7H_1^2 - H_1 E_1)(H_1 - E_1) \\ &= 7H_1^3 - H_1^2 E_1 - 7H_1^2 E_1 + H_1 E_1^2 \\ &= 7 \cdot 4 - 4 = 24 = 8 \cdot 3. \end{aligned}$$

Since the subgroup $\{\, e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M, \mathbb{Z})$ is generated by a single element $3e_1 - 7e_2$, the λ-invariant of M is

$$\begin{aligned} \lambda(M) &= |(3e_1 - 7e_2)^3| = |27e_1^3 - 189e_1^2 e_2 + 441 e_1 e_2^2 - 343 e_2^3| \\ &= |27 \cdot 8 - 189 \cdot 4| = 540. \end{aligned}$$

2.2. **ID** 1**-**17**:** $\mathbb{C}P^3$ **case.** The detailed calculations are written in [Y21]. Hence this subsection only collects the most basic part of computation on the cubic forms and the λ-invariant.

We set $V = \mathbb{C}P^3$, $D \in |\mathcal{O}_V(4)|$, $C \in |\mathcal{O}_D(4)|$ and $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ for $i = 1, 2$, respectively. Then we have $H^2(Y_i) = \mathbb{C}\langle H_i, E_i \rangle$ with $E_i = \pi_i^{-1}(C)$ and $H_i = \pi_i^*(H) \subset Y_i$ for $H \in H^2(V, \mathbb{Z})$. Furthermore, the proper transform D_i of D in Y_i is $4H_i - E_i$ for each i. Then the straightforward computation shows that any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ can be expressed as

$$(a + 4b)(H_1, H_2) - (b + c)(4H_1 - E_1, 0) - c\delta, \qquad \delta := \langle E_1 - 4H_1, 4H_2 - E_2 \rangle .$$

This yields that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (4H_1 - E_1, 0) \rangle$$

up to torsion. Taking $e_1 = (H_1, H_2)$ and $e_2 = (4H_1 - E_1, 0)$ as generators of $H^i(M, \mathbb{Z})$, we see that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 2, \\ e_1^2 e_2 &= (H_1, H_2)^2 (4H_1 - E_1, 0) = 4H_1^3 - H_1^2 E_1 = 4, \\ e_1 e_2^2 &= (H_1, H_2)(4H_1 - E_1, 0)^2 = 16H_1^3 - 8H_1^2 E_1 + H_1 E_1^2 = 0, \\ e_2^3 &= (4H_1 - E_1)^3 = 64H_1^3 - 48H_1^2 E_1 + 12H_1 E_1^2 - E_1^3 = 0. \end{aligned}$$

As we have seen in Section 2.1, the second Chern class of Y_i is $c_2(Y_i) = 22H_i^2 - 4H_i E_i$ for each i. Thus the subgroup $\{\, e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M, \mathbb{Z})$ is generated by $6e_1 - 11e_2$. Then the λ-invariant is $\lambda(M) = |(6e_1 - 11e_2)^3| = 4320$.

3. $(h^{1,1}(M), h^{2,1}(M)) = (2, 44)$ CASE

In this case, the corresponding doubling Calabi-Yau 3-folds are listed in Table 1 with the underlying Fano 3-folds, (a) ID 1-8, (b) ID 1-9 and (c) ID 1-10. We remark that these Fano 3-folds have the following geometric description:

(a) a section of Plücker embedding of $\mathrm{SGr}(3,6)$ by codimension 3 subspace, where $\mathrm{SGr}(3,6)$ is the Lagrangian Grassmannian; $V(1,1,1) \hookrightarrow \mathrm{SGr}(3,6)$,
(b) a section of $\mathrm{G_2Gr}(2,7)$ by codimension 2 subspace; $V(1,1) \hookrightarrow \mathrm{G_2Gr}(2,7)$, and
(c) the zero locus of $\left(\bigwedge^2 \mathcal{V}^\vee\right)^{\oplus 3}$ on $\mathrm{Gr}(3,7)$ where $\mathcal{V} \to \mathrm{Gr}(3,7)$ is the tautological rank 3 vector bundle over the Grassmannian $\mathrm{Gr}(3,7)$.

