On triality relations of certain eight dimensional algebras

– ある 8 次元代数の三対原理–

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この小論では 8 次元代数 (symmetric composition algebra) の三対原理，つまり自己同型群と微分の一般化について論究します．以下のような章です．

§0. Introduction, §1. Gell-Mann's pseudo octonion algebra, §2. Automorphisms and Derivations, §3. Triality relations of algebras, §4. Symmetric composition algebras, §5. Miscellaneous results, References.

この様な内容について，具体的実例を含め述べさせていただきます (又，これらの section は独立に読める様に心がけたつもりです).

§0. Introduction

我々の三対原理の concept を複素数 $\mathbf{C}$ のとき考えます．$\mathbf{C}$ の普通の積を $*$，共役元を $\bar{x} = a - ib (x = a + ib,\ a, b$ は実数 $i = \sqrt{-1})$ とするとき，new product を $xy = \overline{x * y}$ によって定義します．この **new product** で

$$g(xy) = (gx)(gy),\ d(xy) = (dx)y + x(dy)$$

なる自己同型 g と微分 d を考えると，周期 2π で以下のようになります．

$$Aut\ \mathbf{C} = \{e^{i\theta} | \theta = \frac{2\pi}{3}n,\ n : integer\} = \{1, \frac{-1+\sqrt{3}\ i}{2}, \frac{-1-\sqrt{3}\ i}{2}\} \cong S_3$$

$$Der\ \mathbf{C} = \{d \in End\ \mathbf{C} | d = \frac{2\pi}{3}in,\ n : integer\} = \{0, \frac{2\pi i}{3}, \frac{4\pi i}{3}\}$$

が成り立ちます．**product (積)** に注意してください．

$j = 0, 1, 2$ に対して (自然数 j を mod 3 で考えます), $End\ \mathbf{C}$ の元で，

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y),\ d_j(xy) = (d_{j+1}x)y + x(d_{j+2}y)$$

となる $g_j,\ d_j$ を考えると，次のような $Aut\ \mathbf{C}, Der\ \mathbf{C}$ の一般化が得られます．

$$Trig\ \mathbf{C} = \{(g_0, g_1, g_2) | \alpha_0 + \alpha_1 + \alpha_2 = 0,\ e^{i\ \alpha_j} = g_j,\ \alpha_j \in Re\ \mathbf{C}\}$$

$$Trid\ \mathbf{C} = \{(d_0, d_1, d_2) | \alpha_0 + \alpha_1 + \alpha_2 = 0,\ i\ \alpha_j = d_j,\ \alpha_j \in Re\ \mathbf{C}\}.$$

次に $n \times n$ 次の行列代数 A で考察すると，$j = 0, 1, 2$ に対して

$$\sigma_j(a)x = a_j * x *^t a_{j+1},\quad d_j(p)x = p_j * x - x * p_{j+1}.$$

ただし $a_j \in A_0^* := \{b \in A | b *^t b = Id_n\}$, $p_j \in Alt(A) := \{c \in A |\ ^t c = -c\}$,

($x*y$ は standard product of matrix，${}^t x$ は転置行列です．)

と定義する．ここで $xy = {}^t(x*y)$ と **new product** を与えると
$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y)$, $d_j(p)(xy) = (d_{j+1}(p)x)y + x(d_{j+2}(p)y)$, $p_j = (1-a_j)*(1+a_j)^{-1}$ (Cayley transformation) が成り立ちます．簡単な動機ですが，これらの概念を単位元を持たない非結合的代数で以下考えます．

§1. Gell-Mann's pseudo octonian algebra

この節ではノーベル賞を受賞した Gell-Mann の Baryon と Meson の記述に表れる 8 次元代数 [G] を以下，仁科賞と Wigner medal を授与されている S.Okubo の本 [O] より抜粋させていただきます（余談ですが筆者は大久保進氏と多数の共著論文が存在します）．

Let $\lambda_j (j=1,\cdots,8)$ be the 3×3 traceless Hermitian matrices introduced by Gell-Mann. Here denote $i=\sqrt{-1}$. Their explicit forms are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}, \tag{1.1}$$

which satisfy conditions with the standard matrix product $\lambda_j * \lambda_k$ below.

$$\lambda_j = {}^t\overline{\lambda}_j,\ Tr\lambda_j = 0 \text{ and } Tr(\lambda_j * \lambda_k) = 2\delta_{jk}. \tag{1.2}$$

Moreover we have

$$\lambda_j * \lambda_k = \frac{2}{3}\delta_{jk}E + \sum_{l=1}^{8}(d_{jkl} + if_{jkl})\lambda_l \tag{1.3}$$

where $d_{jkl} = \frac{1}{4}Tr(\lambda_j * \lambda_k + \lambda_k * \lambda_j) * \lambda_l$, $f_{jkl} = \frac{-i}{4}Tr(\lambda_j * \lambda_k - \lambda_k * \lambda_j) * \lambda_l$.

