Computer Aided Constructions of Cages

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Abstract

A k-regular graph of girth g and minimal order is called a (k, g)-cage. The orders of cages are determined for only few sets of parameter pairs (k, g), and the general problem of determining these orders and constructing at least one (k, g)-cage for each pair of parameters is called the *Cage Problem.* The voltage lift construction is among the most widely used constructions of small (k, g)-graphs, with the orders of the constructed graphs depending on the choice of a base graph, a voltage group, and a specific voltage assignment. Successful application of the voltage lift construction therefore often requires significant computer aided experimentation with the three fundamental ingredients. We survey some known results concerning the voltage lift construction, and discuss ways to decrease the orders of the smallest known (k, g)-graphs for some specific parameter pairs (k, g).

1 Introduction

In extremal graph theory, one looks for graphs that possess specified properties or parameters and optimize an additional closely related parameter; such as, for example, the order of the graph. Optimal solutions often stem from constructions that rely on connections to various areas of algebra including among others group theory, linear algebra, or theory of finite fields, and almost always depend on extensive computer searches through a large number of candidate structures, e.g., choices of groups to be used together with generating sets or subgroups. Determining the best choices requires a thorough understanding of the interplay between the building blocks of these constructions and the properties of the desired graphs. Limiting the algebraic objects involved in these constructions to those with the most promising properties allows for subsequent efficient use of computing machinery.

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The *Cage Problem* is a computationally demanding optimization problem which, for a given pair of graph parameters, degree k and girth g, requires finding a graph of the smallest order within the infinite set of graphs all of which have the desired parameters, but cannot be efficiently constructed. This makes the brute force approach computationally infeasible for all but the smallest parameter pairs.

More precisely, a (k, g)-*cage* is a k-regular graph of girth g and smallest possible order $n(k, g)$, and solving the Cage Problem means finding a (k, g)-cage and its corresponding order $n(k, g)$ for all parameter pairs (k, g), $3 \leq k, g$. The formulation and first attempts at solving this classical problem go back to Tutte [17] in the 1940s. The Cage Problem is generally accepted to be exceedingly complex and has only been solved for limited parameter classes, with constructions of specific graphs of orders close to theoretical lower bounds seen as the best way to making progress. It is the direction taken by the majority of researchers in the area [9].

Recent renewed interest in the Cage Problem is partially due to its immediate applicability in network design as well as in such areas as Coding Theory where cages give rise to classes of Low Density Parity Check Codes [11, 14, 15] whose decoding efficiency requires a large girth of the corresponding cage while the order of the cage is related to the length of the code and thus to its information rate.

2 The lift construction

Part of the renewed appeal of the search for cages also lies in the introduction of algebraic construction methods with an extensive computational component. One such construction, the main construction discussed in this paper, combines ideas from topological graph theory and group theory, and is known under various names such as *the voltage graph construction* or the *covering graph construction* or simply the *lift construction* [1]. A significant number of *record graphs* (smallest graphs with given parameters (k, g) known to date [9]) has been constructed using this construction which can also be viewed as a generalization of one of the fundamental concepts of Algebraic Graph Theory, the Cayley graphs [1, 2, 5, 6].

Let Γ be a finite group with a generating set X closed under inverses, $X = X^{-1}$, and not containing the identity, $1_\Gamma \notin X$. The *Cayley graph* $C(\Gamma, X)$ is the $|X|$-regular graph on the elements of Γ via the adjacency $g \sim gx$, for all $g \in \Gamma$, and $x \in X$ (equivalently, two vertices $g, h \in \Gamma$ are adjacent if and only if $g^{-1}h \in X$). Since X is closed under inverses, the Cayley graph $C(\Gamma, X)$ is undirected.