In the above description (b), $\mathrm{G_2Gr}(2,7)$ denotes the adjoint G_2-Grassmannian which is the zero locus of the section $s \in \bigwedge^3 \mathbb{C}^7$ corresponding to the G_2-invariant 3-form. See [FG], [IsPr99, Chapter 4], [D08, Section 5] for more details. Systematically, all of these Fano 3-folds are expressed as anticanonically embedded Fano 3-folds $V = V_{2g-2} \subset \mathbb{C}P^{g+1}$ with Picard number 1 and genus g. Moreover, we may assume that $\mathrm{Pic}(V) = H \cdot \mathbb{Z}$ where H is the unique generator of $H^2(V, \mathbb{Z})$ and $H = -K_V$ for each case (a) $g = 9 : V_{16} \subset \mathbb{C}P^{10}$, (b) $g = 10 : V_{18} \subset \mathbb{C}P^{11}$ and (c) $g = 12 : V_{22} \subset \mathbb{C}P^{13}$, respectively.

3.1. **ID 1-9: $V_{18} \subset \mathbb{C}P^{11}$ case.**

Firstly, we consider case (b). Let $V = V_{18} \subset \mathbb{C}P^{11}$ be an anticanonically embedded Fano 3-fold with genus $g = 10$, $\mathrm{Pic}(V) = \mathbb{Z} \cdot H$ and $-K_V = H$. Here and hereafter, we use the same notation as in Section 2. According to [FG], we have $-K_V^3 = 18$ and

$$h^{p,q}(V) = \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 2 & & 2 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} . \tag{3.1}$$

Let $D \in |\mathcal{O}_V(1)|$ be an anticanonical divisor and $C \in |\mathcal{O}_D(1)|$ a smooth curve in D. Setting Y_i to be two copies of the blow-up $\mathrm{Bl}_C(V)$ for $i = 1, 2$, we see that $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ and $H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0)\rangle$ up to torsion. This yields that generators of $H^2(M, \mathbb{Z})$ are given by $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$.

In the same manner as the previous computation in Section 2.1, we find that $H_i^3 = 18$, $H_iL = 0$ and $E_iL = -1$ where L is a fiber over a point on C under the blow-up. Moreover, for $d = \deg C$, we have

$$\begin{aligned} H_iE_i &= dL = (2g-2)L = 18L \qquad \text{and} \\ H_i^2E_i &= H_i(H_iE_i) = 18H_iL = 0. \end{aligned}$$

Let $\tau = 2g$ be the number of branches of the double curve $Y_i \supset \widetilde{C} \xrightarrow{2:1} C \subset V$. By the list in [GH, p.623], we see that

$$\begin{aligned} E_i^2 &= -dH_i^2 + (4d + 2g - 2 - 2\tau)L \\ &= -18H_i^2 + (72 + 20 - 2 - 40)L = -18H_i^2 + 50L, \\ H_iE_i^2 &= H_i(-18H_i^2 + 50L) = -18H_i^3 + 50H_iL = -18 \cdot 18 = -324, \\ E_i^3 &= E_i(-18H_i^2 + 50L) = -18E_iH_i^2 + 50E_iL = -50. \end{aligned}$$

Consequently, we have the following table of the multiplication of the intersection forms on $H^{2*}(Y_i, \mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	18	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$18L$	
E_i	$18L$	$-18H_i^2+50L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic products, we see that

$$\begin{aligned} e_1^3 &= (H_1,H_2)^3 = H_1^3 + H_2^3 = 36, \\ e_1^2e_2 &= (H_1,H_2)^2(H_1-E_1,0) = H_1^3 - H_1^2E_1 = 18, \\ e_1e_2^2 &= (H_1,H_2)(H_1-E_1,0)^2 = H_1^3 - 2H_1^2E_1 + H_1E_1^2 = -306, \\ e_2^3 &= (H_1-E_1,0)^3 = H_1^3 - 3H_1^2E_1 + 3H_1E_1^2 - E_1^3 = -904. \end{aligned}$$