We remark that f_{jkl} defined by (1.3) are the structure constants of the $su(3)$ Lie algebra with respect to the product $[x,y]^* = x*y - y*x$ as follows;

$$[\frac{1}{2}\lambda_j, \frac{1}{2}\lambda_k]^* = i\sum_{l=1}^{8} f_{jkl}(\frac{1}{2}\lambda_l). \tag{1.4}$$

Now we expand any elements X, Y in terms of λ_j by

$$X = \sum_{j=1}^{8}\sqrt{3}x_j\lambda_j,\ Y = \sum_{j=1}^{8}\sqrt{3}y_j\lambda_j \tag{1.5}$$

and define a **new product** by

$$Z = XY = \sum_{j=1}^{8} \sqrt{3} z_j \lambda_j, \text{ where } z_j = \sum_{k,l=1}^{8} (\sqrt{3} d_{jkl} + f_{jkl}) x_k y_l. \tag{1.6}$$

This algebra with the product XY is said to be a *pseudo octonion algebra.*

Then by using a symmetric bilinear form $< X|Y >= \frac{1}{6} Tr(X * Y)$ where $X * Y$ denote the standard product of matrix, we can show that

$$< X|Y >= \sum_{j=1}^{8} x_j y_j, \text{ and } (\sum_{j=1}^{8} x_j^2)(\sum_{j=1}^{8} y_j^2) = \sum_{j=1}^{8} z_j^2. \tag{1.7}$$

That is, in the new product XY, we obtain the composition law

$$< XY|XY >=< X|X >< Y|Y >, \tag{1.8}$$

and the associativity

$$< XY|Z >=< X|YZ > . \tag{1.9}$$

Note that these relations

$$< XY|XY >=< X|X >< Y|Y > \text{ and } \quad < XY|Z >=< X|YZ >$$

$$\Longleftrightarrow_{iff} (XY)X = X(YX) =< X|X > Y, \qquad (\spadesuit)$$

if $< X|Y >$ is a non-derenerate and symmetric bilinear form.

This algebra satisfying $Eqs(1.8)$ *and* (1.9) with non-degenerate symmetric bilinear form $< X|Y >$ is said to be a *symmetric composition algebra* ([K-O.2],[K-O.3]), and it contains a class of the pseudo octonion algebra.

To next section, we will consider the pseudo octonion algebra with respect to the orthogornal basis vectors

$$e_j = \sqrt{3} \lambda_j \tag{1.10}$$

with the product given by

$$e_j e_k = \sum_{l=1}^{8} (\sqrt{3} d_{jkl} + f_{jkl}) e_l \tag{1.11}$$

equipped with $< e_j|e_k >= \delta_{jk}$. Note that the product (1.11) is same as (1.6).

この algebra を今後の為に A と表す．一方,By new product $xy = \mu x * y + \nu y * x - \frac{1}{3} Tr(x * y) E$, where $\mu + \nu = 1,\ 3\mu\nu = 1$, この xy の積で A は ($\spadesuit$) を満たします．そして，$e_1, \cdots, e_8$ を直交基底として (1.11) の積で考えます．$A = span < e_1, \cdots, e_8 >$. A is a nonassociative algebra without unit element.

A は simple ですが，$B = span < e_1, e_8 >$ with $dim\ B = 2$ を考えると，この B は $A = span < e_1, \cdots, e_8 >$ の subalgebra です．又 $C = span < e_3, e_4, e_5, e_8 >$ with $dim\ C = 4$ も subalgebra of A, since $e_4 e_5 = \frac{1}{2} e_3 + \frac{\sqrt{3}}{2} e_8$, $e_3 e_8 = e_3$, etc.

Remark. d を pseudo octonion alg. A defined by (1.6) の任意の derivation とすると，$\forall x,\ y \in A$ に対して $d(xy) = (dx)y + x(dy)$ より，$d[x,y] = [dx,y] + [x,dy]$and $[x,y] = xy - yx = (\mu-\nu)(x*y - y*x) = (\mu-\nu)[x,y]^*$ から $(A,[x,y])$ は 8 次元の A_2 type の simple Lie algebra となり，その derivation は内部微分です．つまり d は (1.11) により定義された積で，8 次元単純リー代数

$$span < ad\ e_1, ad\ e_2, \cdots, ad\ e_8 > (= Inner\ Der(A,[x,y])) = A_2\ type$$