Unlike the above definition, the ingredients of the lift construction are of two different kinds. Namely, the lift construction requires both a finite (multi)graph G (possibly admitting multiple edges and multiple loops) called the *base graph*, and a finite group Γ, called the *voltage group*. If one lets $D(G)$ denote the set of *darts* of G obtained by replacing each edge or loop e of G by a pair of opposing (oriented) darts $\vec{e}, \vec{e}^{-1}$, the *voltage assignment* on G is any mapping $\alpha : D(G) \to \Gamma$ which satisfies the requirement $\alpha(\vec{e}^{-1}) = (\alpha(\vec{e}))^{-1}$, for all $\vec{e} \in D(G)$ (no other algebraic properties are required). Given a pair G, Γ together with a voltage assignment $\alpha : D(G) \to \Gamma$, the *derived regular cover* or the *lift* of G with respect to α is denoted by G^α, and has the vertex set $V(G) \times \Gamma = \{u_g | u \in V(G), g \in \Gamma\}$ and adjacency between two vertices u_g and $v_f \in G^\alpha$ if and only if $\vec{e} = (u, v) \in D(G)$ and $f = g \cdot \alpha(\vec{e})$. In this sense, Cayley graphs are lifts of one-vertex bouquets of cycles and semi-edges, and thus the lift construction is a generalization of the Cayley graph construction.

We illustrate the lift construction with two notorious examples which give rise to cages. The first example demonstrates the construction of the *Petersen graph*, which is well-known to be the $(3, 5)$-cage, from the dumbbell graph using voltages from $\mathbb{Z}_5$. The second example is the $(3, 6)$-cage known under the name of the Heawood graph which is the lift of the θ-graph via voltages from $\mathbb{Z}_7$ [9].

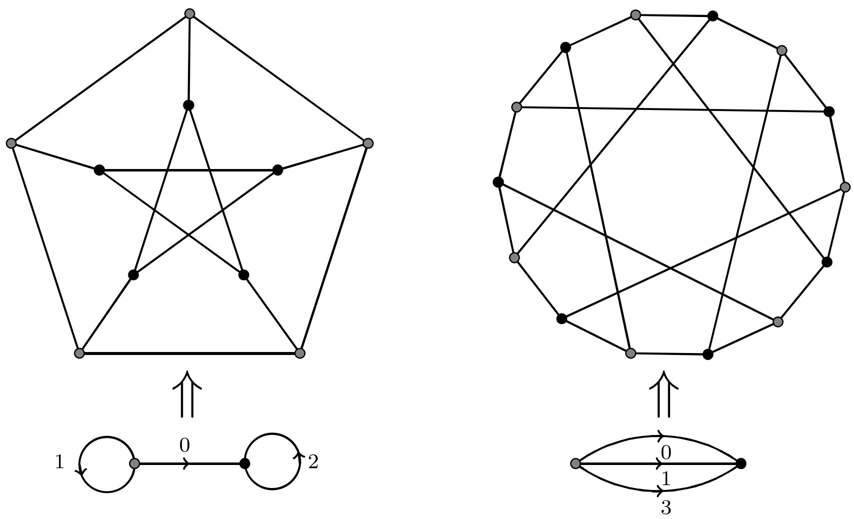

Figure 1: Petersen graph as a lift by $\mathbb{Z}_5$ and Heawood graph as a lift by $\mathbb{Z}_7$

The above examples amply demonstrate the key ingredients behind the efficacy of the lift construction. Both components involved in the construction are relatively small when compared to the order of the resulting lift graph, $|V(G^\alpha)| = |V(G)| \cdot |\Gamma|$. At the same time, all graph properties of G^α are determined by the choice of G, Γ and α, and thus understanding the impact of these choices on the properties of the resulting G^α allows for constructions of large graphs with desired properties from small building blocks. Since we are specifically interested in k-regular graphs of prescribed girth g, it is important to observe that the set of vertices $\{u_g | g \in \Gamma\}$, where u is a fixed vertex of G, called the *fibre* of u in G^α, consists of vertices whose degrees are equal to the degree of u in G, and thus, in particular, a lift of a k-regular graph is necessarily k-regular. It follows from this observation that all the base graphs we will be interested in our paper will be regular. The girth of the lift G^α as a function of G, Γ, and α, is a bit more complicated, and we will devote our next section to determining the girth of the lift graph.

3 The girth of a lift graph

The most important observation about the girth of a lift G^α is that it is determined by the properties of the base graph G and the specific voltage assignments to its edges. Thus, determining the girth of G^α does not necessarily require constructing G^α and determining its girth by inspecting its cycle structure which is a computational task polynomial in the order of G^α. Consequently, one can search for large girth lifts of a base graph G without having to construct the lift for each considered voltage assignment and subsequently determining the girth of the much larger derived graphs.