Next we compute the λ-invariant of the doubling Calabi-Yau 3-fold M. Since $V = V_{18} \subset \mathbb{C}P^{11}$ is an anticanonically embedded Fano 3-fold with $-K_V = H$, we see that the first Chern class of V is given by $c_1(V) = H$. In order to find the second Chern class of V, we use the Riemann-Roch-Hirzebruch formula

$$\sum_{q=0}^{n}(-1)^q \dim H^q(V,\Omega^p) = \int_V td(V)\, ch\Big(\bigwedge^p T^*V\Big) \tag{3.2}$$

for $n=3$ and $p=0$. This yields the equality

$$\begin{aligned} &\sum_{q=0}^{3}(-1)^q \dim H^q(V,\Omega^0) = \int_V \left(1+\frac{1}{2}c_1(V)+\frac{1}{12}(c_1(V)^2+c_2(V))+\frac{1}{24}c_1(V)c_2(V)\right) ch\Big(\bigwedge^0 T^*V\Big) \\ \Leftrightarrow \quad & h^{0,0}-h^{0,1}+h^{0,2}-h^{0,3} = \frac{1}{24}\int_V c_1(V)c_2(V) \end{aligned} \tag{3.3}$$

Suppose that $c_2(V) = aH^2$ for $a \in \mathbb{Q}$. Then the Hodge diamond (3.1) and the equality (3.3) imply that

$$\frac{1}{24}\int_V aH^3 = 1 \quad \Leftrightarrow \quad a = \frac{4}{3}$$

by $\int_V H^3 = (-K_V^3) = 18$. Thus, we find $c_2(V) = \frac{4}{3}H^2$. As we have seen in (2.1), the second Chern classes of Y_i are given by

$$\begin{aligned} c_2(Y_i) &= \pi_i^*(c_2(V)+\eta_C) - \pi_i^*(c_1(V))\cdot E_i \\ &= \pi_i^*\left(\frac{4}{3}H^2 + H^2\right) - H_iE_i = \frac{7}{3}H_i^2 - H_iE_i. \end{aligned}$$

Then the products of $c_2(M)$ and e_i are

$$\begin{aligned} e_1\cdot c_2(M) &= \frac{7}{3}H_1^3 - H_1^2E_1 + \frac{7}{3}H_2^3 - H_2^2E_2 = 84 = 6\cdot 14, \\ e_2\cdot c_2(M) &= (H_1-E_1)c_2(Y_1) = (H_1-E_1)\left(\frac{7}{3}H_1^2 - H_1E_1\right) \\ &= \frac{7}{3}H_1^3 + H_1E_1^2 = \frac{7}{3}\cdot 18 + (-324) = -282 = -6\cdot 47. \end{aligned}$$

Since the subgroup $\{\, e \in \langle e_1, e_2\rangle \mid e\cdot c_2(M) = 0 \,\}$ of $H^2(M,\mathbb{Z})$ is generated by $47e_1 + 14e_2$, we see that the λ-invariant of M is given by

$$\lambda(M) = |(47e_1+14e_2)^3| = |47^3e_1^3 + 3\cdot 47^2\cdot 14\cdot e_1^2e_2 + 3\cdot 47\cdot 14^2e_1e_2^2 + 14^3e_2^3| = 5529560.$$

3.2. **ID** 1-8**:** $V_{16} \subset \mathbb{C}P^{10}$ **case.** Secondly, we shall consider case (a). We refer the reader to [Y21] for details. The most essential part of the calculation can be summarized as follows.

We suppose that $V = V_{16} \subset \mathbb{C}P^{10}$, $g = 9$, $\mathrm{Pic}(V) = \mathbb{Z} \cdot H$ and $-K_V = H$. Furthermore, we have $-K_V^3 = 16$ and

$$h^{p,q}(V) = \begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 0&&3&&3&&0 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix} .$$

Setting $D \in |\mathcal{O}_V(1)|$, $C \in |\mathcal{O}_D(1)|$ and $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ for $i = 1, 2$, we see that $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ and $H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0)\rangle$ up to torsion. Hence two generators of $H^2(M, \mathbb{Z})$ are taken as $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$. Consequently, we find the values of the cubic forms as follows:

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 32, \\ e_1^2 e_2 &= (H_1, H_2)^2 (H_1 - E_1, 0) = H_1^3 - H_1^2 E_1 = 16, \\ e_1 e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2 E_1 + H_1 E_1^2 = -240, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2 E_1 + 3H_1 E_1^2 - E_1^3 = -708. \end{aligned}$$

As we computed in Section 3.1, the second Chern class of V is calculated by the Riemann-Roch-Hirzebruch formula (3.2), from which we conclude that $c_2(V) = \frac{3}{2}H^2$. Thus the second Chern classes of Y_i are

$$c_2(Y_i) = \pi_i^* \left(\frac{3}{2} H^2 + H^2 \right) - H_i E_i = \frac{5}{2} H_i^2 - H_i E_i$$

for $i = 1, 2$. Then the subgroup $\{ e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \}$ of $H^2(M, \mathbb{Z})$ is generated by $27e_1 + 10e_2$. This implies that the λ-invariant is $\lambda(M) = |(27e_1 + 10e_2)^3| = 1672224$.