の元として表され，$ad\ e_j\ (j = 1, \cdots, 8)$ の 1 次結合です．

行列代数の場合，$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ とおくと $exp\ t\ B = \begin{pmatrix} cos\ t & -sin\ t \\ sin\ t & cos\ t \end{pmatrix}$ が成立し，交代行列と回転群の対応が存在します．これと同様に

Pseudo octonion algebra A で $(e_1, \cdots, e_8) \to (e'_1, \cdots, e'_8)$ なる $End\ A$ で特に $\hat{d} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ とおき，$d = \begin{pmatrix} \mathbf{0} & & & \\ & \hat{d} & & \\ & & \hat{d} & \\ & & & 0 \end{pmatrix}$ なる 8×8 行列の 1 次写像を考えます．この時 $\forall x, y \in A$ に対して，for the product xy is defined by (1.11),

$$d(xy) = (dx)y + x(dy)$$

が成り立ちます．一方

$$\hat{g} = \begin{pmatrix} cos\frac{4}{3}\pi & -sin\frac{4}{3}\pi \\ sin\frac{4}{3}\pi & cos\frac{4}{3}\pi \end{pmatrix}$$

とおき，$g = \begin{pmatrix} Id_3 & & & \\ & \hat{g} & & \\ & & \hat{g} & \\ & & & 1 \end{pmatrix}$ なる 1 次変換 g を考えると，$g(xy) = (gx)(gy)$ が成立します．つまり A における derivation と automorphism の一例です．行列と複素数の automorphism and derivation の場合の拡張を考えることができます．

§2. Automorphisms and Derivations of pseudo octonion algebra

From now on, we assume that the algebra A is a symmeric composition algebra ([K-O.2],[K-O.3]) over $ch\ F \neq 2,\ 3$ in this section.

Let

$$\Sigma = \{a = (a_1, a_2, a_3) \in A^3 | \\ a_j a_{j+1} = a_{j+2},\ < a_j | a_j >= 1,\ \forall j = 1,2,3\}. \quad (2.1)$$

For a given $a = (a_1, a_2.a_3) \in \Sigma$, we introduce a notation

$$\Lambda(a) = \{p = (p_1, p_2, p_3) \in A^3 | \\ a_j p_{j+1} + p_j a_{j+1} = p_{j+2}, < p_j | a_j >= 0,\ \forall j = 1,2,3\} \quad (2.2)$$

Note that $\Lambda(a)$ is a vector space over F.

Moreover, we define $q_j \in A$ by

$$q_j = a_{j+1}p_{j+2} = p_j - p_{j+1}a_{j+2}. \tag{2.3}$$

From (Th.3.2 in [K-O.3]) and the notation being as avobe, we have the following.

Theorem 2.1 *For any $a \in \Sigma$ and $p \in \Lambda(a)$, if we introduce $D_j(a,p) \in End\ A$ by*

$$D_j(a,p)x = (p_{j+1}x)a_{j+1} + a_j(xq_j). \tag{2.4}$$

Then they satisfy

$$D_j(a,p)(xy) = (D_{j+1}(a,p)x)y + x(D_{j+2}(a,p)y) \tag{2.5}$$

*i.e., $(D_1(a,p), D_2(a,p), D_3(a,p)) \in Trid\ A$ (see §**3** for this notation).*

Corollary *For $\forall a \in A, s.t\ a^2 = a$ and $< a|a >= 1$, also $\exists p \in A, s.t\ ap + pa = p$ and $< a|p >= 0$. Then $D(a,p)x = (px)a + a(xq)$ is a derivation of A, where $q = ap$. This means the special case of $a = a_1 = a_2 = a_3$ and $p = p_1 = p_2 = p_3$ in Theorem 2.1.*

この $Cor.$ を満たす idempotent elements a を pseudo octonion algebra において explicitness に求める．§1 の $e_1, \cdots, e_8$ なる基底を用いる．

$$(a = -e_8, p = e_4),\ (a = -\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_8,\ p = e_2),$$

$$(a = -\frac{\sqrt{3}}{2}e_2 + \frac{1}{2}e_8, p = e_3)\ (a = -\frac{\sqrt{3}}{2}e_3 + \frac{1}{2}e_8,\ p = e_1)$$

$$(a = ie_4 + e_5 - e_8,\ p = e_6 - ie_7),\ (a = ie_6 + e_7 - e_8,\ p = e_4 - ie_5)$$

$$(a = e_4 - ie_5 + e_6 - ie_7 - e_8,\ p = e_4 - ie_5)$$

$$(a = \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_8,\ p = e_1 - e_2).$$