The key to this observation lies in the concept of a *projection* p from G^α onto G, which is a graph homomorphism mapping the vertices of each fibre $\{u_g | g \in \Gamma\}$, $u \in V(G)$, onto the vertex u. This means, in particular, that each vertex u of G has $|\Gamma|$ pre-images u_g, $p(u_g) = u$, $g \in \Gamma$, and each dart $\vec{e}$ with initial vertex u and terminal vertex v is the image of $|\Gamma|$ different darts in G^α, each starting from a different vertex u_g and terminating at a different vertex $u_{g \cdot \alpha(\vec{e})}$. It is also easy to see that each n-cycle $u_{1,g_{i_1}}, u_{2,g_{i_2}}, \dots, u_{n,g_{i_n}} = u_{1,g_{i_1}}$ in G^α projects onto a *closed walk* $u_1, u_2, \dots, u_n = u_1$ in G of length n having the additional property that it traverses no edge of G 'back and forth' in an immediate sequence, i.e., it does not contain any pair of consecutive darts $\vec{e}, \vec{e}^{-1}$. We call such closed walks *non-reversing*.

In view of the above, let us now consider a non-reversing closed walk $\mathcal{W}$ of length n in the base graph G starting off (and returning to) a vertex $u \in V(G)$. It is again easy to see that G^α contains $|\Gamma|$ different walks of length n, each starting at a different vertex u_g and each projecting onto $\mathcal{W}$. Note that, since $\mathcal{W}$ is assumed to be closed, the end points of all these n-walks projecting onto $\mathcal{W}$ also belong to the fibre of u. It is therefore meaningful to ask whether the lifted walks are closed or not. The answer to this question, i.e., the answer to the question whether the initial vertex u_g of any such a lift is equal to its terminal vertex $u_{g'}$, depends on the so-called net voltage of $\mathcal{W}$. Namely, if $\mathcal{W}$ consists of the sequence of darts $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ (with the initial vertex of $\vec{e}_1$ and the terminal vertex of $\vec{e}_n$ equal to the vertex u), the *net voltage* $\alpha(\mathcal{W})$ of $\mathcal{W}$ is the product $\alpha(\vec{e}_1)\alpha(\vec{e}_2)\dots\alpha(\vec{e}_n)$ in Γ. Starting of $\vec{e}_1$ and traversing $\mathcal{W}$ one edge at a time yields that $u_{g'} = u_{g \cdot \alpha(\mathcal{W})}$, and thus a lift of $\mathcal{W}$ starting at u_g (and also all the other lifts) forms a closed walk if and only if $g = g \cdot \alpha(\mathcal{W})$, which is equivalent to $g \cdot \alpha(\mathcal{W}) = 1_\Gamma$. In summary, the pre-images of the closed walk $\mathcal{W}$ are closed walks if and only if the net voltage of $\mathcal{W}$ in Γ is equal to 1_Γ, and are cycles if and only if $\mathcal{W}$ contains no proper closed sub-walk of net voltage 1_Γ. These observations yield the following lemma.

Lemma 3.1 ([10]). *Let G be a finite graph and $\alpha : G \to \Gamma$ be a voltage assignment of G. The girth of the voltage graph lift G^α is equal to the length of a shortest closed non-reversing walk $\mathcal{W}$ in G of net voltage 1_Γ.*

Let us point out again that the above lemma provides us with a justification of the main computational advantage of the use of the lift construction. Namely, even though the lift G^α contains many more cycles than G, they all project onto non-reversing closed walks in G of net voltage 1_Γ, and hence the girth of G^α can be determined via considering all closed non-reversing walks in G and their net voltages. To conclude this section we include one more observation concerning the lifts of closed non-reversing walks in G.

Lemma 3.2. *Let G be a finite graph, let $\alpha : G \to \Gamma$ be a voltage assignment of*

G, and let $\mathcal{W}$ be a closed non-reversing walk in G of length n and net voltage $\alpha(\mathcal{W})$ of order r in Γ. Let us also assume that $\mathcal{W}$ starts from u. The lift graph G^α contains $|\Gamma|/r$ different closed walks of length nr starting in the fibre of u and projecting onto $\mathcal{W}$. Moreover, if $\mathcal{W}$ does not contain a proper closed sub-walk of net voltage 1, all the lifts of $\mathcal{W}$ are (nr)-cycles in G^α.