3.3. **ID** 1-10**:** $V_{22} \subset \mathbb{C}P^{13}$ **case.** Finally, we consider case (c), that is, $V = V_{22} \subset \mathbb{C}P^{13}$ is an anticanonically embedded Fano 3-fold with genus $g = 12$, $\mathrm{Pic}(V) = \mathbb{Z} \cdot H$ and $-K_V = H$. Note that the unique such 3-fold with $\mathrm{Aut}(V) = \mathrm{PGL}(2, \mathbb{C})$ is called the Mukai-Umemura 3-fold, and we refer the reader to [D08, Ti97] and references therein for more details.

As one can see in [FG], the Hodge diamond of V is

$$h^{p,q}(V) = \begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 0&&0&&0&&0 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix} \tag{3.4}$$

and $-K_V^3 = 22$. Let $D \in |\mathcal{O}_V(1)|$ be an anticanonical divisor, $C \in |\mathcal{O}_D(1)|$ a smooth curve in D and Y_i two copies of the blow-up $\mathrm{Bl}_C(V)$ as usual. Then we see that $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ and $H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0)\rangle$ up to torsion. Hence two generators of $H^2(M, \mathbb{Z})$ are given by $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$. The straightforward computation shows that $H_i^3 = 22$, $H_i L = 0$ and $E_i L = -1$. Furthermore, we have

$$\begin{aligned} H_i E_i &= dL = (2g - 2)L = 22L \qquad \text{and} \\ H_i^2 E_i &= H_i (H_i E_i) = 22 H_i L = 0. \end{aligned}$$

Again, let $\tau = 2g$ be the number of branches of the double curve $\widetilde{C} \xrightarrow{2:1} C \subset V$. Then we see that

$$\begin{aligned} E_i^2 &= -dH_i^2 + (4d + 2g - 2 - 2\tau)L \\ &= -22H_i^2 + (88 + 24 - 2 - 48)L = -22H_i^2 + 72L, \\ H_iE_i^2 &= H_i(-22H_i^2 + 72L) = -22H_i^3 + 72H_iL = -22 \cdot 22 = -484, \qquad \text{and} \\ E_i^3 &= E_i(-22H_i^2 + 72L) = -22E_iH_i^2 + 72E_iL = -72. \end{aligned}$$

Consequently, we have the following table of the multiplication of the intersection forms on $H^{2*}(Y_i, \mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	22	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$22L$	
E_i	$22L$	$-22H_i^2 + 72L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic products, we see that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 44, \\ e_1^2e_2 &= (H_1, H_2)^2(H_1 - E_1, 0) = H_1^3 - H_1^2E_1 = 22, \\ e_1e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2E_1 + H_1E_1^2 = -462, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2E_1 + 3H_1E_1^2 - E_1^3 = -1358. \end{aligned}$$

Now, we compute the λ-invariant. As we have seen in Section 3.1, the first Chern class of V is given by $c_1(V) = H$. In order to calculate the second Chern class of V, we use (3.2) for $n = 3$ and $p = 0$. Then we obtain

$$h^{0,0} - h^{0,1} + h^{0,2} - h^{0,3} = \frac{1}{24}\int_V c_1(V)c_2(V). \tag{3.5}$$

Suppose that $c_2(V) = aH^2$ for $a \in \mathbb{Q}$. Since the left hand side of (3.5) is 1 by (3.4), we see that

$$\frac{1}{24}\int_V aH^3 = 1 \quad \Leftrightarrow \quad a = \frac{12}{11}$$

where we used $\int_V H^3 = (-K_V^3) = 22$. Thus, we find $c_2(V) = \frac{12}{11}H^2$. By (2.1), the second Chern classes of Y_i are

$$\begin{aligned} c_2(Y_i) &= \pi_i^*(c_2(V) + \eta_C) - \pi_i^*(c_1(V)) \cdot E_i \\ &= \pi_i^*\left(\frac{12}{11}H^2 + H^2\right) - H_iE_i = \frac{23}{11}H_i^2 - H_iE_i. \end{aligned}$$