これらの対 (a,p) が Cor の仮定を満たす．これらの 8 個の対より作られた derivations $D(a,p)$ を $D(a_j,p_j)$ $(j = 1, \cdots 8)$ と表す．つまり $D(a_1,p_1) = D(-e_8, e_4) = D_1$ 等と書くことにする．§ 1 の結果 (Remark) より $D(a_j,p_j)$ は次が成り立つ．

$$Der\ A = span < D(a_1,p_1), \cdots, D(a_8,p_8) > \cong A_2 \text{ (simple Lie algebra of } dim\ 8),$$

$$< D(a_j,p_j)x|y >= - < x|D(a_j,p_j)y > .$$

Next we put (by notation of $D_j = D(a_j,p_j)$, $j = 1, \cdots, 8$)

$$\xi_j = exp\ D(a_j,p_j)(= Id + D_j + \frac{D_j^2}{2!} + \cdots) \tag{2.6}$$

and

$$\eta_j = \xi_1^{-1}\xi_j (j = 1, \cdots, 8),\ \text{(that is } \eta_1 = Id).$$

Furthermore, we set

$$G =< Id, \eta_2, \cdots, \eta_8 >_{span} \ \text{(as a group)}, \ then \ \forall \ g \in G, < gx|gy > = < x|y > .$$

On the other hand, for 3×3 matrix $M(3, C)$ over the complex number field C, we set

$$GL(3, C) := \{A \in M(3, C) | det \ A \neq 0\} \ \text{(general linear group)},$$

$$Z(GL(3, C)) := \{\lambda Id_3 | \lambda \neq 0\} \ \text{(center)},$$

$$PGL(3, C) := GL(3, C)/Z(GL(3, C)) \ \text{(projective general linear group)}.$$

Then we obtain

$$PGL(3, C) \cong G.$$

Indeed, if $f : GL(3, C) \to G$ を次の様な準同型とするならば，$\lambda Id_3 \mapsto Id$, and $f^{-1}(\eta_j) \mapsto \eta_j$ $(i = 2, \cdots, 8)$ より上の同型を求めることが可能です.

§3. Triality relations of algebras (代数における三対原理)

少し長くなりますが，この節で非結合的代数系における一般的な triality relations (local and global の cases) を筆者の視点で述べさせていただきます.

Let A be a nonassociative algebra over $ch \ F \neq 2$ (以下において，algebra は有限次元そして必ずしも単位元をもたない場合も仮定する）.

Following ([K-O.2] or [K-O.3]), suppose that a triple $g = (g_1, g_2, g_3) \in (Epi \ A)^3$ satisfies a global triality relation

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y) \tag{3.1}$$

where the index j is defined by modulo 3, so that $g_{j\pm3} = g_j$ (this is said to be a triality group). We denote

$$Trig \ A = \{g = (g_1, g_2, g_3) \in (Epi \ A)^3 | g_j(xy) = (g_{j+1}x)(g_{j+2}y), \forall j = 1, 2, 3\} \tag{3.2}$$

This is a generalization of the automorphism group of A.

In constrast to the algebra triality relations (3.1), we may also consider the local triality relation

$$t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y). \tag{3.3}$$

Analogously to (3.2), we introduce

$$Trid \ A = \{t = (t_1, t_2, t_3) \in (End \ A)^3 | t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y), \forall j = 1, 2, 3\}. \tag{3.4}$$

Then, it defines a Lie algebra with component wise commutation relation. Also if $(t_1, t_2, t_3) \in Trid \ A$, it is easy to verify that we have for any $\alpha_j \in F, \alpha_{j\pm3} = \alpha_j$

$$t' = (t'_1, t'_2, t'_3) \in Trid \ A, \ \text{where} \ t'_j = \sum_{k=1}^{3} \alpha_{j-k} t_k \ (j = 1, 2, 3).$$

Furtheremore, if the exponential map $t_j \to \xi_j$ is given by

$$\xi_j = exp\ t_j = \sum_{n=0}^{\infty} \frac{1}{n!}(t_j)^n \tag{3.5}$$

is well-defined, then we can show that

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y), \tag{3.6}$$

provided that $t = (t_1, t_2, t_3) \in Trid\ A$ and vice-versa.

Next we introduce multiplication operators of A, $L(x), R(x) \in End\ A$ by

$$L(x)y = xy \text{ and } R(x)y = yx.$$

Def.3.1. Let $d_j(x, y) \in End\ A$, for $x, y \in A$ $(j = 1, 2, 3)$ be to satisfy
(i)

$$d_1(x, y) = R(y)L(x) - R(x)L(y) \tag{3.7a}$$

$$d_2(x, y) = L(y)R(x) - L(x)R(y). \tag{3.7b}$$

(ii) The explicit form for $d_3(x, y)$ is unspecified except for

$$d_3(x, y) = -d_3(y, x), \tag{3.7c}$$

(iii)

$$(d_1(x, y), d_2(x, y), d_3(x, y)) \in Trid\ A.$$

We call the algebra A satisfying these conditions to be a *regular triality algebra.*

Remark Any Lie (resp. Jordan) algebra with product $[x, y]$ (resp. xy) is an example of the regular (in particular, normal) triality algebra w.r.t. $L([x, y]) = d_j(x, y)$ (resp.$[L(x), L(y)]$) for $j = 1, 2, 3$.

