# Visiting Old, Learn New *

Yuji Kobayashi

Laboratory of Mathematics and Games
(https://math-game-labo.com)

## 1 History

The story started from a question asked by my friend. There are seven associative algebras of dimension 2 up to isomorphism over the complex field, but how about in the case of dimension 3 ? I took the time to calculate, and finally got the result that there are 24 species on the complex number field and 27 on the real number field. This theme is so basic that I thought maybe someone had already done it. So I searched for literature, but it was not easy to find such papers. While searching for various things, I learned that in the 19th century, many attempts were made to extend the complex number field, and it was called the hypercomplex number system. I searched for literature with this keyword, then various documents came out.

Apparently, such research started with the discovery of quaternions by W.R. Hamilton [2] in 1843. And, it was B. Peirce who first studied associative algebras systematically ([3]). Reading his paper, it looks like a literary book, essay or poetry ? His proofs are ambiguous and literary explanations continue in them. Unfortunately, I could not understand them. However, the result seems to be basically correct. They, the mathematicians of the 19th century, can see the right results without proof. Perhaps we are running too far into abstraction and generalization and are losing intuition like them.

Our paper [1] gives a modern and rigorous proof of the classification of three-dimensional algebra over the real and complex fields. We gave the subtitle 温故知新 to it, a word from the Analects of Confucius, which means 'visiting old, learn new'. Our paper is a homage to the works by mathematicians in the 19th century, particularly to Pierse's pioneering work. The paper was published in Asian European Journal of Mathematics and has been ranked as the "Most Read Paper" in the journal. It has been downloaded more than 7000 times.

*This is a summary of our results [1].

## 2 Strategy

Let $K$ be an algebraically closed field and $A$ be an associative algebra over $K$ of dimension 3. Let $E = \{e, f, g\}$ be a linear basis of $A$ over $K$. We say that $A$ is *curled*, if $x$ and $x^2$ are linearly dependent for all $x \in A$, that is, for any $x \in A$, $x^2 = kx$ for some $k \in K$. It is *waved* if it is not curled but $x, x^2$ and $x^3$ are linearly dependent for any $x \in A$, Otherwise, it is *straight*, that is there is, $x \in A$ such that $x, x^2$ and $x^3$ forms a basis of $A$.

(1) Suppose that $A$ is unital. Assume, moreover, that $\{1, h, h^2\}$ forms a basis of $A$ for some $h \in A$. Then, $h^3 = ah^2 + bh + c$ for some $a, b, c \in K$. Hence, $A$ is isomorphic to the quotient algebra of the polynomial algebra $K[X]$ by the ideal generated by $X^3 - aX^2 - bX - c$. So, a classical method can be applied ([4]). If there is not such $h$, we can find a basis $\{1, f, g\}$ of $A$ such that $f^2 = 0$ or $f^2 = 1$.

(2) Suppose that $A$ is curled. We divide cases depending on whether $\{e, f, ef\}$ or $\{e, f, fe\}$ forms a basis of $A$ for some $f, g \in A$ or not.

(3) Suppose that $A$ is straight, that is $\{h, h^2, h^3\}$ forms a basis of $A$ for some $h \in A$. We have $h^4 = ah^2 + bh^2 + ch$ for some $a, b, c \in A$. If $c \neq 0$, then $\frac{1}{c}(h^3 - sh^2 - bh)$ is the identity element and thus $A$ is unital. Otherwise, we have $h^4 = ah^3 + bh^2$.

(4) Suppose that $A$ is waved. We can choose $f \in A$ such that $\{f, f^2\}$ is linearly independent and $\{f, f^2, f^3\}$ is linearly dependent. Let $A'$ be the subalgebra of $A$ generated by $\{f, f^2\}$. We can take $g \in A$ such that $\{g, g^2\}$ is linearly independent and $\{f, f^2, g\}$ forms a basis of $A$. Let $A''$ be the subalgebra generated by $\{g, g^2\}$. Then, $A' \cap A$ is 1-dimensional algebra isomorphic to either the zero algebra $K_0$ or the unit algebra $K_1$. For subalgebras $A'$ and $A''$ we have 4 possibilities, four 2-dimensional algebras defined by

$$(a) \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, (b) \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, (c) \begin{pmatrix} e & f \\ f & e \end{pmatrix}, (d) \begin{pmatrix} e & f \\ f & 0 \end{pmatrix}.$$

We analyze 12 combinations (aa), (ab), ..., (dd) individually.