Then the products of $c_2(M)$ and e_i are

$$\begin{aligned} e_1 \cdot c_2(M) &= \frac{23}{11}H_1^3 - H_1^2E_1 + \frac{23}{11}H_2^3 - H_2^2E_2 = 92 = 2 \cdot 46, \\ e_2 \cdot c_2(M) &= (H_1 - E_1)c_2(Y_1) = (H_1 - E_1)\left(\frac{23}{11}H_1^2 - H_1E_1\right) \\ &= \frac{23}{11}H_1^3 + H_1E_1^2 = \frac{23}{11} \cdot 22 + (-484) = -438 = -2 \cdot 219. \end{aligned}$$

Since the subgroup $\{\, e \in \langle e_1, e_2\rangle \mid e \cdot c_2(M) = 0 \,\}$ of $H^2(M, \mathbb{Z})$ is generated by $219e_1 + 46e_2$, we see that

$$\lambda(M) = |(219e_1 + 46e_2)^3| = |219^3e_1^3 + 3 \cdot 219^2 \cdot 46 \cdot e_1^2e_2 + 3 \cdot 219 \cdot 46^2e_1e_2^2 + 46^3e_2^3| = 122507896.$$

4. $(h^{1,1}(M), h^{2,1}(M)) = (2, 58)$ CASE

Now we consider the case where the doubling Calabi-Yau 3-folds have the same Hodge numbers $(h^{1,1}(M), h^{2,1}(M)) = (2, 58)$, that is, the underlying Fano 3-folds are (a) ID 1-4, (b) ID 1-12 and (c) 1-14. These Fano 3-folds are described as follows:

(a) a complete intersection of three quadrics in $\mathbb{C}P^6$; $V(2,2,2) \subset \mathbb{C}P^6$,
(b) a hypersurface of degree 4 in the weighted projective space $\mathbb{C}P(1,1,1,1,2)$; $V(4) \subset \mathbb{C}P^4(1^4, 2)$, and
(c) a complete intersection of two quadrics in $\mathbb{C}P^5$; $V(2,2) \subset \mathbb{C}P^5$.

4.1. **ID 1-14:** $V(2,2) \subset \mathbb{C}P^5$ **case.** Let V be a smooth complete intersection of 3 quadrics in $\mathbb{C}P^5$, which is the Fano 3-fold with $-K_V^3 = 32$ and

$$h^{p,q}(V) = \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 2 & & 2 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} .$$

By the adjunction formula, we see that

$$K_{V(2)} \cong (K_{\mathbb{C}P^5} + [V(2)])\big|_{V(2)} = -4H, \quad \text{and}$$
$$K_V \cong \big(K_{V(2)} + [V]\big)\big|_V = (-4+2)H = -2H$$

where $H \in H(V, \mathbb{Z})$ is the ample generator and $V(2) \subset \mathbb{C}P^5$ is a smooth quadric hypersurface in $\mathbb{C}P^5$. Let $D = 2H \in |-K_V|$ be an anticanonical divisor and $C \in |\mathcal{O}_D(2)|$ a smooth curve in D representing the intersection class of $D \cdot D$. For $i = 1, 2$, we take the blow-ups $Y_i = \mathrm{Bl}_C(V)$ which have the cohomology rings $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$. Then the proper transforms D_i of D in Y_i are $2H_i - E_i$. Thus we set δ by $\langle -D_1, D_2\rangle = \langle E_1 - 2H_1, 2H_2 - E_2\rangle$. We observe that any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ is written as

$$(aH_1 + bE_1, cE_2 + (a + 2b - 2c)H_2) = (a + 2b)(H_1, H_2) - (b + c)(2H_1 - E_1, 0) - c\delta.$$

Consequently, we find that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (2H_1 - E_1, 0)\rangle$$

up to torsion. This implies that two generators of $H^2(M, \mathbb{Z})$ can be taken as $e_1 = (H_1, H_2)$ and $e_2 = (2H_1 - E_1, 0)$.