Proposition 3.2 ([K-O.3]) *Let A be a regular triality algebra satisfying either the condition (B) or (C); (B) $AA = A$, (C) if some $b \in A$ satisfies either $L(b) = 0$ or $R(b) = 0$, then $b = 0$. Then we obtain the following.*
(i) For any $t = (t_1, t_2, t_3) \in Trid\ A$, we have

$$[t_j, d_k(x, y)] = d_k(t_{j-k}x, y) + d_k(x, t_{j-k}y) \tag{3.8a}$$

Especially, if we choose $t_j = d_j(x, y)$ it yields

$$[d_j(u, v), d_k(x, y)] = d_k(d_{j-k}(u, v)x, y) + d_k(x, d_{j-k}(u, v)y). \tag{3.8b}$$

(ii) For any $g = (g_1, g_2, g_3) \in Trig\ A$, we have

$$g_j d_k(x, y) g_j^{-1} = d_k(g_{j-k}x, y) + d_k(x, g_{j-k}y). \tag{3.8c}$$

Let A be a regular triality algebra with either (B) or (C), and set

$$L_0 = span < d_j(x, y), \forall x, y \in A > . \tag{3.9}$$

Then L_0 is a Lie algebra by (3.8b). Moreover, it is an ideal of the large Lie algebra $Trid\ A$ by (3.8a), denoted by $L_0 \triangleleft Trid\ A$. We call an "inner triality derivation"(naming of the author) this L_0.

Def.3.3 If a regular triality algebra satisfies Eqs.(3.7) as well as

$$d_3(x,y)z + d_3(y,z)x + d_3(z,x)y = 0, \tag{3.10a}$$

$$[d_j(u,v), d_k(x,y)] = d_k(d_{j-k}(u,v)x, y) + d_k(x, d_{j-k}(u,v)y)], \tag{3.10b}$$

then we call a *pre normal triality algebra* ([K-O.1]). Furthermore, if we have

$$Q(x,y,z) := d_1(z,xy) + d_2(y,zx) + d_3(x,yz) = 0, \tag{3.11}$$

then A is called a *normal triality algebra* ([K-O.2]). Next we introduce the second bilinear product in the same vector space A with involution $x \to \overline{x}$ by

$$x * y = \overline{xy} = \overline{y}\ \overline{x}. \tag{3.12}$$

Then the resulting algebra $(A, x*y)$ is said to be a *conjugation algebra* of A, for the new product $x * y$, by means of $\overline{Qx} = \overline{Q}\ \overline{x}$ and $Q \in End\ A$, we have

$$\overline{g}_j(x*y) = (g_{j+1}x)*(g_{j+2}y),\ \overline{d}_j(x*y) = (d_{j+1}x)*y + x*(d_{j+2}y). \tag{3.13}$$

Remark ([K-O.1]) The conjugation algebra of a structurable algebra which contains an alternative algebra is a normal triality algebra.

Note that the vector space $\mathfrak{A}_0 \otimes \mathfrak{J}_0$ with 182 dimension ([S]) appeared Tits second construction of E_8 is a normal triality algebra.

Theorem 3.4 ([K-O.2]) *The symmetric composition algebra, Lie and Jordan algebras are a normal triality algebra.*

Theorem 3.5 *For a normal triality algebra A, if we define*

$$\xi_j = exp\ d_j\ (j = 1,2,3),\ (\ assuming\ \ the\ well-defined)$$

then we have

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y),$$

$$[\frac{d}{dt}(exp\ td_j)d_k(exp\ td_j)^{-1}]_{t=0} = [d_j, d_k] \in Trid\ A.$$

That is, this means that $([d_j,d_k],[d_{j+1},d_{k+1}],[d_{j+2},d_{k+2}]) \in Trid\ A$.

Corollary. *For the pseudo octonion or para Hurwitz algebras, the same result in Theorem 3.5 holds, as these algebras are a symmetric composition algebra and so a normal triality algebra.*

Remark. In the normal triality algebra A, if we define an endomorphism by $D(x,y) := d_1(x,y) + d_2(x,y) + d_3(x,y)$, then we have the relations $D(x,y) = -D(x,y)$, $D(xy,z) + D(yz,x) + D(zx,y) = 0$ and $D(x,y)$ is a derivation satisfying $[D(x,y), D(u,v)] = D(D(x,y)u,v) + D(u, D(x,y)v)$, thus this algebra A is a generalized structurable algebra ([K.1]).