The argument used in the proof of the above lemma should now be relatively clear: When tracing simultaneously the sequence of darts in $\mathcal{W}$ starting in u and the (connected) sequence of the lifts of these darts in G^α starting in a specific vertex u_g, reaching the end-point u of $\mathcal{W}$ in G means reaching the vertex $u_{g\cdot\alpha(\mathcal{W})}$ in G^α. If $\alpha(\mathcal{W}) \neq 1_\Gamma$, the walk starting in u_g and projecting on $\mathcal{W}$ is not closed and can therefore be concatenated with the walk starting in $u_{g\cdot\alpha(\mathcal{W})}$ and also projecting onto $\mathcal{W}$. Repeating the process of attaching a number of connected non-closed walks in G^α will only result in a completed closed walk at the point when $u_g = u_{g\cdot(\alpha(\mathcal{W}))^r}$, or equivalently, when $(\alpha(\mathcal{W}))^r = 1_\Gamma$. Thus, each of the different copies of the *closed* walks starting in the fibre of u and projecting onto $\mathcal{W}$ contains different r vertices from the fibre, $u_g, u_{g\cdot(\alpha(\mathcal{W}))}, \ldots, u_{g\cdot(\alpha(\mathcal{W}))^{r-1}}$, and the fibre, which consists of $|\Gamma|$ vertices, splits into $|\Gamma|/r$ disjoint groups belonging to $|\Gamma|/r$ different closed walks of length nr starting from the fibre of u and projecting onto $\mathcal{W}$.

Let us revisit the construction of the Petersen graph via the voltage assignment described in Figure 1 in view of the above observations. The dumbbell graph contains two non-reversing closed walks of length 1, the loops, and neither one of them has net voltage 0. It contains two closed non-reversing walks of length 2 of non-zero net voltage consisting of one of the loops travelled twice in the same direction. It contains four closed non-reversing walks of length 3 containing the handle and one of the loops and also four visiting the same loop in the same direction three times, none of which are of net voltage 0. As it also contains no closed non-reversing walks of length 4 of non-zero net voltage, the girth of the lift is necessarily larger than 4 (and is in fact 5).

In the following sections we consider examples which unlike the construction of the Petersen graph from the dumbbell graph yield record graphs which have not yet been shown to be minimal with regard to their parameters (k, g), and therefore it may be the case that they are not the smallest possible. In each case, we discuss potential approaches to improving these constructions.

4 The record $(3, 14)$-graph

The smallest known trivalent graph of girth 14 is of order 384 and was constructed by Exoo [7]. It is a lift of the trivalent multigraph G_8 of order 8 shown in Figure 2. The voltage group used in its construction is a semidirect product of the cyclic group of order 3 by the generalized quaternion group of order 16, more specifically, the SmallGroup(48,18) in the Small Group Library of GAP [12] generated by the following two permutations:

χ = (1, 2, 5, 9)(3, 18, 12, 32)(4, 21, 14, 8)(6, 24, 16, 38)
(7, 25, 19, 11)(10, 31, 23, 17)(13, 44, 27, 48)(15, 46, 29, 37)
(20, 47, 34, 40)(22, 30, 36, 42)(26, 35, 39, 45)(28, 33, 41, 43)

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ψ = (1, 28, 16, 12, 17, 13, 5, 41, 6, 3, 31, 27)
    (2, 35, 23, 19, 24, 20, 9, 45, 10, 7, 38, 34)
    (4, 47, 29, 11, 30, 39, 14, 40, 15, 25, 42, 26)
    (8, 48, 36, 18, 37, 43, 21, 44, 22, 32, 46, 33)
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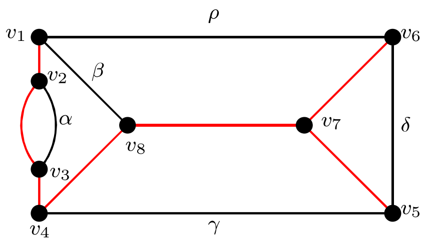