In order to compute the cubic forms in $H^6(M, \mathbb{Z})$, we first see that the Fano genus g of V is

$$g = \frac{-K_V^3}{2} + 1 = \frac{32}{2} + 1 = 17.$$

Then the straightforward computation shows that $H_i^3 = 32$, $H_i L = 0$ and $E_i L = -1$ where L is a fiber over a point on C under the blow-up. Furthermore, for $d = \deg C$, we have

$$H_i E_i = dL = (2g-2)L = 32L \qquad \text{and}$$
$$H_i^2 E_i = H_i(H_i E_i) = 32 H_i L = 0.$$

In the same manner as in Section 3, let us denote the number of branches of the double curve $\widetilde{C}$ by τ. Then we find that

$$E_i^2 = -dH_i^2 + (4d + 2g - 2 - 2\tau)L = -32H_i^2 + (128 + 34 - 2 - 68)L = -32H_i^2 + 92L,$$
$$H_i E_i^2 = H_i(-32H_i^2 + 92L) = -32H_i^3 + 92H_i L = -32 \cdot 32 = -1024, \qquad \text{and}$$
$$E_i^3 = E_i(-32H_i^2 + 92L) = -32E_i H_i^2 + 92E_i L = -92.$$

In the following table, we summarize the values of the multiplication of the intersection forms on $H^{2*}(Y_i,\mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	32	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$32L$	
E_i	$32L$	$-32H_i^2+92L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic forms, we find that

$$\begin{aligned} e_1^3 &= (H_1,H_2)^3 = H_1^3+H_2^3 = 64,\\ e_1^2e_2 &= (H_1,H_2)^2(2H_1-E_1,0) = 2H_1^3 - H_1^2E_1 = 64,\\ e_1e_2^2 &= (H_1,H_2)(2H_1-E_1,0)^2 = 4H_1^3-4H_1^2E_1+H_1E_1^2 = 4\cdot 32-1024=-896,\\ e_2^3 &= (2H_1-E_1,0)^3 = 8H_1^3-12H_1^2E_1+6H_1E_1^2-E_1^3 = 8\cdot 32+6\cdot(-1024)-(-92) = -5796. \end{aligned}$$

Next we compute the λ-invariant. Since V is a complete intersection of two quadrics in $\mathbb{C}P^5$, the total Chern classes of V are given by the formula

$$\begin{aligned} \frac{(1+H)^6}{(1+2H)^2} &= \left(1+6H+\binom{6}{2}H^2\right)\left(1+2H\right)^{-2}+O(H^3)\\ &= (1+6H+15H^2)(1-4H+12H^2)+O(H^3) = 1+2H+3H^2+O(H^3). \end{aligned}$$

Hence the second Chern classes of Y_i are computed as

$$\begin{aligned} c_2(Y_i) &= \pi_i^*(c_2(V)+\eta_C)-\pi_i^*(c_1(V))\cdot E_i\\ &= \pi_i^*(3H^2+4H^2)-2H_iE_i = 7H_i^2-2H_iE_i. \end{aligned}$$

Then the products of $c_2(M)$ and e_i are given by

$$\begin{aligned} e_1\cdot c_2(M) &= 7H_1^3-2H_1^2E_1+7H_2^3-2H_2^2E_2 = 448 = 2^6\cdot 7,\\ e_2\cdot c_2(M) &= (2H_1-E_1)(7H_1^2-2H_1E_1)\\ &= 14H_1^3-4H_1^2E_1-7H_1^2E_1+2H_1E_1^2\\ &= 14\cdot 32-2\cdot 2^{10} = -1600 = 2^6\cdot(-25). \end{aligned}$$

Since the subgroup $\{\, e\in\langle e_1,e_2\rangle \mid e\cdot c_2(M)=0 \,\}$ of $H^2(M,\mathbb{Z})$ is generated by a single element $25e_1+7e_2$, the λ-invariant of M is

$$\begin{aligned} \lambda(M) &= |(25e_1+7e_2)^3| = |25^3e_1^3+3\cdot 25^2\cdot 7e_1^2e_2+3\cdot 25\cdot 7^2e_1e_2^2+7^3e_2^3|\\ &= |25^3\cdot 64+3\cdot 25^2\cdot 7\cdot 64+3\cdot 25\cdot 7^2\cdot(-896)+7^3\cdot(-5796)| = 3440828. \end{aligned}$$

4.2. **ID 1-12: $V(4)\subset \mathbb{C}P(1^4,2)$ case.** Let V be a smooth hypersurface of degree 4 in the weighted projective space $\mathbb{C}P^4(1^4,2)$, which is the Fano 3-fold with $-K_V^3=16$ and

$$h^{p,q}(V) = \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 10 & & 10 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}\ .$$