Remark.([K-O.1]) The conjugation algebra $(A, x * y)$ of normal triality algebra (A, xy) with a para unit e (i.e., $ex = xe = \overline{x}$) is a structurable algebra with the unit $e * x = x * e = x$, since $x * y = \overline{xy}$ and $\overline{\overline{x}} = x$.

Remark. Let $(A, x * y)$ be an associative algebra. Then $(A, x \cdot y)$ is a Jordan algebra with new product and involution difined by $x \cdot y = x * y + y * x$ and $\overline{x} = x$, since they satisfy the identities $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2)$.

Remark. Let A be a normal triality alg. For $(\xi_1, \xi_2, \xi_3) = (exp\ d_1, exp\ d_2, exp\ d_3) \in exp\ L_0$, provided that the exponential map is well-defined,

$$\forall g = (g_1, g_2, g_3) \in Trig\ A \implies$$

$$g_j \xi_k g_j^{-1} = g_j(exp\ d_k)g_j^{-1} = exp(g_j d_k g_j^{-1}) \in exp\ L_0. \text{ (by (3.8) and (3.9))}$$

Therefore $G_0 =< \xi_1, \xi_2, \xi_3 >_{span}$ is an invariant subgroup of $Trig\ A$. We call an "inner triality group"(naming of the author) this G_0.

For the details of this section, we would like to refer ([K-O.1],[K-O.2]and [K-O.3]), that is, for the concept of normal triality algebras and related topics.

§4. Symmetric compostion algebras

Algebra が次の条件を満たすとき，symmetric composition algebra という．(必ずしも単位元を持たない非結合的代数系で考えています)

$$x(yx) = (xy)x =< x|x > y, \quad \text{bilinear form} < x|y >=< y|x > \text{ is nondeg.} \quad (\spadesuit)$$

これは $< x|yz >=< x|yz >$ をもつ composition law と同値です (cf.§1).

Proposition 4.1([O]) *Any symmetric composition algebra with eight dimension over a field of ch* $F \neq 2,\ 3$ *is limited to be either*

(i) a para-Hurwitz algebra, or

(ii) an eight dimensional pseudo octonion algebra,

where the para-Hurwitz algebra is the conjugation algebra of Hurwitz algebra (i.e., **R**, **C**, **H** (*quaternion*), **O** (*octonion*) *if ch* $F = 0$*).*

By (Example 2.2 in [K-O.2]), we have results as follows.

Symmetric composition algebras satisfy the triality relation for the choice of

$$d_0(x, y)z = 2\{[L(x), L(y)] - R([x, y])\}z = 4\{< x|z > y- < y|z > x\}. \qquad (4.1)$$

Remark. The pseudo octonion algebra has neither unit nor para-unit. But the para-Hurwitz algebra has a para-unit, since $ex = \overline{e * x} = \overline{x}$.

Next, ([T] or [S]), and 日本語としては ([To]) における a principle of triality (local triality relation) として知られている Cayley algebra **O** の三対原理を我々の notation で表すと次の様になります．$x*y$ は standard product of **O**, xy は the product of para octonion algebra (or para Hurwitz algebra with 8 dim) です:

$$\overline{d}_j(x, y)(u * v) = (d_{j+1}(x, y)u) * v + u * (d_{j+2}(x, y)v) \qquad (\sharp)$$

ただし $\overline{d}_j(x, y) = d_{3-j}(\overline{x}, \overline{y})$, そして the involution $\overline{x}$ (i.e., $\overline{x * y} = \overline{y} * \overline{x}$ and $\overline{\overline{x}} = x$) of **O**, と the product $x * y = \overline{xy}$ のもとで ($x * y$ は conjugation algebra's product xy です)

$$l(x)y = x * y,\ r(x)y = y * x \qquad (4.2)$$

の記号において (by means of Eqs (3.7), (4.1) and Eq (4.2)) 変形すると

$$\overline{d}_0(x,y) = l(\overline{y} * x - \overline{x} * y) + r(\overline{y})r(x) - r(\overline{x})r(y) (= \overline{d}_3) = d_0(\overline{x}, \overline{y})$$