Figure 2: Base graph G_8

As we already explained in the previous sections, in order to construct a $(3, 14)$-graph of order smaller than 384 using the lift construction, we need to decrease the order of the base graph and/or the order of the voltage group. In view of the fact that the base graph used in Exoo's construction consists of only 8 vertices, we have decided to keep this graph as the base graph and experiment with groups of order smaller than 48. This means that in order to answer *the question whether there exists a smaller lift of* G_8 *of girth* 14 *than that of order* 384, one can consider all groups of orders larger than $33 = \lfloor \frac{258}{8} \rfloor$ (as 258 is a known lower bound on the order of a $(3, 14)$-graph [9]) and smaller than 48. For each such group Γ, $33 \leq |\Gamma| < 48$, one can consider all possible voltage assignments from Γ onto the 12 edges of G_8, and for each of these voltage assignments construct the corresponding lift graph and determine its order. If none of the constructions produced a graph of girth 14 (or bigger), the conclusion would be that there is no lift of G_8 smaller than 384. If, on the other hand, one was lucky enough to find a graph of girth at least 14, it would constitute a new record. Thus, determining the existence of a lift of G_8 of girth 14 and order smaller than 384 is a computational task proportional to the number of groups Γ of orders between 33 and 47 multiplied by the number of possible voltage assignments for the edges of G_8 which is equal to $12^{|\Gamma|}$. Without further restrictions on the possible voltage assignments, such exhaustive search is still computationally infeasible. In the forthcoming paragraphs we will outline some basic improvements to such search.

Probably the most significant improvement is based on the fact that without loss of generality one may choose a spanning tree $\mathcal{T}$ of G_8 and assign the identity 1_Γ of the voltage group Γ to all edges of $\mathcal{T}$ (every voltage lift is isomorphic to a lift in which the edges of a spanning tree of the base graph have all been assigned the identity voltage [13, p. 91]). This limits the number of voltage assignments that need to be considered for a specific Γ to $(12 - 7)^{|\Gamma|} = 5^{|\Gamma|}$, where 5 is the number of non-tree edges in G_8; sometimes also called the *Betti number* of G_8.

The second most significant improvement is based on our arguments presented in Section 3 concerning the girths of the lift graphs, where we argued that in order to determine whether a specific voltage gives rise to a graph of girth greater than 14, one only needs to consider the net voltages of closed non-reversing walks of length less than 14 and to determine whether any of these closed walks has the net voltage 1_Γ. To take advantage of this insight, we used a DFS algorithm to find all non-reversing closed walks of lengths between 7 and 13 (any shorter walk of net voltage 1_Γ doubles into a closed walk consisting of a concatenation of such a walk with itself and also having the net voltage 1_Γ, and so it will not be overlooked). For each voltage assignment for the non-tree edges of G_8 one needs to check all the walks on this precalculated list to see whether any of them received the net voltage 1_Γ. Finding the first such voltage assignment terminates the search as the existence of such a walk means that the girth of the lift graph is smaller than 14. Since Exoo in [7] does not list the voltages used in his construction, we first applied our algorithm to voltages from SmallGroup(48,18). We found that 192 different voltage assignments produced graphs of girth at least 14. After a closer inspection we determined that the 192 voltage lift graphs are all isomorphic, and the girth of all of them is equal to 14.

One possible voltage assignment to the non-tree edges of G_8 is as follows:

α = (1, 7, 14, 32, 5, 19, 4, 18)(2, 12, 21, 11, 9, 3, 8, 25)
(6, 35, 29, 48, 16, 45, 15, 44)(10, 41, 36, 40, 23, 28, 22, 47)
(13, 37, 39, 24, 27, 46, 26, 38)(17, 20, 42, 43, 31, 34, 30, 33)

β = (1, 38, 5, 24)(2, 17, 9, 31)(3, 48, 12, 44)(4, 37, 14, 46)
(6, 23, 16, 10)(7, 40, 19, 47)(8, 42, 21, 30)(11, 45, 25, 35)
(13, 43, 27, 33)(15, 22, 29, 36)(18, 28, 32, 41)(20, 26, 34, 39)

γ = (1, 34, 14, 33, 5, 20, 4, 43)(2, 13, 21, 39, 9, 27, 8, 26)
(3, 36, 25, 23, 12, 22, 11, 10)(6, 19, 29, 18, 16, 7, 15, 32)
(17, 45, 42, 44, 31, 35, 30, 48)(24, 28, 46, 47, 38, 41, 37, 40)

δ = (1, 47, 16, 11, 17, 39, 5, 40, 6, 25, 31, 26)
(2, 48, 23,18 ,24, 43, 9, 44, 10, 32, 38, 33)
(3, 30, 27, 14, 28, 15, 12, 42, 13, 4, 41, 29)
(7, 37, 34, 21, 35, 22, 19, 46, 20, 8, 45, 36)