## 3 Classification

(1) We have 24 families of associative algebras of dimension 3 over $\mathbb{C}$, precisely 5 unital algebras $U_0, U_1, U_2, U_3, U_4$ defined on $E = \{e, f, g\}$ by the multiplication table

$$\begin{pmatrix} e & f & g \\ f & 0 & 0 \\ g & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & f & g \\ f & 0 & f \\ g & -f & e \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & f & g \\ 0 & g & 0 \end{pmatrix}, \begin{pmatrix} e & f & g \\ f & g & 0 \\ g & 0 & 0 \end{pmatrix},$$

5 non-unital curled algebras $C_0, C_1, C_2, C_3, C_4$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & -e & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ e & f & 0 \\ 0 & g & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e & f & g \end{pmatrix}, \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & 0 & g \end{pmatrix},$$

4 straight algebras $S_1, S_2, S_3, S_4$:

$$\begin{pmatrix} f & g & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} e & f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and 10 families of non-unital waved algebras $W_1$, $W_2$, $W_4$, $W_5$, $W_6$, $W_7$, $W_8$, $W_9$, $W_{10}$, an infinite family $\{W_3(k)\}_{k \in H}$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e & 0 \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f & g \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & g \end{pmatrix},$$
$$\begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f & g \end{pmatrix}, \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & g \end{pmatrix}, \begin{pmatrix} 0 & e & 0 \\ e & f & 0 \\ 0 & g & 0 \end{pmatrix}, \begin{pmatrix} 0 & e & 0 \\ e & f & g \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & ke & e \end{pmatrix},$$

where $H$ is the half plane $\{x + y\mathbf{i} \,|\, x > 0 \text{ or } x = 0, y \geq 0\} \subset \mathbb{C}$.

(2) Over $\mathbb{R}$, we add 3 families to the above list: A unital algebra $U_{2^-}$, a non-unital straight algebra $S_{3^-}$, non-unital waved algebras: $\{W_{3^-}(k)\}_{k \geq 0}$:

$$\begin{pmatrix} e & 0 & 0 \\ 0 & f & g \\ 0 & g & -f \end{pmatrix}, \begin{pmatrix} e & f & 0 \\ f & -e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & ke & -e \end{pmatrix}.$$

# 4 Remarks

Our classification theorem holds even when $K$ is a general algebraically close field of characteristic 0 and when $K$ is a real closed field, that is, $K(\sqrt{-1})$ is algebraically closed field of characteristic 0.

Let $S$ be a semigroup with zero element $z$. We call the quotient algebra of the semigroup algebra $KS$ by the ideal $Kz$ generated by $z$ a *pseudo-semigroup algebra*. It is interesting that over $\mathbb{C}$ all the algebras except for $U_1, C_1$ and $W_3(k)$ $(k \neq 0, 1)$ are pseudo-semigroup algebras.

# References


[1] Y. Kobayashi, K. Shirayanagi, M. Tsukada and S.-E. Takahasi, *A complete classification of three-dimensional algebras over* $\mathbb{C}$ *and* $\mathbb{R}$ (温故知新) Asian-European J. Math. **14** (2021).

[2] W.R. Hamilton, "Letter", *London, Edinburgh and Dublin Phil. M. and J. Science* **xxv** (1844), 489–495.

[3] B. Peirce, Linear associative algebra, *Amer. J. Math.* **4** (1881), 97–229.

[4] E. Study, Über systeme complexer zahlen und ihre anwendung in der theorie der transformationsgruppen, *Monatsh. Math. u. Phisik* **1** (1890), 283–354.