$\overline{d}_0(x,y) = r(x * \overline{y} - y * \overline{x}) + l(\overline{y})l(x) - l(\overline{x})l(y)$, and furthermore
$d_1(x,y) = l(\overline{y})l(x) - l(\overline{x})l(y) = \overline{d}_2$, $d_2(x,y) = r(\overline{y})r(x) - r(\overline{x})r(y) = \overline{d}_1$ です.
Using the notation $l(\mathbf{O}_0) = \{l(a) | \overline{a} = -a\}$, $r(\mathbf{O}_0) = \{r(a) | \overline{a} = -a\}$, その時 $d_0 - d_1 \in r(\mathbf{O}_0)$, $d_0 - d_2 \in l(\mathbf{O}_0)$, $d_1 - d_2 \in r(\mathbf{O}_0) + l(\mathbf{O}_0)$ and ($\sharp$) は

$$(\overline{d_0 - d_1})(u * v) = ((d_1 - d_2)u) * v + u * ((d_2 - d_0)v) \qquad (\sharp')$$

等のように表示されます. $\sharp$ and $\sharp'$ が初期の local "principle of triality"です.

勿論, $< d_j(x,y)u | v > = - < u | d_j(x,y)v >$, and $D = d_1 + d_2 + d_3$ は derivation です. そして $Der\ \mathbf{O} \cong G_2$(Lie algebra of 14 dim) です, ただし $< x | y > = \frac{1}{2}(x * \overline{y} + y * \overline{x}) = \frac{1}{2}Tr(x * \overline{y})$ で内積を定義します.

($\sharp$) は $xy = \overline{x * y}$ の積 ($x * y$ is the product of the conjugate algebra of the para octonion algebra with a product xy) では

$$d_j(xy) = (d_{j+1}x)y + x(d_{j+2}y) \qquad (\sharp\sharp)$$

となり, そして $L_0 = span < d_j > \cong D_4$(Lie algebra of 28 dim) $\triangleleft\, Trid\ \mathbf{O}$ です.

($\sharp$) と ($\sharp\sharp$) を比較すると, ($\sharp\sharp$) の公式の方が自然のような原理と思えます.

($\sharp\sharp$) *is a local principle of triality in the para octonion algebra.*

Remark.([K-O.3]) A を symmetric compostion algebra, $a = (a_1, a_2, a_3) \in \Sigma$ とする. そのとき $g_j(a) := R(a_{j+1})R(a_{j+2})$ $(j = 1, 2, 3)$ と定義すると

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y) \text{ and } g_{j+2}(a)g_{j+1}(a)g_j(a) = Id$$

が成り立つ. つまり global triality relation の例です. i.e., $(g_1, g_2, g_3) \in Trig\ A$.

一方, [K-O.1] において, structurable algebra についての三対原理も論究しています. そこではジョルダン代数, 交代代数についても論じました. つまり symmetric compotion algebra 以外の非結合的代数系でも我々の概念 triality relations（三対群, 三対原理）が適用可能であることを示しています. 従って Freudenthal による 56 次元の meta symplectic geometry に表れる代数系でも適用可能です. For the triality relation of Lie algebras, we discuss in another paper (see [K-O.4]).

§5. Miscellaneous results

この節では色々な事柄について, 将来の研究課題を含め論究します.

◎ Symmetric composition algebra A のとき, $a \in A$ を idempotent $(a^2 = a)$ かつ $< a | a > = 1$ として $g(a) = R(a)R(a)$ と定義する $\Longrightarrow g(a)$ は自己同型写像, $< g(a)x | g(a)y > = < x | y >$ and $g(a)^3 = Id$. Furthemore, following Theorem 2.5 in [K-O.3], by using (2.1), $\forall a, b \in \Sigma$, then $G = < \sigma_j(a)\theta_j(b)\ and\ \theta_j(a)\sigma_j(b) >_{span}$ is an invariant subgroup of $Trig(A)$, where $\theta_j(a) = L(a_{j+2})L(a_{j+1}), \sigma_j(b) = R(b_{j+1})R(b_{j+2})$. Also if $F = \mathbf{R}$, we obtain $G \cong so(8)$, $Trid\ L_0 \cong D_4$, since $< g_j x | g_j y > = < x | y >$, $\forall g_j \in G$, $< d_j x | y > + < x | d_j y > = 0$, $\forall d_j \in Trid\ L_0$.