ρ = (1, 27, 31, 3, 6, 41, 5, 13, 17, 12, 16, 28)
(2, 34, 38, 7, 10, 45, 9, 20, 24, 19, 23, 35)
(4, 26, 42, 25, 15, 40, 14, 39, 30, 11, 29, 47)
(8, 33, 46, 32, 22, 44, 21, 43, 37, 18, 36, 48)

To conclude this section, let us mention one more limitation on the considered voltages. Since we were specifically looking for lifts of girth at least 14, the order of the voltage assignment α cannot be selected to be smaller than $14/2 = 7$ (being a part of a closed walk of length 2, a voltage assignment of order smaller than 7 would yield a cycle in the lift of length smaller than 14). Similar limitations on the orders of the voltages of the non-tree edges of G_8 are listed in the following table. Regardless of the choice of the voltage group, these limitations can be used to further limit the possible voltage choices that need to be considered.

non-reversing closed walk of length l	l	desirable condition
$v_2v_3v_2$	2	$\|\alpha\| \geq 14/2 = 7$
$v_1v_2v_3v_4v_8v_1$	5	$\|\beta\| \geq 14/5 = 2.8$
$v_4v_5v_7v_8v_4$	4	$\|\gamma\| \geq 14/4 = 3.5$
$v_5v_6v_7v_5$	3	$\|\delta\| \geq 14/3 = 4.6$

5 The record $(3, 30)$-graph

The last example we include in our paper is a lift of the θ-graph of order 1143408 using voltages from the group $SL(2, 83)$ that appeared in [10]. It was a record holder for more than ten years, and was only recently replaced by a graph constructed by another computer aided group based construction.

Since the θ-graph is the smallest truly trivalent graph, lifts of the θ-graph are in some sense the closest to Cayley graphs (which are lifts of the single vertex bouquet of cycles and semiedges). Any lift of the θ-graph is necessarily bipartite and contains exclusively cycles of even length. Moreover, all closed non-reversing walks in the θ-graph are of even length and are particularly easy to obtain.

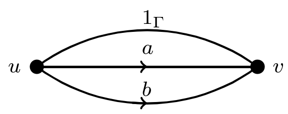

Figure 3: The θ-graph

Without loss of generality, we may assume the voltage assignment pictured in Figure 3 (with the top edge constituting our selected spanning tree). All closed walks in the θ-graph contain both of its vertices, and thus it is sufficient to consider closed walks that start with either vertex of the two. If we select for the starting vertex the left vertex u in Figure 3, all closed walks start with one of the three left-to-right darts and can be associated with their voltages, $1_\Gamma, a, b$. Each of the starting darts can only be followed by two of the opposing darts, namely the dart 1_Γ can only be followed by the darts a^{-1}, b^{-1}, the dart a can only be followed by the darts $1_\Gamma, b^{-1}$, and the dart b can only be followed by the darts $1_\Gamma, a^{-1}$. It follows that all non-reversing closed walks starting in u can be constructed recursively from the six closed walks $1_\Gamma a^{-1}, 1_\Gamma b^{-1}, a1_\Gamma, ab^{-1}, b1_\Gamma, ba^{-1}$ composed according to the above exclusion rules. This yields the immediate observation that the number of closed non-reversing walks of length $2n$ in the θ-graph that start from u is bounded from above by the product $3 \cdot 2^{2n-1}$, and is very easy to obtain recursively.

Observe further that selecting a voltage assignment using a finite group Γ simply means choosing the elements $a, b \in \Gamma$. Since cages are necessarily connected, we may assume that $\langle a, b \rangle = \Gamma$. In addition, we may also assume regular permutation representation of Γ on the set $\{1, 2, \ldots, |\Gamma|\}$, in which case the vertices of the lift graph can be labeled as u_i, v_j, $i, j \in \{1, 2, \ldots, |\Gamma|\}$, and all the darts are of the form $(u_i, v_{x(i)})$, $x \in \{1_\Gamma, a, b\}$, or of the form $(v_j, u_{y(j)})$, $y \in \{1_\Gamma, a^{-1}, b^{-1}\}$. The elements a and b (as permutations) consist of cycles of length $|a|$ and $|b|$, respectively, and any product $x_{i_1} x_{i_2} \cdots x_{i_{2n}}$, of elements from $\{1_\Gamma, a, b, a^{-1}, b^{-1}\}$ is equal to 1_Γ if only if the image of the element 1 under the permutation $x_{i_1} x_{i_2} \cdots x_{i_{2n}}$ is equal to 1. Thus, determining the girth of the lift of the θ-graph with voltages $a, b \in \Gamma$ is equivalent to determining the shortest closed walk constructed recursively in the previous paragraph which represents a permutation fixing the element 1 (and all the other elements as well).