By the adjunction formula, we find that

$$K_V \cong (K_{\mathbb{P}}+[V])\big|_V = (\mathcal{O}_{\mathbb{P}}(-6)+\mathcal{O}_{\mathbb{P}}(4))\big|_V = \mathcal{O}_{\mathbb{P}}(-2)\big|_V = \mathcal{O}_V(-2)$$

where we denote the weighted projective space $\mathbb{C}P^4(1^4,2)$ by $\mathbb{P}$. Let $D=2H\in|-K_V|$ be a smooth anticanonical divisor and $C\in|\mathcal{O}_D(2)|$ a smooth curve in D. Let $Y_i=\mathrm{Bl}_C(V)$ be the blow-ups of V along C and $H^2(Y_i) = \mathbb{C}\langle H_i, E_i\rangle$ the cohomology rings of Y_i for $i = 1, 2$. For the proper transforms $D_i = 2H_i - E_i$ of D in Y_i, we set δ by $\langle -D_1, D_2\rangle = \langle E_1 - 2H_1, 2H_2 - E_2\rangle$. Repeating the same computation in Section 4.1, we see that two generators of $H^2(M,\mathbb{Z})$ are $e_1 = (H_1, H_2)$ and $e_2 = (2H_1 - E_1, 0)$.

Now we compute the cubic products of e_i in $H^6(M,\mathbb{Z})$. Firstly, the genus of the Fano 3-fold V is given by

$$g = \frac{-K_V^3}{2} + 1 = \frac{16}{2} + 1 = 9.$$

Secondly, we readily see that

$$\begin{aligned} H_i^3 &= 16, \qquad H_iL = 0, \qquad E_iL = -1 \\ H_iE_i &= dL = (2g-2)L = 16L, \qquad \text{and} \\ H_i^2E_i &= H_i(H_iE_i) = 16H_iL = 0. \end{aligned}$$

Let $\tau = 2g$ be the number of branches of the double curve $\widetilde{C}$. Then we find that

$$\begin{aligned} E_i^2 &= -dH_i^2 + (4d + 2g - 2 - 2\tau)L = -16H_i^2 + (64 + 18 - 2 - 36)L = -16H_i^2 + 44L, \\ H_iE_i^2 &= H_i(-16H_i^2 + 44L) = -16H_i^3 + 44H_iL = -16 \cdot 16 = -256, \qquad \text{and} \\ E_i^3 &= E_i(-16H_i^2 + 44L) = -16E_iH_i^2 + 44E_iL = -44. \end{aligned}$$

The following table collects the values of the multiplication of the intersection forms on $H^{2*}(Y_i,\mathbb{Z})$:

	H_i^2	L	$H^4(Y_i,\mathbb{Z})$
H_i	16	0	
E_i	0	-1	
$H^2(Y_i,\mathbb{Z})$			

	H_i	E_i	$H^2(Y_i,\mathbb{Z})$
H_i	H_i^2	$16L$	
E_i	$16L$	$-16H_i^2 + 44L$	
$H^2(Y_i,\mathbb{Z})$			

Substituting these values into the cubic forms, we find that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 32, \\ e_1^2e_2 &= (H_1, H_2)^2(2H_1 - E_1, 0) = 2H_1^3 - H_1^2E_1 = 32, \\ e_1e_2^2 &= (H_1, H_2)(2H_1 - E_1, 0)^2 = 4H_1^3 - 4H_1^2E_1 + H_1E_1^2 = 4 \cdot 16 - 256 = -192, \\ e_2^3 &= (2H_1 - E_1, 0)^3 = 8H_1^3 - 12H_1^2E_1 + 6H_1E_1^2 - E_1^3 = 8 \cdot 16 + 6 \cdot (-256) - (-44) = -1364. \end{aligned}$$

Let us compute the λ-invariant. Since V is a hypersurface of degree 4 in the weighted projective space $\mathbb{C}P^4(1^4, 2)$, the total Chern classes of V are given by

$$\begin{aligned} \frac{(1+H)^4(1+2H)}{(1+4H)} &= \big(1 + 4H + \binom{4}{2} H^2\big)\big(1+2H\big)\big(1+4H\big)^{-1} + O(H^3) \\ &= (1 + 4H + 6H^2)(1+2H)(1 - 4H + 16H^2) + O(H^3) \\ &= 1 + 2H + 6H^2 + O(H^3). \end{aligned}$$