◎ Theorem 2.1 in §2 の実例として，For the pseudo octonion alg. A,

$$D_j(a,p)x = (p_{j+1}x)a_{j+1} + a_j(xq_j), \quad for\ j = 1,2,3$$

$$and\ a = (a_1,a_2,a_3) \in \Sigma,\ p = (p_1,p_2,p_3) \in \Lambda(a),\ q_j = a_{j+1}p_{j+2}$$

の記号を用いるとき，A の基底 $e_1,\cdots,e_8$ によって Σ, $\Lambda(a)$ の具体例を

$$a = (a_1,a_2,a_3) = (-e_8,-e_8,-e_8) \in \Sigma,\ p = (e_2-e_3, e_3-e_1, e_1-e_2) \in \Lambda(a)$$

と explicitly に与えると，この (a,p) で $(D_1(a,p), D_2(a,p), D_3(a,p)) \in Trid\ A$.
別の例として $a' = (a'_1,a'_2,a'_3) = (-e_8,-e_8,-e_8), p' = (0,e_2,-e_2)$ によっても

$$(D_1(a',p'), D_2(a',p'), D_3(a',p')) \in Trid\ A.$$

(A における local triality relation の実例). $g_j(a) = R(a_{j+1})R(a_{j+2})$ is an element of the triality group. i.e., $(g_1,g_2,g_3) \in Trig\ A$. (global triality relation の実例). Also, $a = (e_1,e_2,e_3),\ p = (-e_2-e_8, e_1, -e_2)$ is an example.
これらについての詳しい事柄と実例は別の機会に述べたいと考えます. (勿論 para octonion algebra の triality relations の実例についても論及したいと思います.)

◎ Let A_1, and A_2 be two independent symmetric composition algebras. Then their tensor product $A_1 \otimes A_2$ is a normal triality algebra with

$$D_j(x_1\otimes x_2, y_1\otimes y_2) = d_j^{(1)}(x_1,y_1)\otimes < x_2|y_2 >_2 id + < x_1|y_1 > id \otimes d_j^{(2)}(x_2,y_2) \tag{5.1}$$

for $x_1,y_1 \in A_1, x_2,y_2 \in A_2$. We note that this leads to the Freudenthal magic square for the Lie algebras of G_2, F_4, E_6, E_7 and E_8 types (see [S], [K-O.2]).

◎論文 [K.2] and [K.3] の 2 次代数 $Z_p[\sqrt{q}]$ においては自然数 p と q に依存して

$$Trig(Z_p[\sqrt{q}]) \cong S_4(\text{Symmetric group}) \text{ or } K_4^\theta \tag{5.2}$$

です. ただし $K_4^\theta = \{(g_1,g_2,g_3) \in (Epi(Z_p[\sqrt{q}]))^3 | (Id,Id,Id), (Id,-Id,-Id), (-Id,Id,-Id), (-Id,-Id,Id), (\theta,\theta,\theta), (\theta,-\theta,-\theta), (-\theta,\theta,-\theta), (-\theta,-\theta,\theta)\}$ and θ is the involution. knot theory における quandle の実例も考察しています.
我々の pseudo octonian algebra A においては $\{-e_8,e_8\}$, $\{-e_8,e_8,-e_1,e_1\}$ は乗法で閉じています, $\{-e_8\}$ は quandle ですが, $\{-e_8,e_8,-e_1,e_1\}$ は quandle ではないです. 実際 $e_1(e_8e_8) \neq (e_1e_8)(e_1e_8)$ です. しかし $\{-e_8, \frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_8, -\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_8\}$ は quandle です.　$a^2 = a$ なる idempotent を見つけることが重要だと考えます.

◎ベクトル空間 A を $\{X | X = 3\times 3\ \ matrix,\ TrX = 0\}$ とする. $X*Y$ を行列の普通の積, そして $\mu = \frac{3-\sqrt{3}i}{6}, \nu = \frac{3+\sqrt{3}i}{6}$ とおくとき,

$$XY = \mu X*Y + \nu Y*X - \frac{1}{3}Tr(X*Y)E \tag{5.3}$$

と new product を定義する (E は単位行列).

$$\Longrightarrow < X|X > Y = X(YX) = (XY)X, (symmetric\ compostion\ alg.).$$

ただし $< X|Y >= \frac{1}{6}Tr(X*Y)$ and $Tr(XY) = 0$, for new product XY.

$\omega^3 = 1$ を用いても $XY = \omega X*Y - \omega^2 Y*X - \frac{\omega-\omega^2}{3}Tr(X*Y)E$ と定義する．

$$\Longrightarrow < X|X > Y = X(YX) = (XY)X.$$

この関係式 (5.3) の定義と概念は S.Okubo の１９８０年代の論文等 ([O]) に Gell-Mann の代数を拡張したものとして表れています．我々はそれを一般化した代数で triality relations　(triality group and triality derivation) を考察しています．

最後にこの小論では証明なしで survey 的な部分を含む new idea と具体例を主に述べさせていただきましたので，興味のある方々は [K-O.1],[K-O2],[K-O.3] により一般論を参照して下さい．筆者の力量の少なさの為と時間的節約の理由により，この小論が英語と日本語の混在した論文になったことをお許しください．そして最後に，

This note is a homage to Prof. Okubo (1930-2015).

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