The process of determining the length of the shortest such walk can be sped up further by observing that the net voltages corresponding to the six 2-

walks, $1_\Gamma a^{-1}, 1_\Gamma b^{-1}, a1_\Gamma, ab^{-1}, b1_\Gamma, ba^{-1}$, are the products $a^{-1}, b^{-1}, a, ab^{-1}, b, ba^{-1}$, respectively. It follows that the net voltage of any non-reversing closed walk of length $2n$ starting in u is equal to a product of n elements from the list $a^{-1}, b^{-1}, a, ab^{-1}, b, ba^{-1}$ subject to the condition that the product cannot contain an element immediately followed by its inverse (such products are called *reduced*). In analogy to the concluding remark of the previous section, we note that in order for the lift graph to be of girth at least 30, the net voltages $a^{-1}, b^{-1}, a, ab^{-1}, b, ba^{-1}$ of the closed 2-walks in the θ-graph must all be of order at least $\frac{30}{2} = 15$.

In summary, finding a lift of the θ-graph of girth at least 30 and of order smaller than the graph constructed in [10], one needs to consider groups Γ of orders smaller than the order of $SL(2, 83)$. When considering a specific group and a specific voltage assignment $a, b \in \Gamma$, determining the girth of the lift graph requires checking for all the products described in the above paragraph whether they are equal to 1_Γ; starting from products of length 1 and gradually increasing the length of the considered products up to the length 14. Finding any product equal to 1_Γ (or equivalently, mapping 1 to 1) immediately terminates the process as the girth of the resulting lift is necessarily smaller than $2 \times 14 = 28$. Not finding any product equal to 1_Γ of length smaller than 15 for a specific group Γ and a pair of its elements a, b would yield a cubic graph of girth at least 15 and order smaller than 1143408; possibly a new record graph.

References

[1] N.L. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge (1974).

[2] N.L. Biggs, Girth and residual finiteness, *Combinatorica* 8 (1988) 307-312.

[3] N.L. Biggs, Constructions for cubic graphs of large girth, *Electron. J. Combin.* 5 (1998).

[4] N.L. Biggs and M.J. Hoare, The sextet construction for cubic graphs, *Combinatorica* 3 (1983) 153-165.

[5] J. Bray, C. Parker and P. Rowley, Cayley type graphs and cubic graphs of large girth, *Discrete Math.* 214 (2000) 113-121.

[6] M. Conder, G. Exoo and R. Jajcay, On the limitations of the use of solvable groups in Cayley graph cage constructions, *Europ. J. of Comb.* 31 (2010), 1819-1828.

[7] G. Exoo, A small trivalent graph of girth 14, *Electron. J. Combin.* 9 (2002), N3.

[8] G. Exoo, Regular graphs of given degree and girth. (`http://ginger.indstate.edu/ge/CAGES`).

[9] G. Exoo and R. Jajcay, Dynamic cage survey, *Electron. J. Combin.*, Dynamic Survey 16, September 2008.

[10] G. Exoo and R. Jajcay, On the girth of voltage graph lifts, *Europ. J. of Comb.* 32 (2011) 554-562.

[11] R. G. Gallager, *Low Density Parity-Check Codes*, MIT Press, Cambridge (1963).

[12] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.9, 2006, (`http://www.gap-system.org`).

[13] J.L.Gross and T.W. Tucker, *Topological graph theory*, Dover Mineola, New York (2001).

[14] S. Lin and D.J. Costello, *Error Control Coding*, 2nd Ed., Prentice Hall (2004).

[15] G. Malema and M. Liebelt, High girth column-weight-two LDPC codes based on distance graphs, *EURASIP Journal on Wireless Communications and Networking* 2007 (2007) 6-6.

[16] R. Tanner, A recursive approach to low complexity codes, *IEEE Transactions on Information Theory* 27 (1981) 533-547.

[17] W.T. Tutte, A family of cubical graphs, *Proc. Cambridge Philos. Soc.* 43 (1947) 459-474.