Thus the second Chern classes of Y_i are

$$c_2(Y_i) = \pi_i^*(6H^2 + 4H^2) - 2H_iE_i = 10H_i^2 - 2H_iE_i.$$

Then we see that the products of $c_2(M)$ and e_i are

$$\begin{aligned} e_1 \cdot c_2(M) &= 10H_1^3 - 2H_1^2E_1 + 10H_2^3 - 2H_2^2E_2 = 320 = 2^6 \cdot 5, \\ e_2 \cdot c_2(M) &= (2H_1 - E_1)(10H_1^2 - 2H_1E_1) \\ &= 20H_1^3 - 4H_1^2E_1 - 10H_1^2E_1 + 2H_1E_1^2 \\ &= 20 \cdot 16 + 2 \cdot (-256) = -192 = 2^6 \cdot (-3). \end{aligned}$$

Since the subgroup $\{ e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \}$ of $H^2(M, \mathbb{Z})$ is generated by a single element $3e_1 + 5e_2$, the λ-invariant of M is

$$\begin{aligned}\lambda(M) &= |(3e_1 + 5e_2)^3| = |3^3 e_1^3 + 3 \cdot 3^2 \cdot 5 e_1^2 e_2 + 3 \cdot 3 \cdot 5^2 e_1 e_2^2 + 5^3 e_2^3| \\ &= |27 \cdot 32 + 3 \cdot 27 \cdot 5 \cdot 32 + 9 \cdot 25 \cdot (-192) + 125 \cdot (-1364)| = 208516.\end{aligned}$$

4.3. **ID** 1-4**:** $V(2,2,2) \subset \mathbb{C}P^6$ **case.** We refer the reader to [Y21] for the detailed computation of this example. This subsection collects the minimum amount of calculation necessary to see the values of the cubic forms and the λ-invariants.

Let $V = V(2,2,2) \subset \mathbb{C}P^6$ be a complete intersection of three quadrics in $\mathbb{C}P^6$. As usual, we set $D \in |\mathcal{O}_V(1)|$, $C \in |\mathcal{O}_D(1)|$ and $\pi_i : Y_i = \mathrm{Bl}_C(V) \dashrightarrow V$ for $i = 1, 2$. Then we see that the proper transform D_i of D in Y_i is $H_i - E_i$ and $H^2(Y_i) = \mathbb{C} \langle H_i, E_i \rangle$ for each i. Thus any element in $H^2(Y_1, \mathbb{Z}) \times H^2(Y_2, \mathbb{Z})$ can be written as

$$(a+b)(H_1, H_2) - (b+c)(H_1 - E_1, 0) - c\delta, \qquad \delta := \langle E_1 - H_1, H_2 - E_2 \rangle .$$

This implies that

$$H^2(M, \mathbb{Z}) \cong \langle (H_1, H_2), (H_1 - E_1, 0) \rangle$$

up to torsion. Setting $e_1 = (H_1, H_2)$ and $e_2 = (H_1 - E_1, 0)$ as generators of $H^i(M, \mathbb{Z})$, we find that

$$\begin{aligned} e_1^3 &= (H_1, H_2)^3 = H_1^3 + H_2^3 = 16, \\ e_1^2 e_2 &= (H_1, H_2)^2 (H_1 - E_1, 0) = H_1^3 - H_1^2 E_1 = 8, \\ e_1 e_2^2 &= (H_1, H_2)(H_1 - E_1, 0)^2 = H_1^3 - 2H_1^2 E_1 + H_1 E_1^2 = -56, \\ e_2^3 &= (H_1 - E_1, 0)^3 = H_1^3 - 3H_1^2 E_1 + 3H_1 E_1^2 - E_1^3 = -164. \end{aligned}$$

In the same manner as the previous calculation in Section 4.1, the second Chern class of Y_i is $c_2(Y_i) = 4H_i^2 - H_i E_i$ for each i. Consequently, the subgroup $\{ e \in \langle e_1, e_2 \rangle \mid e \cdot c_2(M) = 0 \}$ of $H^2(M, \mathbb{Z})$ is generated by $e_1 + 2e_2$. Hence we conclude that the λ-invariant is $\lambda(M) = |(e_1 + 2e_2)^3| = 1920$.

Acknowledgement. The author would like to thank the organizers of the conference "Singularity theory of smooth maps and its applications" for giving him an opportunity to publish this note. This work was partially supported by JSPS KAKENHI Grant Number 18K13406 and 22K03316.

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